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ORBITS, RINGS OF INVARIANTS AND WEYL GROUPS FOR CLASSICAL Θ-GROUPS

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Abstract. In this paper, we study the invariant theory of Viberg's Θ-groups in classical cases. For a classical Θ-group naturally contained in a general linear group, we show the restriction map, from the ring of invariants of the Lie algebra of the general linear group to that of the Θ-representation defined by the Θ-group, is surjective. As a consequence, we obtain explicitly algebraically independent generators of the ring of invariants of the Θ-representation. We also give a description of the Weyl groups of the classical Θ-groups.

0. Introduction. In this paper, we study the invariant theory of Viberg's Θ-groups. To be precise, let G be a complex reductive algebraic group with Lie algebra $\mathfrak{g}$ and $\theta : G \to G$ an automorphism of order m. We also denote by $\theta : \mathfrak{g} \to \mathfrak{g}$ the Lie algebra automorphism defined by θ. Let $\mathfrak{g}_1$ be the eigenspace of θ with eigenvalue $e^{2\pi\sqrt{-1}/m}$. Then the isotropy subgroup $G_0 := G^\theta$ acts on $\mathfrak{g}_1$ by the adjoint action. We call (G, θ) a Θ-group of order m and $(G_0, \mathfrak{g}_1)$ the Θ-representation defined by (G, θ). If G is $GL(V)$, $O(V)$ or $Sp(V)$ and $\theta : G \to G$ is an automorphism of classical type, we call $(G_0, \mathfrak{g}_1)$ a classical Θ-representation. Here we call that θ is of classical type if θ is an inner automorphism of G or an outer automorphism of $G = GL(V)$. By the fact that the automorphism group of a simple Lie algebra is a semidirect product of the inner automorphism group and the automorphism group of the Dynkin diagram, we know that a finite order automorphism of non-classical type exists only for $G = O(V)$ with $\dim V = 8$. We call $(G_0, \mathfrak{g}_1)$ a Θ-representation of type (A-I) (resp. (BCD-I)) if $G = GL(V)$ (resp. $G = O(V),\ Sp(V)$) and θ is an inner automorphism. If $G = GL(V)$ and θ is an outer automorphism, we call $(G_0, \mathfrak{g}_1)$ a Θ-representation of type (A-O).

For a classical symmetric pair (G, K) with (-1)-eigenspace $\mathfrak{p}$ (a Θ-representation of order 2), it is known by Helgason and other mathematicians, that the restriction map rest : $\boldsymbol{C}[\mathfrak{g}]^G \to \boldsymbol{C}[\mathfrak{p}]^K$ is surjective (cf. [H]). It is also mentioned in [H] that the restriction map is not surjective for four cases of type E.

In [Pa], Panyushev also give a similar results for N-regular Θ-representations. That is, for an N-regular Θ-representation $(G_0, \mathfrak{g}_1)$, the restriction map rest : $\boldsymbol{C}[\mathfrak{g}]^G \to \boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ is surjective. Here a Θ-representation $(G_0, \mathfrak{g}_1)$ is called N-regular if the regular nilpotent G-orbit in $\mathfrak{g}$ meets $\mathfrak{g}_1$.

Suppose that a reductive group $\tilde{H} \subset GL(V)$ acts on a vector subspace $\tilde{L} \subset \mathfrak{gl}(V)$ by the adjoint action, and a reductive subgroup H of $\tilde{H}$ acts on a subspace L of $\tilde{L}$. In [O3], based

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on the theory of Luna [L], we studied a sufficient condition on $(H, L) \hookrightarrow (\tilde{H}, \tilde{L})$ for the restriction map rest : $\boldsymbol{C}[\tilde{L}]^{\tilde{H}} \to \boldsymbol{C}[L]^H$ to be surjective. The purpose of this paper is to prove the following theorem by applying the above results of [O3] to a classical Θ-representation $(G_0, \mathfrak{g}_1)$ included in $(GL(V), \mathfrak{gl}(V))$.

THEOREM 0.1. *For a classical Θ-representation $(G_0, \mathfrak{g}_1)$ naturally included in $(GL(V), \mathfrak{gl}(V))$, the restriction map*

$$\text{rest} : \boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)} \to \boldsymbol{C}[\mathfrak{g}_1]^{G_0}, \quad f \mapsto f|_{\mathfrak{g}_1} \tag{0.1}$$

is surjective.

We also determine algebraically independent generators of $\boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ explicitly. Since the map $\boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)} \to \boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ decomposes as

$$\boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)} \to \boldsymbol{C}[\mathfrak{g}]^G \to \boldsymbol{C}[\mathfrak{g}_1]^{G_0},$$

we know that the restriction map $\boldsymbol{C}[\mathfrak{g}]^G \to \boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ is also surjective. Thus we obtain the following generalization of the surjectivity which is known for classical symmetric pairs and N-regular Θ-representations.

COROLLARY 0.2. *For any classical Θ-representation $(G_0, \mathfrak{g}_1)$, the restriction map $\boldsymbol{C}[\mathfrak{g}]^G \to \boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ is surjective.*

Based on [O3], the surjectivity of the map (0.1) is proved by using the fact that the map

$$\mathfrak{g}_1^{ss}/G_0 \to \mathfrak{gl}(V)^{ss}/GL(V), \quad \mathcal{O} \mapsto \mathrm{Ad}(GL(V)) \cdot \mathcal{O}, \tag{0.2}$$

from the set of semisimple orbits in $\mathfrak{g}_1$ to that in $\mathfrak{gl}(V)$, is injective. The injectivity of the map (0.2) is shown in Sections 2 and 3, with the proof based on a classification of semisimple G_0-orbits.

The injectivity of the map (0.2) can be used not only for showing the surjectivity of the map (0.1), but also for computation of the Weyl groups of Θ-representations.

In [V], Vinberg introduced the notions of Cartan subspaces and Weyl groups of Θ-representations $(G_0, \mathfrak{g}_1)$ and determined them for classical Θ-representations.

