

K3 SURFACES WITH AN AUTOMORPHISM OF ORDER 11

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Abstract. This paper concerns K3 surfaces with automorphisms of order 11 in arbitrary characteristic. Specifically we study the wild case and prove that a generic such surface in characteristic 11 has Picard number 2. We also construct K3 surfaces with an automorphism of order 11 in every characteristic, and supersingular K3 surfaces whenever possible.

1. Introduction. Automorphisms of K3 surfaces have gained considerable attention in recent years. Over \mathbf{C} they usually lend themselves to lattice theoretic investigations while positive characteristic often requires more delicate considerations. Order 11 is special in the sense that by [6] it is the first prime order which over \mathbf{C} does not allow symplectic automorphisms (leaving the regular 2-form invariant). This result carries over to any positive characteristic except for $p = 11$. In fact, it is this wild case in which our primary interest lies. The wild case was studied in great detail in [4], and our main aim is to solve the problem of the Picard number which was left open there (see [4, Remark 3.6]):

THEOREM 1.1. *Let k denote an algebraically closed field of characteristic 11. Then a generic K3 surface with an automorphism of order 11 over k has Picard number $\rho = 2$.*

K3 surfaces with a wild automorphism of order 11 come in 1-dimensional families where each surface is a torsor of a jacobian elliptic K3 surface (see Section 2). This allows us to reduce the proof of Theorem 1.1 to the family of jacobian elliptic K3 surfaces with a wild automorphism of order 11 as studied in [4]. Then it suffices to exhibit a member of this family with Picard number $\rho = 2$. Our method to achieve this is arguably elementary as we simply compute the zeta functions of the elliptic K3 surfaces over \mathbf{F}_{11} in question. In contrast, the way to derive the zeta function in an effective fashion is quite non-elementary, as it makes essential use of the automorphisms. As a by-product we also compute the heights of the K3 surfaces over \mathbf{F}_{11} which attain the finite maximum $h = 10$ (Corollary 3.2).

The paper is organised as follows. After reviewing the background, particularly from [4], in Section 2, we give the proof of Theorem 1.1 in Section 3. The paper concludes with considerations for other characteristics in Section 4: Proposition 4.1 states that any characteristic p admits K3 surfaces with automorphisms of order 11, and whenever possible (i.e., $p = 11$ or there is some ν such that $p^\nu \equiv -1 \pmod{11}$), there are supersingular examples. This answers two questions from [4, Remark 6.6] in a surprisingly uniform manner.

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2. Background. Let k denote an algebraically closed field of characteristic $p = 11$. Let X be a K3 surface over k admitting an automorphism φ of order 11. Then by [4, Theorem 1.1] X admits an elliptic fibration that is compatible with φ . That is, there is an automorphism φ' of \mathbf{P}^1 such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{\varphi'} & \mathbf{P}^1. \end{array}$$

In fact, X is $\langle \varphi \rangle$ -equivariantly isomorphic to a torsor over a specific jacobian elliptic K3 surface X_ε which can be given as

$$(1) \quad X_\varepsilon : y^2 = x^3 + \varepsilon x^2 + t^{11} - t \quad (\varepsilon \in k).$$

The K3 surface X_ε admits an automorphism of order 11 given by

$$(2) \quad \varphi : (x, y, t) \mapsto (x, y, t + 1).$$

Generically the elliptic fibration over \mathbf{P}^1_t has the following singular fibres: a cuspidal cubic at $t = \infty$ and 22 nodal cubics at the roots of the affine discriminant

$$\Delta = -(t^{11} - t)(5t^{11} - 5t + 4\varepsilon^3).$$

Visibly, for $\varepsilon = 0$, the discriminant degenerates to a square, and the nodal cubics degenerate to cuspidal ones. In fact, the special member X_0 is isomorphic to the Fermat quartic over k . Here's a conceptual line of argument to see this which we will refer to again in the proof of Proposition 4.1. To start with, X_0 arises from the singular K3 surface

$$X : y^2 = x^3 + t^{11} - 11t^6 - t$$

over \mathbf{Q} by way of reduction. Here the term *singular* refers to the Picard number attaining the maximum of 20 over \mathbf{C} , and it was proved in [12] that X is in fact singular of discriminant -300 . In particular, this implies that X is modular, i.e., associated to X there is a modular form f with complex multiplication by $\mathbf{Q}(\sqrt{-3})$. By virtue of the vanishing of the Fourier coefficients of f , X has supersingular reduction at any prime $p_0 \equiv -1 \pmod 3$:

$$\rho(X \otimes \bar{\mathbf{F}}_{p_0}) = 22, \quad \text{and in particular, } \rho(X_0) = 22.$$

Then by [9, Proposition 1.0.1] any of the supersingular reductions has Artin invariant 1. Another such example is the Fermat quartic in any characteristic $p_1 \equiv -1 \pmod 4$. But then supersingular K3 surfaces with Artin invariant 1 are unique up to isomorphism over an algebraically closed field by [8]. Hence X_0 is isomorphic to the Fermat quartic over k .