Let $\mathfrak{c} \subset \mathfrak{g}_1$ be a Cartan subspace of $\mathfrak{g}_1$, i.e., a maximal abelian subspace of $\mathfrak{g}_1$ which consists of semisimple elements. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{gl}(V)$ which contains $\mathfrak{c}$. Let us consider the following groups:

$$W(G_0, \mathfrak{c}) = N_{G_0}(\mathfrak{c})/Z_{G_0}(\mathfrak{c}) \subset GL(\mathfrak{c}), \quad W = N_{GL(V)}(\mathfrak{t})/Z_{GL(V)}(\mathfrak{t}).$$

Here the former is called the Weyl group of the Θ-representation $(G_0, \mathfrak{g}_1)$ and the latter is the Weyl group of $(GL(V), \mathfrak{gl}(V))$ isomorphic to the symmetric group of degree $\dim V$. Then $W(G_0, \mathfrak{c})$ is naturally identified with a subgroup of $N_W(\mathfrak{c})|_{\mathfrak{c}}$. The injectivity of the map (0.2) simplifies the computation of $W(G_0, \mathfrak{c})$, since it implies $W(G_0, \mathfrak{c}) = N_W(\mathfrak{c})|_{\mathfrak{c}}$. Thus we can compute $W(G_0, \mathfrak{c})$ as the normalizer of $\mathfrak{c}$ in the symmetric group W. As a consequence, we know that the Weyl group $W(G_0, \mathfrak{c})$ is isomorphic to the complex reflection group $G(k, 1, r)$ (in the notation of [ST]), where $r = \dim \mathfrak{c}$ and k is a number which depends on the Θ-representation $(G_0, \mathfrak{g}_1)$. Vinberg already computed the Weyl groups of classical

Θ-representations under the setting that $G = SL(V)$, $SO(V)$ or $Sp(V)$ and $G_0 = (G^\theta)^0$ (the identity component of G^θ). In some cases in types (BD-I) and (A-O), his Weyl groups are $G(k, 2, r)$ (cf. [ST]). Since our method of computation is different from that of Vinberg, the author thinks that there is some meaning to present a computation of the Weyl groups by the method which use the injectivity of the correspondence of semisimple orbits.

Now we are going to explain the contents of this paper briefly.

In Section 1, we see that any Θ-representation $(G_0, \mathfrak{g}_1)$ of type (BCD-I) or (A-O) is naturally contained in a Θ-representation $(\tilde{G}_0, \tilde{\mathfrak{g}}_1)$ of type (A-I) (cf. (1.1)) and show that the map $\mathfrak{g}_1/G_0 \to \tilde{\mathfrak{g}}_1/\tilde{G}_0$ of adjoint orbits is injective. By [O3], we know that $\boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ is the integral closure of $\boldsymbol{C}[\tilde{\mathfrak{g}}_1]^{\tilde{G}_0}|_{\mathfrak{g}_1}$ in its quotient field.

In Section 2, we give a classification of general orbits of the Θ-representation $(\tilde{G}_0, \tilde{\mathfrak{g}}_1) \hookrightarrow (GL(V), \mathfrak{gl}(V))$ of type (A-I) by means of $\boldsymbol{Z}_m$-labeled Young diagrams with eigenvalues. The classification of nilpotent orbits of Θ-representations of type (A-I) was given in Kempken [Ke] by using $\boldsymbol{Z}_m$-labeled Young diagrams (called "words" in [Ke]). $\boldsymbol{Z}_m$-labeled Young diagrams with eigenvalues are a generalization of $\boldsymbol{Z}_m$-labeled Young diagrams. By using this classification, we know that the map $\tilde{\mathfrak{g}}_1^{\mathrm{ss}}/\tilde{G}_0 \to \mathfrak{gl}(V)^{\mathrm{ss}}/GL(V)$ between the sets of semisimple orbits is injective. We also know, by the inclusion $\mathfrak{g}_1/G_0 \hookrightarrow \tilde{\mathfrak{g}}_1/\tilde{G}_0$, that general orbits of Θ-representations of types (BCD-I) and (A-O) can be classified by $\boldsymbol{Z}_m$-labeled Young diagrams with eigenvalues and that the map (0.2) is injective.

In Section 3, we give a classification of semisimple orbits of Θ-representations of types (BCD-I) and (A-O) as a preparation of Section 4.

In Section 4, we first show the surjectivity of the map (0.1) by using the injectivity of the map (0.2) for a Θ-representation of type (A-I). From the fact that $\boldsymbol{C}[\tilde{\mathfrak{g}}_1]^{\tilde{G}_0} = \boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\tilde{\mathfrak{g}}_1}$, we know that $\boldsymbol{C}[\tilde{\mathfrak{g}}_1]^{\tilde{G}_0}|_{\mathfrak{g}_1} = \boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\mathfrak{g}_1}$ for a Θ-representation $(G_0, \mathfrak{g}_1)$ of type (BCD-I) or (A-O). By using the classification of semisimple orbits in Section 3, we know that the ring $\boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\mathfrak{g}_1}$ is a polynomial ring. Since $\boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ is the integral closure of $\boldsymbol{C}[\tilde{\mathfrak{g}}_1]^{\tilde{G}_0}|_{\mathfrak{g}_1}$, we have

$$\boldsymbol{C}[\mathfrak{g}_1]^{G_0} = \boldsymbol{C}[\tilde{\mathfrak{g}}_1]^{\tilde{G}_0}|_{\mathfrak{g}_1} = \boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\mathfrak{g}_1}\,,$$

and the surjectivity of the restriction map (0.1) is shown for a Θ-representation of type (BCD-I) or (A-O).

In Section 5, we determine the Weyl groups of classical Θ-representations.

1. Inclusion theorem for orbits in the classical Θ-representations. Let G be a complex reductive algebraic group with the Lie algebra $\mathfrak{g}$ and m a positive integer. Let $\theta : G \to G$ be an automorphism of G such that $\theta^m = \mathrm{id}_G$ and $\theta^k \neq \mathrm{id}_G$ $(1 \leq k < m)$. We write $\theta : \mathfrak{g} \to \mathfrak{g}$ the induced automorphism. We put $\zeta := e^{2\pi\sqrt{-1}/m}$,

$$G_0 = \{g \in G\,;\, \theta(g) = g\} \text{ and } \mathfrak{g}_j := \{X \in \mathfrak{g}\,;\, \theta(X) = \zeta^j X\}\ (j \in \boldsymbol{Z}_m = \boldsymbol{Z}/m\boldsymbol{Z})\,.$$

Then $\mathfrak{g}$ is decomposed as

$$\mathfrak{g} = \oplus_{j \in \boldsymbol{Z}_m} \mathfrak{g}_j$$

and we obtain a $\boldsymbol{Z}_m$-graded Lie algebra. We call the pair (G,θ) a Θ-group of order m. For each $j \in \boldsymbol{Z}_m$, the isotropy group G_0 acts on $\mathfrak{g}_j$ by the adjoint action. In this paper, we mainly consider the adjoint representation $(G_0, \mathfrak{g}_1)$ of G_0 on $\mathfrak{g}_1$ and call it the Θ-representation defined by (G,θ).

(1.1) Classical Θ-representations. In this paper, we call the following Θ-representations, defined by finite order automorphisms of $GL(V)$, $O(V)$ or $Sp(V)$, classical Θ-representations.

Type (A-I). Let V be a finite dimensional vector space over $\boldsymbol{C}$ and $S \in GL(V)$ a linear transformation of V such that $S^m = \mathrm{id}_V$ and $\mathrm{Ad}(S^k) \neq \mathrm{id}_{GL(V)}$ for any $1 \leq k \leq m-1$. We call such a transformation S an m-automorphism of V and such a pair (V, S) a vector space with m-automorphism.