Note how we derived in a rather indirect way the Picard number of X_0 . In comparison, the Picard number of X_ε does not seem to be known for $\varepsilon \neq 0$. In this paper, we will prove that generically

$$\rho(X_\varepsilon) = 2$$

by verifying the claim for all $\varepsilon \in \mathbf{F}_p^\times$. Our proof will make use of Lefschetz' fixed point formula for l -adic étale cohomology ($l \neq p$). In the remainder of this section, we review some basic results from [4] and set up the notation.

A key step in [4] is to show that

$$H_{\text{ét}}^2(X_\varepsilon, \mathbf{Q}_l)^{\varphi^*=1} = U \otimes \mathbf{Q}_l,$$

where U is the hyperbolic plane generated by general fibre F and zero section O . In particular, the orthogonal complement with respect to cup-product,

$$V = U^\perp \subset H_{\text{ét}}^2(X_\varepsilon, \mathbf{Q}_l),$$

decomposes into 10 equidimensional eigenspaces for φ^* . With ζ a primitive eleventh root of unity in $\bar{\mathbf{Q}}_l$, we obtain the 2-dimensional $\mathbf{Q}_l(\zeta)$ -vector spaces

$$V_i = H_{\text{ét}}^2(X_\varepsilon, \mathbf{Q}_l)^{\varphi^*=\zeta^i} \quad (i = 0, \dots, 10)$$

such that $V = \bigoplus_{i=1}^{10} V_i$ and $V_0 = U \otimes \mathbf{Q}_l$. This eigenspace decomposition can be used to show that $\rho(X_\varepsilon) = 2, 12$ or 22 . In fact, there is a simple argument to prove that $\rho \neq 12$ for infinitely many $\varepsilon \in k$. Namely X_ε admits a k -isomorphic model

$$(3) \quad X_\gamma : y^2 = x^3 + \gamma x + t^{11} - t$$

which comes equipped with the automorphism φ of order 11 as in (2) and with an automorphism ι of order 4:

$$(4) \quad \iota : (x, y, t) \mapsto (-x, \sqrt{-1}y, -t).$$

Since the automorphism ι does not commute with φ , its impact does not translate directly onto the cohomology eigenspaces V_i . There is, however, an indirect argument for any X_γ defined over some field \mathbf{F}_q with $q = p^r$, r odd. Namely, because of the symmetry encoded in (4), the pair $(x, t) \in \mathbf{F}_q^2$ gives a non-zero square in \mathbf{F}_q upon substituting in the right-hand side of (3) if and only if $(-x, -t)$ gives a non-square. That is to say, the numbers of \mathbf{F}_q -rational points on the fibres F_t and F_{-t} of X_γ add up to a constant number:

$$\#F_t(\mathbf{F}_q) + \#F_{-t}(\mathbf{F}_q) = 2q + 2 \quad (t \in \mathbf{P}^1(\mathbf{F}_q)).$$

As a consequence, we deduce that

$$\#X_\gamma(\mathbf{F}_q) = (q + 1)^2.$$

In comparison, Lefschetz' fixed point formula gives, with the decomposition $H_{\text{ét}}^2(\bar{X}_\varepsilon, \mathbf{Q}_l) = U \otimes \mathbf{Q}_l \oplus V$,

$$(5) \quad \#X(\mathbf{F}_q) = 1 + \text{tr Frob}_q^*(H_{\text{ét}}^2(\bar{X}_\varepsilon, \mathbf{Q}_l)) + q^2 = 1 + 2q + \text{tr Frob}_q^*(V) + q^2.$$

Thus we deduce that

$$(6) \quad \text{tr Frob}_q^*(V) = 0 \quad \text{for any } q = p^r, r \text{ odd}.$$

Denote by $\mu_q(T)$ the characteristic polynomial of Frob_q^* on V . Then (6) implies

$$\mu_q(T) = \nu_q(T^2), \quad \nu_q(T) \in \mathbf{Z}[T], \quad \text{deg}(\nu_q) = 10.$$

In fact we deduce right away that

$$\mu_{q^2}(T) = v_q(T)^2.$$

Now one can argue with the roots of this characteristic polynomial as in the following section and appeal to the Tate conjecture to rule out the alternative $\rho(X_\gamma) = 12$.