For a vector space (V, S) with m-automorphism, by putting $G = GL(V)$ and $\theta(g) = SgS^{-1}$ $(g \in G)$, we obtain a Θ-group (G,θ) of order m. We call (G,θ) the Θ-group of type (A-I) defined by (V, S), since θ is an inner automorphism of a group $G = GL(V)$ of type A. Also, we call the corresponding $(G_0, \mathfrak{g}_1)$ a Θ-representation of type (A-I).

Type (BCD-I). Let V be a finite dimensional vector space over $\boldsymbol{C}$ and $(\ ,\ )$ a non-degenerate ε-symmetric form on V, where $\varepsilon = \pm 1$. An ε-symmetric form means a bilinear form such that $(u, v) = \varepsilon(v, u)$ $(u, v \in V)$. For $X \in \mathrm{End}(V)$, we denote by X^* the adjoint of X with respect to $(\ ,\ )$. Put

$$G := \{g \in GL(V);\, g^* = g^{-1}\} = \begin{cases} O(V) & (\varepsilon = 1) \\ Sp(V) & (\varepsilon = -1)\,. \end{cases}$$

Let $a \in G$ be an element of G such that the automorphism $\theta : G \to G$ defined by $\theta(g) = aga^{-1}$ $(g \in G)$ has finite order m. Then we easily see that $a^m = \pm\mathrm{id}_V$. We put $\zeta = e^{2\pi\sqrt{-1}/m}$ and $\xi = e^{\pi\sqrt{-1}/m}$. Let us define $\omega \in \{0, 1\}$ and $S \in GL(V)$ by

$$\omega = \begin{cases} 0 & (a^m = \mathrm{id}_V) \\ 1 & (a^m = -\mathrm{id}_V)\,, \end{cases} \qquad S := \xi^\omega a\,.$$

Then we see easily the following.

LEMMA 1.1. (i) $S^m = \mathrm{id}_V$ *and* $\theta(g) = SgS^{-1}$ $(g \in GL(V))$.
(ii) $S^* = \zeta^\omega S^{-1}$, *in particular* $(Su, Sv) = \zeta^\omega(u, v)$ $(u, v \in V)$.

DEFINITION 1.2. (i) For $(\varepsilon, \omega) \in \{\pm 1\} \times \{0, 1\}$ and a positive integer m, if a triple $(V, (\ ,\ ), S)$ consisting of a finite dimensional vector space V, a non-degenerate ε-symmetric form $(\ ,\ )$ on V and $S \in GL(V)$ satisfies the following conditions (a) and (b), we call $(V, (\ ,\ ), S)$ an (ε, ω)-space with m-automorphism:
(a) $S^m = \mathrm{id}_V$ and $\mathrm{Ad}(S^k) \neq \mathrm{id}_G$ $(1 \leq k \leq m-1)$.
(b) $S^* = \zeta^\omega S^{-1}$.
This notion is a generalization of (ε, ω)-spaces in [O1], which define symmetric pairs of type B, C, and D.

(ii) For the above $(V, (\ ,\ ), S)$, by putting $G := \{g \in GL(V);\, g^* = g^{-1}\}$ and defining $\theta : G \to G$ by $\theta(g) = SgS^{-1}$, we obtain a Θ-group (G,θ). We call it the Θ-group of type

(BCD-I) defined by the (ε, ω)-space $(V, (\ ,\ ), S)$ with m-automorphism, since θ is an inner automorphism of a group G of type B, C or D.

(iii) For the Θ-group (G, θ) defined by $(V, (\ ,\ ), S)$, we call $(GL(V), \mathrm{Ad}(S))$ the associated Θ-group of type (A-I).

Type (A-O). Let V be a finite dimensional vector space over $\boldsymbol{C}$ and $\langle\ ,\ \rangle$ a non-degenerate bilinear form on V. For $X \in \mathrm{End}(V)$, we denote by X^* the adjoint of X with respect to the bilinear form $\langle\ ,\ \rangle$ defined by $\langle Xu, v\rangle = \langle u, X^*v\rangle$ $(u, v \in V)$. We put $G = GL(V)$ and consider the automorphism $\theta : G \to G$ defined by $\theta(g) = (g^*)^{-1}$.

LEMMA 1.3. *Define an element* $a \in GL(V)$ *by* $\langle u, v\rangle = \langle v, au\rangle$ $(u, v \in V)$. *Then we have the following.*

(i) $a^* = a^{-1}$.

(ii) $\theta^2(g) = aga^{-1}$ $(g \in G)$. *In particular*, θ *has finite order if and only if so does* $\mathrm{Ad}(a) : G \to G$.

(iii) *If* $\mathrm{Ad}(a)$ *has finite order* m, *then* $a^m = \pm\mathrm{id}_V$.

PROOF. (i) Since $\langle u, v\rangle = \langle v, au\rangle = \langle au, av\rangle$ $(u, v \in V)$, we have $a^* = a^{-1}$.

(ii) For $X \in \mathrm{End}(V)$, we see

$$\langle u, (X^*)^*v\rangle = \langle X^*u, v\rangle = \langle v, aX^*u\rangle = \langle Xa^{-1}v, u\rangle = \langle u, aXa^{-1}u\rangle$$

and hence $(X^*)^* = aXa^{-1}$. In particular, we have $\theta^2(g) = [\{(g^*)^{-1}\}^*]^{-1} = (g^*)^* = aga^{-1}$. Thus (ii) holds.

(iii) Since a^m is a scalar matrix, we put $a^m = c\ \mathrm{id}_V$ $(c \in \boldsymbol{C}^\times)$. Then $c\ \mathrm{id}_V = (c\ \mathrm{id}_V)^* = (a^m)^* = a^{-m} = (c\ \mathrm{id}_V)^{-1} = c^{-1}\mathrm{id}_V$ and we have $c^2 = 1$. □

As before, we define $\omega \in \{0, 1\}$ and $S \in GL(V)$ by

$$\omega = \begin{cases} 0 & (a^m = \mathrm{id}_V) \\ 1 & (a^m = -\mathrm{id}_V)\,, \end{cases} \qquad S := \xi^\omega a\,.$$

Then we easily see the following.

LEMMA 1.4. (i) $S^m = \mathrm{id}_V$ *and* $\theta^2(g) = SgS^{-1}$ $(g \in G)$.

(ii) $\langle u, v\rangle = \xi^{-\omega}\langle v, Su\rangle$ $(u, v \in V)$.

(iii) $S^* = \zeta^\omega S^{-1}$.

DEFINITION 1.5. (i) Let ω be an element of $\{0, 1\}$ and m a positive integer. A pair $(V, \langle\ ,\ \rangle)$ of a finite dimensional vector space V and a non-degenerate bilinear form $\langle\ ,\ \rangle$ on V is called a vector space with (ω, m)-bilinear form, if there exists an element $S \in GL(V)$ satisfying the following conditions (a) and (b).

(a) $\langle u, v\rangle = \xi^{-\omega}\langle v, Su\rangle$ $(u, v \in V)$.

(b) $(X^*)^* = SXS^{-1}$ $(X \in \mathrm{End}(V))$, $S^m = \mathrm{id}_V$ and $\mathrm{Ad}(S^k) \neq \mathrm{id}_{GL(V)}$ $(1 \leq k \leq m-1)$.

We call S the (ω, m)-automorphism of V corresponding to $(V, \langle\ ,\ \rangle)$.

(ii) For the above $(V, \langle\ ,\ \rangle)$, by defining $G := GL(V)$ and $\theta : G \to G$ by $\theta(g) = (g^*)^{-1}$, we obtain a Θ-group (G, θ) of order $2m$. We call this the Θ-group of type (A-O) defined by the vector space $(V, \langle\ ,\ \rangle)$ with (ω, m)-bilinear form, since θ is an outer automorphism of a group $G = GL(V)$ of type A.

(iii) Let (G, θ) be a Θ-group of type (A-O). Then $\theta^2 = \mathrm{Ad}(S)$ for the above S, and (G, θ^2) is called the associated Θ-group of type (A-I). If (G, θ) is of order $2m$, then (G, θ^2) is of order m.

REMARK 1.6. (i) Let (G, θ) be one of the above Θ-groups and put $H := \{g \in G; \det(g) = 1\}$, $\mathfrak{h} := \mathrm{Lie}(H)$. In [V], Vinberg called (H, θ) the classical Θ-group and studied the adjoint action $((H^\theta)^0, \mathfrak{h}_1)$, where $(H^\theta)^0$ is the identity component of H^θ. But from the viewpoint of giving a classification of orbits and the ring of invariants in a unified manner, we call (G, θ) the classical Θ-group and study it.

(ii) For the above H, any finite order automorphism of $\mathfrak{h}$ can be obtained as θ which we have described above, except for automorphisms of $\mathfrak{so}(8, \boldsymbol{C})$ coming from the automorphism of the Dynkin diagram of order 3.

(1.2) Embedding of orbits into those in a Θ-representation of type (A-I). We conclude this section with showing that the set of G_0-orbits of a Θ-representation of type (BCD-I) or (A-O) can be embedded injectively to those of a Θ-representation of type (A-I). We first treat a Θ-representation of type (BCD-I).

Let (G, θ) be a Θ-group of type (BCD-I) defined by an (ε, ω)-space $(V, (\ ,\ ), S)$ with m-automorphism (cf. Definition 1.2), and $(\tilde{G}, \theta) = (GL(V), \mathrm{Ad}(S))$ the associated Θ-group of type (A-I). We put $\zeta = e^{2\pi\sqrt{-1}/m}$ and write X^* the adjoint of $X \in \mathrm{End}(V)$ with respect to $(\ ,\ )$. Thus we obtain a $\boldsymbol{C}$-linear anti-automorphism $\sigma : \mathrm{End}(V) \to \mathrm{End}(V)$ defined by $\sigma(X) := X^*$. Then $\tilde{G}_0, G_0, \tilde{\mathfrak{g}}_j, \mathfrak{g}_j$ $(j \in \boldsymbol{Z}_m)$ can be written as

$$\tilde{G}_0 = \{g \in GL(V); SgS^{-1} = g\}, \quad G_0 = \{g \in \tilde{G}_0; \sigma(g) = g^{-1}\},$$

$$\tilde{\mathfrak{g}}_j = \{X \in \mathrm{End}(V); SXS^{-1} = \zeta^j X\}, \quad \mathfrak{g}_j = \{X \in \tilde{\mathfrak{g}}_j; \sigma(X) = -X\}.$$

We have the following.

PROPOSITION 1.7. *For any $j \in \boldsymbol{Z}_m$, the map*

$$\mathfrak{g}_j/G_0 \to \tilde{\mathfrak{g}}_j/\tilde{G}_0, \ \mathcal{O} \mapsto \mathrm{Ad}(\tilde{G}_0) \cdot \mathcal{O}$$

is injective.

The proof is given by applying the following proposition to the case when $\tilde{H} = \tilde{G}_0$, $H = G_0$, $\tilde{L} = \tilde{\mathfrak{g}}_j$, $L = \mathfrak{g}_j$ and $\alpha(X) = -X$ $(X \in \tilde{L})$.

PROPOSITION 1.8 ([O3, Theorem 1]). *Let V be a finite dimensional vector space over $\boldsymbol{C}$ and $\sigma : \mathrm{End}(V) \to \mathrm{End}(V)$ a $\boldsymbol{C}$-linear anti-automorphism of the associative algebra. Let $\tilde{H}$ be a subgroup of $GL(V)$ such that*

(a) $\langle \tilde{H} \rangle_{\boldsymbol{C}} \cap GL(V) = \tilde{H}$, *where $\langle \tilde{H} \rangle_{\boldsymbol{C}}$ denotes the subspace of* $\mathrm{End}(V)$ *spanned by $\tilde{H}$.*

(b) $\sigma(\tilde{H}) = \tilde{H}$ *and $\sigma^2|_{\tilde{H}} = \mathrm{id}_{\tilde{H}}$.*

Let $\tilde{L}$ be an $\mathrm{Ad}(\tilde{H})$-stable and σ-stable subspace of $\mathrm{End}(V)$, and α an element of $GL(\tilde{L})$ such that $\alpha(\mathrm{Ad}(g)X) = \mathrm{Ad}(g)\alpha(X)$ for any $g \in \tilde{H}$ and $X \in \tilde{L}$, i.e., $\alpha \in Z_{GL(\tilde{L})}(\mathrm{Ad}_{\tilde{L}}(\tilde{H}))$. Define a subgroup $H := \{g \in \tilde{H}; \sigma(g) = g^{-1}\}$ of $\tilde{H}$ and a subspace $L := \{X \in \tilde{L}; \sigma(X) = \alpha(X)\}$ of $\tilde{L}$. Then the map $L/H \to \tilde{L}/\tilde{H}$ of adjoint orbits defined by $\mathcal{O} \mapsto \tilde{\mathcal{O}} := \mathrm{Ad}(\tilde{H}) \cdot \mathcal{O}$ is injective.