3. Proof of Theorem 1.1. Essentially our proof of Theorem 1.1 amounts to computing the characteristic polynomials μ_p of Frob_p^* on V for (any of) the surfaces $X = X_\gamma$ or X_ε for $\gamma, \varepsilon \in \mathbf{F}_p^\times$. Then we use that the Galois group always acts through a finite group on $\text{NS}(X)$, so that, via the cycle class map $\text{NS}(X) \otimes \mathbf{Q}_l(-1) \hookrightarrow H_{\text{ét}}^2(\bar{X}, \mathbf{Q}_l)$,

$$(7) \quad \rho(X) \leq \#\{\text{zeroes of } \mu_p(T) \text{ of the shape } \xi p \text{ for some root of unity } \xi\}.$$

Here equality is subject to the Tate conjecture [13] which holds for elliptic K3 surfaces by [2]. Without having to appeal to the Tate conjecture, we will find that the right-hand side of (7) gives an upper bound of 2 for $\rho(X)$, thus inferring $\rho(X) = 2$ by virtue of the classes of fibre and zero section generating U . This will enable us to prove Theorem 1.1 for the family of jacobian elliptic K3 surfaces with an automorphism of order 11. From this we will infer the full statement of Theorem 1.1 in 3.4.

3.1. Strategy. We shall now outline the strategy how to compute the characteristic polynomials of Frob_p^* quite easily. We will use Lefschetz' fixed point formula as in (5). Without further machinery, it would require to count points as deep as down to the field $\mathbf{F}_{p^{10}}$ to find $\mu_p(T)$. Presently we circumvent this by using the automorphism φ . The crucial property is that φ commutes with Frob_p . Hence the induced linear transformations on $H_{\text{ét}}^2(\bar{X}, \mathbf{Q}_l)$ can be diagonalised simultaneously, and we can define the relative trace of Frobenius:

$$a_i(q) = \text{tr} \text{Frob}_q^*(V_i) \quad (i = 0, \dots, 10).$$

Note that $a_0(q) = 2q$ always and $\text{tr} \text{Frob}_q^*(H_{\text{ét}}^2(\bar{X}, \mathbf{Q}_l)) = \sum_{i=0}^{10} a_i(q)$. Since φ and Frob_p commute, we have by definition

$$\text{tr}(\varphi^n \circ \text{Frob}_q)^*(V_i) = \zeta^{ni} a_i(q).$$

As a consequence, we find

$$(8) \quad \text{tr}_n(q) = \text{tr}(\varphi^n \circ \text{Frob}_q)^*(H_{\text{ét}}^2(\bar{X}, \mathbf{Q}_l)) = \sum_{i=0}^{10} \zeta^{ni} a_i(q) \quad (n = 0, \dots, 10).$$

Note that $\text{tr}_n(q) \in \mathbf{Z}$ for any $n \in \{0, \dots, 10\}$, and this integer can be computed by Lefschetz' fixed point formula again. Conversely, we can calculate each $a_i(q)$ from the collection of $\text{tr}_n(q)$'s.

We continue by discussing how to efficiently compute $\text{tr}_n(q)$. Lefschetz' fixed point formula gives

$$\#\text{Fix}(\varphi^n \circ \text{Frob}_q) = 1 + \text{tr}_n(q) + q^2.$$

For $n = 0$, the fixed locus is simply $X(\mathbf{F}_q)$. We shall now analyse the fixed locus for $n > 0$ and reduce the computations again to the field \mathbf{F}_q . Clearly $\text{Fix}(\varphi^n \circ \text{Frob}_q)$ contains $q + 1$

rational points on the zero section and at the fibre at ∞ exactly the F_q -rational points; these contribute $2q + 1$ rational points in total. It remains to study the fixed locus on the affine chart given in (1) or (3). Suppose that (x, y, t) is a fixed point of $\varphi^n \circ \text{Frob}_q$. Keeping in mind $n > 0$, the resulting conditions are first of all

$$x^q = x, \quad y^q = y, \quad t^q + n = t \implies x, y \in F_q, \quad t \notin F_q.$$

But then we infer from the defining equation (over F_q) that $t^p - t = c \in F_q$, and any such c occurs for exactly p fibres. Directly, this implies

$$(9) \quad -n = t^q - t = c + c^p + \dots + c^{p^{r-1}} = \text{tr}_{F_q/F_p} c.$$

We are now ready to compute all the quantities $\#\text{Fix}(\varphi^n \circ \text{Frob}_q)$ at once. Fix ε or γ and pick any $(x, y) \in F_q^2$. Then this determines $c = t^p - t$ by (1) or (3). Compute n by (9). Then the affine point $(x, y) \in F_q^2$ contributes exactly p -times to $\#\text{Fix}(\varphi^n \circ \text{Frob}_q)$, namely once for each fibre F_t at a root of $t^p - t = c$. Thus we can easily distribute all $(x, y) \in F_q^2$ over all the fixed loci $\text{Fix}(\varphi^n \circ \text{Frob}_q)$.