Next we consider a Θ-group of type (A-O). Let (G, θ) be a Θ-group of order $2m$ of type (A-O) defined by a vector space $(V, \langle\ ,\ \rangle)$ with (ω, m)-bilinear form, and S the (ω, m)-automorphism of V corresponding to $(V, \langle\ ,\ \rangle)$ (cf. Definition 1.5). Let $(\tilde{G}, \theta^2) = (GL(V), \mathrm{Ad}(S))$ the associated Θ-group of order m of type (A-I). We put $\xi = e^{\pi\sqrt{-1}/m}$, $\zeta = \xi^2 = e^{2\pi\sqrt{-1}/m}$. We note that $\sigma : \mathrm{End}(V) \to \mathrm{End}(V)$ defined by $\sigma(X) := X^*$ is a $\boldsymbol{C}$-linear anti-automorphism. Then $\tilde{G}_0$, G_0, $\tilde{\mathfrak{g}}_i$, $\mathfrak{g}_j$ can be written as

$$\tilde{G}_0 = \{g \in GL(V); SgS^{-1} = g\}, \quad G_0 = \{g \in \tilde{G}_0; \theta(g) = g\ (\Leftrightarrow g^* = g^{-1})\},$$

$$\tilde{\mathfrak{g}}_i = \{X \in \mathrm{End}(V); SXS^{-1} = \zeta^i X\ (\Leftrightarrow \theta^2(X) = \xi^{2i} X)\}\ (i \in \boldsymbol{Z}_m),$$

$$\mathfrak{g}_j = \{X \in \tilde{\mathfrak{g}}_j; \theta(X) = \xi^j X\ (\Leftrightarrow X^* = -\xi^j X)\}\ (j \in \boldsymbol{Z}_{2m}).$$

Apply Proposition 1.8 to $\sigma(X) = X^* = -\theta(X)$ $(X \in \mathrm{End}(V))$, $\tilde{H} = \tilde{G}_0$, $H = G_0$, $\tilde{L} = \tilde{\mathfrak{g}}_j$, $L = \mathfrak{g}_j$ $(j \in \boldsymbol{Z}_{2m})$ and $\alpha(X) = -\xi^j X$ $(X \in \tilde{L})$. Then we obtain the following.

PROPOSITION 1.9. *For any $j \in \boldsymbol{Z}_{2m}$, the map*

$$\mathfrak{g}_j/G_0 \to \tilde{\mathfrak{g}}_j/\tilde{G}_0,\ \mathcal{O} \mapsto \mathrm{Ad}(\tilde{G}_0) \cdot \mathcal{O}$$

is injective.

2. Classification of orbits of Θ-representations of type (A-I). Let (G, θ) be a Θ-group of type (A-I) defined by a vector space (V, S) with an m-automorphism. We put $\zeta = e^{2\pi\sqrt{-1}/m}$ and $V^j := \{v \in V; Sv = \zeta^j v\}$ for $j \in \boldsymbol{Z}_m$. Then G_0 and $\mathfrak{g}_1$ can be written as

$$G_0 = \{g \in GL(V); gV^j = V^j,\ j \in \boldsymbol{Z}_m\},\ \mathfrak{g}_1 = \{X \in \mathfrak{gl}(V); XV^j \subset V^{j+1},\ j \in \boldsymbol{Z}_m\}.$$

A classification of nilpotent G_0-orbits in $\mathfrak{g}_1$ was already given in Kempken [Ke] (see also [O2]) by means of $\boldsymbol{Z}_m$-labeled Young diagrams defined in [Ke] which we call $\langle\zeta\rangle$-signed diagrams in [O2]. A similar classification of nilpotent orbits is also given in [DKP] in the category of color Lie algebras. We may say that classifications of nilpotent orbits of Θ-representation of types (A-I), (BCD-I) and (A-O) are given in [DKP]. A classification of nilpotent orbits by means of weighted Dynkin diagrams is also given in [Ka].

In this section, we give a classification of general orbits of Θ-representations of type (A-I) by means of $\boldsymbol{Z}_m$-labeled Young diagrams with eigenvalues. By Propositions 1.7 and 1.9, we know that general orbits of Θ-representations of types (BCD-I) and (A-O) can also be classified by $\boldsymbol{Z}_m$-labeled Young diagrams with eigenvalues.

The classification is mainly based on the following proposition.

PROPOSITION 2.1. *For any $A \in \mathfrak{g}_1$, V is represented as a direct sum $V = V_1 \oplus V_2 \oplus \cdots \oplus V_p$ of A-stable and S-stable subspaces with one of the following properties*:

(i) *$A|_{V_k}$ is nilpotent, and there exists a basis $\{v_0, v_1, \dots, v_l\}$ of V_k contained in $\cup_{j\in \mathbf{Z}_m} V^j$ such that $Av_i = v_{i+1}$ $(0 \le i \le l-1)$ and $Av_l = 0$. We denote such an operation of A by $A : v_0 \to v_1 \to \cdots \to v_l \to 0$.*

(ii) *$A|_{V_k}$ is isomorphic. Denote by $A = A_s + A_n$ the Jordan decomposition of A in $\mathfrak{gl}(V)$ with the semisimple part A_s and the nilpotent part A_n. Since $\theta(A) = \zeta A$, we know $A_s,\ A_n \in \mathfrak{g}_1$ by the uniqueness of the Jordan decomposition. Then there exist $\alpha \in \mathbf{C}^\times$ and a basis $\{v_i^j;\ j \in \mathbf{Z}_m,\ 0 \le i \le l\}$ of V_k such that $\alpha^{-1}A_s$ and A_n map this basis in the following manner:*

$$\begin{array}{ccccccccccc}
v_0^0 & \to & v_1^0 & \to & v_2^0 & \to & \cdots & \to & v_l^0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
v_0^1 & \to & v_1^1 & \to & v_2^1 & \to & \cdots & \to & v_l^1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
v_0^{m-1} & \to & v_1^{m-1} & \to & v_2^{m-1} & \to & \cdots & \to & v_l^{m-1} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
v_0^0 & \to & v_1^0 & \to & v_2^0 & \to & \cdots & \to & v_l^0 & \to & 0
\end{array},$$

where $\downarrow$ (resp. $\to$) denotes the operation of $\alpha^{-1}A_s$ (resp. A_n) on this basis.

We introduce two lemmas before the proof of Proposition 2.1. Let A be an element of $\mathfrak{g}_1$. For an S-stable and A-stable subspace W of V and $\alpha \in \mathbf{C}$, we write

$$W_A(\alpha) := \{v \in W;\ (\alpha\ \mathrm{id}_W - A)^k v = 0 \text{ for } k >> 0\}.$$

LEMMA 2.2. *If α is an eigenvalue of $A|_W$, so is $\zeta^{-1}\alpha$ and it holds $SW_A(\alpha) = W_A(\zeta^{-1}\alpha)$.*

PROOF. For $k \ge 0$, we see

$$\begin{aligned}(\zeta^{-1}\alpha\ \mathrm{id}_V - A)^k S &= SS^{-1}(\zeta^{-1}\alpha\ \mathrm{id}_V - A)^k S = S(\zeta^{-1}\alpha\ \mathrm{id}_V - S^{-1}AS)^k \\ &= S(\zeta^{-1}\alpha\ \mathrm{id}_V - \zeta^{-1}A)^k = \zeta^{-k} S(\alpha\ \mathrm{id}_V - A)^k .\end{aligned}$$