3.2. Execution. For $\varepsilon, \gamma \in F_p$, the above procedure enables us to calculate the characteristic polynomials of Frobenius on $H_{\text{ét}}^2(\bar{X}, \mathcal{Q}_l)$ solely from computations over F_p and F_{p^2} . Below, we tabulate the results for $V(1)$, i.e., we give the characteristic polynomial

$$\tilde{\mu}_p(T) = \mu_p(pT)/p^{20}$$

on the 20-dimensional vector space V with Galois representation shifted so that all eigenvalues have absolute value 1. Due to the shape of the equation, the characteristic polynomials only depend on the property whether respectively ε or γ is a square mod p or not.

ε	$\tilde{\mu}(T)$
1, 3, 4, 5, 9	$T^{20} + T^{19} + 2T^{18} + T^{17} + T^{16} + 2T^{15} + 3T^{14} + 5T^{13} + 4T^{12} + 3T^{11} + \frac{23}{11}T^{10} + 3T^9 + 4T^8 + 5T^7 + 2T^5 + 3T^6 + T^4 + T^3 + 2T^2 + T + 1$
2, 6, 7, 8, 10	$T^{20} - T^{19} + 2T^{18} - T^{17} + T^{16} - 2T^{15} + 3T^{14} - 5T^{13} + 4T^{12} - 3T^{11} + \frac{23}{11}T^{10} - 3T^9 + 4T^8 - 5T^7 - 2T^5 + 3T^6 + T^4 - T^3 + 2T^2 - T + 1$
γ	$\tilde{\mu}(T)$
1, 3, 4, 5, 9	$1 - 4T^2 + 10T^4 - 18T^6 + 25T^8 - \frac{307}{11}T^{10} + 25T^{12} - 18T^{14} + 10T^{16} - 4T^{18} + T^{20}$
2, 6, 7, 8, 10	$1 - 5T^2 + 12T^4 - 18T^6 + 20T^8 - \frac{219}{11}T^{10} + 20T^{12} - 18T^{14} + 12T^{16} - 5T^{18} + T^{20}$

All four characteristic polynomials are irreducible over \mathcal{Q} . Since they are not integral, they are not cyclotomic. Hence there are no roots of unity as zeroes, and we deduce that $\rho(X) = 2$ by (7). This proves Theorem 1.1 for the family $\{X_\varepsilon\}$ (or equivalently $\{X_\gamma\}$), i.e.,

for those K3 surfaces with a jacobian elliptic fibration compatible with the automorphism of order 11.

3.3. Easy consequences. Before continuing with the proof of Theorem 1.1 for all K3 surfaces, we note two corollaries. The first is straight-forward:

COROLLARY 3.1. *Any elliptic K3 surface over \mathcal{C} reducing mod p to some X_ε or X_γ for $\varepsilon, \gamma \in \mathbf{F}_p^\times$ has Picard number 2.*

For the second corollary, we recall the notion of height for a K3 surface X in positive characteristic. Generally this is defined in terms of the formal Brauer group $\widehat{\text{Br}}(X)$ and ranges through $\{1, 2, \dots, 10\}$ and ∞ where the last case was coined *supersingular* by Artin [1]. Note that in case of finite height h , there is an inequality

$$\rho \leq b_2 - 2h.$$

If the K3 surface is defined over some finite field, then the height translates into arithmetic through the Newton polygon of the characteristic polynomial of Frobenius. This collects the π -adic valuations of all eigenvalues at some prime π above p in the splitting field of the characteristic polynomial. For instance, height 1 corresponds to the so-called *ordinary* case where Hodge and Newton polygon coincide, i.e., some eigenvalue is a π -adic unit. Equivalently, $\text{tr} \text{Frob}_q^*(H_{\text{ét}}^2(\bar{X}, \mathbf{Q}_l)) \not\equiv 0 \pmod{p}$. Presently we find that all pairs of complex conjugate eigenvalues on the 20-dimensional subspace $V \subset H_{\text{ét}}^2(\bar{X}, \mathbf{Q}_l)$ have evaluations 9/10, 11/10. Hence we get the following corollary:

COROLLARY 3.2. *The K3 surfaces X_ε and X_γ for $\varepsilon, \gamma \in \mathbf{F}_p^\times$ have height 10.*

It is instructive to note that while our K3 surfaces are not supersingular (in either sense, $h = \infty$ or $\rho = 22$), in terms of height they are as close to being supersingular as possible. We should also point out that we are not aware of any other explicit examples of K3 surfaces of height 10.