If $v \in W_A(\alpha)$, there exists $k \ge 0$ such that $(\alpha\ \mathrm{id}_V - A)^k v = 0$. Hence $(\zeta^{-1}\alpha\ \mathrm{id}_V - A)^k Sv = \zeta^{-k}S(\alpha\ \mathrm{id}_V - A)^k v = 0$. Therefore $Sv \in W_A(\zeta^{-1}\alpha)$. □

For $\alpha \in \mathbf{C}^\times$, we put

$$W_A(\langle\zeta\rangle\alpha) := \bigoplus_{j \in \mathbf{Z}_m} W_A(\zeta^j \alpha),$$

where $\langle\zeta\rangle$ denotes the subgroup of $\mathbf{C}^\times$ generated by ζ and $\langle\zeta\rangle\alpha$ denotes the set $\{\zeta^j\alpha;\ j \in \mathbf{Z}_m\}$. Then W is decomposed as $W = W_A(0) \oplus (\bigoplus_{i=1}^q W_A(\langle\zeta\rangle\alpha_i))$ for some non-zero eigenvalues $\alpha_1, \dots, \alpha_q$ of A. If A is semisimple, by decomposing each $W_A(\langle\zeta\rangle\alpha_i)$ into indecomposable S-stable and A-stable subspaces, we obtain the following.

LEMMA 2.3. *Suppose that $A \in \mathfrak{g}_1$ is semisimple and W is an S-stable and A-stable subspace of V. Then there exists a decomposition $W = W_A(0) \oplus W_1 \oplus W_2 \oplus \cdots \oplus W_p$ of W into A-stable and S-stable subspaces such that each direct summand W_k has the following properties.*

For any eigenvalue $\alpha \in \boldsymbol{C}^\times$ of $A|_{W_k}$, there exists a basis $v^0, v^1, \ldots, v^{m-1}$ of W_k with $v^j \in V^j$ ($j \in \boldsymbol{Z}_m$) such that $\alpha^{-1}Av^j = v^{j+1}$. We denote such an operation of $\alpha^{-1}A$ by

$$\alpha^{-1}A : v^0 \to v^1 \to \cdots \to v^{m-1} \to v^0 .$$

In particular, the eigenvalues of $A|_{W_k}$ are $\alpha, \zeta\alpha, \zeta^2\alpha, \ldots, \zeta^{m-1}\alpha$ each of which appears with multiplicity one.

PROOF OF PROPOSITION 2.1. If $A \in \mathfrak{g}_1$ is nilpotent (resp. semisimple), V has a decomposition of components belong to Proposition 2.1, (i) (resp. (ii)) by [O2, Proposition 1.2] (resp. Lemma 2.3). Therefore we assume that A is neither nilpotent nor semisimple.

Then there exist non-zero eigenvalues $\beta_1, \beta_2, \ldots, \beta_q$ of A such that

$$V = V_A(0) \oplus V_A(\langle\zeta\rangle\beta_1) \oplus \cdots \oplus V_A(\langle\zeta\rangle\beta_q) .$$

Thus it is sufficient to show that $V_A(0)$ and $V_A(\langle\zeta\rangle\beta_k)$ ($1 \le k \le q$) have the direct sum decomposition of Proposition 2.1. Again by [O2, Proposition 1.2], $V_A(0)$ has such a decomposition.

Let $A = A_s + A_n$ be the Jordan decomposition of A. As mentioned before, A_s and A_n are in $\mathfrak{g}_1$. We write $x := A_n$. Since A_s is semisimple, the centralizer $\mathfrak{z}_{\mathfrak{g}}(A_s)$ is reductive. Since $SA_sS^{-1} = \zeta A_s$, $\mathfrak{z}_{\mathfrak{g}}(A_s)$ is $\theta = \mathrm{Ad}(S)$-stable and we obtain $\boldsymbol{Z}_m$-graded Lie algebra $\mathfrak{z}_{\mathfrak{g}}(A_s) = \bigoplus_{j\in \boldsymbol{Z}_m} \mathfrak{z}_{\mathfrak{g}_j}(A_s)$. Since $x \in \mathfrak{z}_{\mathfrak{g}_1}(A_s)$ is nilpotent, there exist $h \in \mathfrak{z}_{\mathfrak{g}_0}(A_s)$ and $y \in \mathfrak{z}_{\mathfrak{g}_{-1}}(A_s)$ such that (h, x, y) is an $\mathfrak{sl}_2$-triple as in the proof of [KrP, Lemma 7.3], i.e., $[h, x] = 2x$, $[h, y] = -2y$ and $[x, y] = h$. We write $\mathfrak{h}$ the 3-dimensional subalgebra spanned by h, x, y.

Let α be a nonzero eigenvalue of A. Then, clearly, $W := V_A(\langle\zeta\rangle\alpha) = V_{A_s}(\langle\zeta\rangle\alpha)$ is an S-stable $\mathfrak{h}$-submodule of V. For an integer $p \ge 0$, we write $K^p := \{v \in W; yv = 0,\ hv = -pv\}$. Since h, y are in $\mathfrak{z}_{\mathfrak{g}}(A_s)$ and $ShS^{-1} = h$, $SyS^{-1} = \zeta^{-1}y$, K^p is A_s-stable and S-stable.

Let W^p be the $\mathfrak{h}$-submodule of V generated by K^p. Clearly, W^p is also A_s-stable and S-stable, and W is equal to $\bigoplus_{p\ge 0} W^p$ by the representation theory of $\mathfrak{sl}_2$.

Since A_s is semisimple, K^p has a decomposition $K^p = \bigoplus_k K^p_k$ in Lemma 2.3 with respect to A_s. Then α is an eigenvalue of A_s restricted to each K^p_k, and K^p_k has a basis $\{v^j; j \in \boldsymbol{Z}_m\}$ with $\alpha^{-1}A_s : v^0 \to v^1 \to \cdots v^{m-1} \to v^0$. Since each v^j is an h-lowest weight vector of weight $-p$, we have $x^p v^j \ne 0$ and $x^{p+1}v^j = 0$. Denote by W^p_k the $\mathfrak{h}$-submodule of V generated by K^p_k. Then $\{x^i v^j; j \in \boldsymbol{Z}_m,\ 1 \le i \le p\}$ is a basis of W^p_k, and $\alpha^{-1}A_s$ and $x = A_n$ map this basis as in Proposition 2.1, (ii). By the representation theory of $\mathfrak{sl}_2$, we have $W^p = \bigoplus_k W^p_k$. □

DEFINITION 2.4 (cf. [O2, Definition 1.1]). (i) A Young diagram η for which an element of $\boldsymbol{Z}_m$ is placed in each box is called a $\boldsymbol{Z}_m$-labeled Young diagram (called "word" in

[Ke]) if the attached number in $\boldsymbol{Z}_m$ of each box is $+1$ of that of the left adjacent box if exists.