3.4. Proof of Theorem 1.1. Recall that K3 surfaces with an automorphism φ of order 11 come in families consisting of torsors over the family $\{X_\varepsilon\}$. In fact, any such K3 surface X comes with an elliptic fibration whose jacobian $\text{Jac}(X)$ equals X_ε for some $\varepsilon \in k$. Generally one has by [3, Corollary 5.3.5]

$$\rho(X) = \rho(\text{Jac}(X)).$$

Hence the result from 3.2 that a generic member of the family $\{X_\varepsilon\}$ has Picard number 2 carries over directly to all other families. This proves Theorem 1.1.

3.5. Remark. We should like to emphasise the conceptual difference to the standard case of symplectic automorphisms over \mathcal{C} . Given a finite group G of automorphisms acting symplectically on a complex K3 surface X , Nikulin proved in [6] that there is an abstract lattice Ω_G of fairly large rank such that

$$(H^2(X, \mathbf{Z})^G)^\perp = \Omega_G.$$

Since the transcendental lattice $T(X)$ is contained in $H^2(X, \mathbf{Z})^G$ by assumption, we deduce that X has fairly large Picard number. In contrast, symplectic automorphisms of order 11 (necessarily wild) imply generally $\rho = 2$ as we have seen.

4. Other characteristics. In [4, Remark 6.6], the authors give an elliptic K3 surface over \mathcal{Q} with a non-symplectic automorphism of order 11 and good reduction outside $\{2, 3, 11\}$. They ask whether there is a K3 surface with an automorphism of order 11 in characteristic 2 and 3 as well. Moreover, they pose the problem whether there is a supersingular example in any possible characteristic, i.e., by [7], whenever there is some v such that $p^v \equiv -1 \pmod{11}$. Equivalently p is a non-square in \mathbf{F}_{11} . Here we present a uniform answer in terms of the extended Weierstrass form

$$(10) \quad y^2 + xy = x^3 + t^{11}.$$

PROPOSITION 4.1. *Let k be an algebraically closed field and X the elliptic K3 surface given by the Weierstrass form (10) over k . Then X admits an automorphism of order 11. Moreover, X is supersingular with $\rho(X) = 22$ whenever $\text{char}(k)$ either equals 11 or is a non-square modulo 11.*

PROOF. It is easily computed that (10) gives indeed an elliptic K3 surface X of discriminant $t^{11}(432t^{11} + 1)$. There are singular fibres of Kodaira type II at ∞ , I_{11} at 0 and, depending on $p = \text{char}(k)$,

- 11 fibres of type I_1 at the roots of $432t^{11} + 1$ if $p \neq 2, 3, 11$;
- 1 fibre of type I_{11} at $t = 7$ if $p = 11$;
- no other singular fibres if $p = 2, 3$ as there is wild ramification of index 11 at ∞ .

On the one hand, if $p \neq 11$, then the automorphism of order 11 thus is given by $t \mapsto \zeta t$ for a primitive 11th root of unity $\zeta \in k$. On the other hand, if $p = 11$, then we read off $\text{NS}(X) = U + A_{10}^2$. In particular, $\rho(X) = 22$ and X is a supersingular K3 surface of Artin invariant $\sigma = 1$. As such, it is unique up to isomorphism by [8]. As in Section 2, X is therefore isomorphic to X_0 from (1). We conclude that X admits an automorphism of order 11.

To prove Proposition 4.1 it remains to justify the claim about supersingularity for non-squares p modulo 11. For this purpose, we note that X is a Delsarte surface covered by the Fermat surface S of degree 11. Writing affinely

$$S = \{u^{11} + v^{11} + w^{11} + 1 = 0\} \subset \mathbf{P}^3,$$

the dominant rational map is given by

$$\begin{aligned} S & \dashrightarrow X \\ (u, v, w) & \mapsto (x, y, t) = (-u^{11}v^{11}, -u^{22}v^{11}, -wu^3v^2). \end{aligned}$$

Since S is unirational over $\bar{\mathbf{F}}_p$ for any non-square p modulo 11 by [5], so is X . Thus the supersingularity of X follows from Shioda’s theorem [10]. □

REMARK 4.2. For $\text{char}(k) = 0$ or a non-zero square modulo 11, the argument with the covering Fermat surface can be used to show that $\rho(X) = 2$ by [11].

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