For example, $\eta =$

1	2	3	0	1	2	3
0	1	2	3	0	1	
3	0	1	2			

is a $\boldsymbol{Z}_4$-labeled Young diagram.

(ii) For a $\boldsymbol{Z}_m$-labeled Young diagram η and $j \in \boldsymbol{Z}_m$, we denote by $n_j(\eta)$ the number of j's which occur in η. We write $\mathrm{YD}_m(n_0, n_1, n_2, \dots, n_{m-1})$ for the set of $\boldsymbol{Z}_m$-labeled Young diagrams η such that $n_j(\eta) = n_j$ $(j \in \boldsymbol{Z}_m)$.

For example, $\eta =$

3	0	1	2	3	0	1	2	3
2	3	0	1	2	3			
3	0	1	2					

is in $\mathrm{YD}_4(4, 4, 5, 6)$.

Write $n_j := \dim V^j$ $(j \in \boldsymbol{Z}_m)$. It is known that nilpotent G_0-orbits in $\mathfrak{g}_1$ are classified by $\mathrm{YD}_m(n_0, n_1, n_2, \dots, n_{m-1})$ ([Ke], see also [O2] and [DKP]).

To give the classification of general G_0-orbits in $\mathfrak{g}_1$, we generalize this notion as follows.

DEFINITION 2.5. (i) For $l \geq 0$ and $\alpha \in \boldsymbol{C}^\times$, we denote by $\Delta_l^m(\langle\zeta\rangle\alpha)$ a pair $(\delta_l^m, \langle\zeta\rangle\alpha)$ of the $\boldsymbol{Z}_m$-labeled Young diagram

$\delta_l^m :=$

0	1	2	$\cdots$	l
1	2	3	$\cdots$	$l+1$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$m-2$	$m-1$	0	$\cdots$	$l+m-2$
$m-1$	0	1	$\cdots$	$l+m-1$

and the set $\langle\zeta\rangle\alpha$ of complex numbers. For $j \in \boldsymbol{Z}_m$ and $l \geq 0$, we denote by $\Delta_l^m(j, \{0\})$ a pair $(\nu_l^m(j), \{0\})$ of the $\boldsymbol{Z}_m$-labeled Young diagram

$\nu_l^m(j) :=$

j	$j+1$	$j+2$	$\cdots$	$j+l-1$	$j+l$

and the set $\{0\}$.

(ii) We call a formal sum of the components $\Delta_l^m(\langle\zeta\rangle\alpha)$ and $\Delta_l^m(j, \{0\})$ for various l, α and j a $\boldsymbol{Z}_m$-labeled Young diagram with eigenvalues (abbreviated $\boldsymbol{Z}_m$-YDE).

(iii) For a $\boldsymbol{Z}_m$-YDE Δ and $j \in \boldsymbol{Z}_m$, we denote by $n_j(\Delta)$ the number of j's which occur in Δ. We write $\mathrm{YDE}_m(n_0, n_1, n_2, \dots, n_{m-1})$ the set of $\boldsymbol{Z}_m$-YDE's Δ such that $n_j(\Delta) = n_j$ for each $j \in \boldsymbol{Z}_m$.

For any $A \in \mathfrak{g}_1$, let us attach a $\boldsymbol{Z}_m$-YDE $\Delta(A)$ to A as follows. Take the decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_p$ given in Proposition 2.1. To a component V_k for which $A|_{V_k}$ is nilpotent and $v_0 \in V^j$, we attach the $\boldsymbol{Z}_m$-YDE $\Delta(A, V_k) := \Delta_l^m(j, \{0\})$. For a component V_k in Proposition 2.1, (ii), let us define a basis $\{u_h^j\,;\, j \in \boldsymbol{Z}_m, 0 \leq h \leq l\}$ of V_k by $u_h^j = \sum_{i \in \boldsymbol{Z}_m} (\zeta^{-j})^i v_h^i$. Then we easily see that

$$A_s u_h^j = (\alpha\zeta^j) u_h^j \ \text{ and } \ A_n : u_0^j \to u_1^j \to \cdots u_l^j \to 0\,,$$

and know the set of eigenvalues of $A|_{V_k}$ is $\langle\zeta\rangle\alpha$. Thus, to a component V_k for which $A|_{V_k}$ is isomorphic, let us attach the $\mathbf{Z}_m$-YDE $\Delta(A, V_k) := \Delta_l^m(\langle\zeta\rangle\alpha)$. In such a way, we obtain a $\mathbf{Z}_m$-YDE $\Delta(A)$ which is the sum of $\Delta(A, V_k)$ for $1 \le k \le p$, i.e., $\Delta(A) := \sum_{k=1}^{p} \Delta(A, V_k)$. Then we easily see the following.

LEMMA 2.6. $\Delta(A)$ $(A \in \mathfrak{g}_1)$ *is independent of the choice of the decomposition* $V = V_1 \oplus V_2 \oplus \cdots \oplus V_p$ *nor that of the basis of each* V_k.

Let $V = V^0 \oplus V^1 \oplus \cdots \oplus V^{m-1}$ be the $\mathbf{Z}_m$-gradation of V defined by S and put $n_j := \dim V^j$ for $j \in \mathbf{Z}_m$. Then, for an element $A \in \mathfrak{g}_1$, we can define an element $\Delta(A) \in \mathrm{YDE}_m(n_0, n_1, \ldots, n_{m-1})$ which we call the $\mathbf{Z}_m$-YDE of A.

THEOREM 2.7. (i) *Suppose that* $A, B \in \mathfrak{g}_1$ *are mutually conjugate under* $\mathrm{Ad}(G_0)$. *Then we have* $\Delta(A) = \Delta(B)$. *Thus we obtain a map*

$$\mathfrak{g}_1/G_0 \to \mathrm{YDE}_m(n_0, n_1, \ldots, n_{m-1}), \quad \mathrm{Ad}(G_0)\cdot A \mapsto \Delta(A)\,.$$

We write $\Delta(\mathrm{Ad}(G_0)\cdot A) := \Delta(A)$ *and call it the* $\mathbf{Z}_m$*-YDE of the orbit* $\mathrm{Ad}(G_0)\cdot A$.

(ii) *The map in* (i) *is bijective*: $\mathfrak{g}_1/G_0 \simeq \mathrm{YDE}_m(n_0, n_1, \ldots, n_{m-1})$.

PROOF. Since (i) is clear, we only show (ii). Suppose $A, B \in \mathfrak{g}_1$ satisfy $\Delta(A) = \Delta(B)$. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_p$ be a decomposition for A in Proposition 2.1 and $V = U_1 \oplus U_2 \oplus \cdots \oplus U_p$ for B. We can assume that $\Delta(A, V_k) = \Delta(B, U_k)$ for $1 \le k \le p$. Then we can take $g \in GL(V)$ which maps the basis of each V_k to that of U_k. Then clearly $g \in G_0$ and $B = gAg^{-1}$. Hence the map in (i) is injective.

Let Δ be any element of $\mathrm{YDE}_m(n_0, n_1, \ldots, n_{m-1})$. Suppose that $\Delta = \sum_k \Delta_k$, where each Δ_k is a $\mathbf{Z}_m$-YDE in Definition 2.5, (i). By corresponding the boxes of Δ with the attached number $j \in \mathbf{Z}_m$ to linearly independent vectors of V^j, we can construct a basis $\mathcal{B}$ of V. Let us construct an element $A \in \mathfrak{gl}(V)$ as follows.

Let $\Delta_k = \Delta_l^m(j, \{0\})$ be a diagram which appears in Δ and $v_0, v_1, \ldots, v_l$ the vectors in $\mathcal{B}$ corresponding to $\Delta_l^m(l, \{0\})$. We put $V_k := \langle v_0, v_1, \ldots, v_l\rangle_{\mathbf{C}}$ and define $A_k \in \mathfrak{gl}(V_k)$ by $A_k : v_0 \to v_1 \to \cdots \to v_l \to 0$.

Suppose that $\Delta_k = \Delta(\langle\zeta\rangle\alpha, l)$ and $\{v_i^j\ ;\ j \in \mathbf{Z}_m,\ 0 \le i \le l\}$ are the vectors in $\mathcal{B}$ corresponding to Δ_k. We put $V_k := \langle v_i^j\ ;\ j \in \mathbf{Z}_m,\ 0 \le i \le l\rangle_{\mathbf{C}}$ and define $s_k, x_k \in \mathfrak{gl}(V_k)$ by the operations on the basis $\{v_i^j\ ;\ j \in \mathbf{Z}_m\}$ similar to those of A_s, A_n of Proposition 2.1. For each k, we put $A_k := s_k + x_k \in \mathfrak{gl}(V_k)$. Then $V = \oplus_k V_k$. We define $A := \sum_k A_k \in \mathfrak{gl}(V)$. Then by the construction, A is in $\mathfrak{g}_1$ and clearly we have $\Delta = \Delta(A)$. Therefore, the map is surjective. □

Let us consider the classification given by Theorem 2.7 in the special case $m = 1$. Suppose that $S = \mathrm{id}_V$. Then we have

$$m = 1,\ \ \mathbf{Z}_1 = \{0\},\ \ \zeta = 1,\ \ V^0 = V,\ \ G_0 = GL(V),\ \ \mathfrak{g}_1 = \mathfrak{g}_0 = \mathfrak{gl}(V)\,,$$

and the $\boldsymbol{Z}_1$-YDE's given in Definition 2.5, (i) can be written as the sum of components of the form

$$\Delta_l^1(\{\alpha\}) = (\overbrace{\boxed{}\boxed{}\boxed{\cdots}\boxed{}\boxed{}}^{l+1}, \{\alpha\}) \quad \text{and} \quad \Delta_l^1(0, \{0\}) = (\overbrace{\boxed{}\boxed{}\boxed{\cdots}\boxed{}\boxed{}}^{l+1}, \{0\}) ,$$

where we omit the number $0 \in \boldsymbol{Z}_1 = \{0\}$ which appears in the Young diagrams. Then, by Proposition 2.1, $\Delta_l^1(\{\alpha\})$ (resp. $\Delta_l^1(0, \{0\})$) is considered as a diagram which corresponds to the Jordan block of size $l + 1$ with the eigenvalue α (resp. 0). Therefore the classification of G_0-orbits in $\mathfrak{g}_1$ given by Theorem 2.7 is considered as a generalization of that of $GL(V)$-orbits in $\mathfrak{gl}(V)$ by Jordan normal forms.

Now let us describe the map

$$\gamma : \mathfrak{g}_1/G_0 \to \mathfrak{g}/G = \mathfrak{gl}(V)/GL(V), \quad \mathcal{O} \mapsto \mathrm{Ad}(G) \cdot \mathcal{O}$$

by means of Young diagrams with eigenvalues. We write $n = \sum_{j \in \boldsymbol{Z}_m} n_j = \dim V$. Under the identifications

$$\mathfrak{g}_1/G_0 = \mathrm{YDE}_m(n_0, n_1, \dots, n_{m-1}) \quad \text{and} \quad \mathfrak{gl}(V)/GL(V) = \mathrm{YDE}_1(n) ,$$

the map $\gamma : \mathrm{YDE}_m(n_0, n_1, \dots, n_{m-1}) \to \mathrm{YDE}_1(n)$ is described as follows.

For a $\boldsymbol{Z}_m$-YDE $\Delta_l^m(\langle\zeta\rangle\alpha)$, let us define the $\boldsymbol{Z}_1$-YDE $[\Delta_l^m(\langle\zeta\rangle\alpha)]_1$ by

$$[\Delta_l^m(\langle\zeta\rangle\alpha)]_1 = \sum_{j \in \boldsymbol{Z}_m} \Delta_l^1(\{\zeta^j\alpha\}), \quad \text{while we define} \quad [\Delta_l^m(j, \{0\})]_1 = \Delta_l^1(0, \{0\}) .$$

For $A \in \mathfrak{g}_1$, let us consider the decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_p$ given in Proposition 2.1 and the $\boldsymbol{Z}_m$-YDE $\Delta(A) = \sum_{k=1}^p \Delta(A, V_k) \in \mathrm{YDE}_m(n_0, n_1, \dots, n_{m-1})$ defined after Definition 2.5. Then we easily see that the $\boldsymbol{Z}_1$-YDE of $A \in \mathfrak{g}$ is given by $\sum_{i=1}^k [\Delta(A, V_k)]_1$. Hence we know that, for $\Delta = \sum_{i=1}^k \Delta_i \in \mathrm{YDE}_m(n_0, n_1, \dots, n_{m-1})$ which is a sum of components Δ_i in Definition 2.5, (i), the corresponding $\boldsymbol{Z}_1$-YDE $\gamma(\Delta)$ is give by $\gamma(\Delta) = \sum_{i=1}^k [\Delta_i]_1$.

By considering the case when A is semisimple, we obtain the following.

COROLLARY 2.8. (i) *The eigenvalues of any semisimple element of* $\mathfrak{g}_1$ *can be written as*

$$\alpha_1, \zeta\alpha_1, \dots, \zeta^{m-1}\alpha_1,\ \alpha_2, \zeta\alpha_2, \dots, \zeta^{m-1}\alpha_2, \dots,\ \alpha_q, \zeta\alpha_q, \dots, \zeta^{m-1}\alpha_q,\ \overbrace{0, \dots, 0}^{\dim V - mq}$$

for some $\alpha_1, \alpha_2, \dots, \alpha_q \in \boldsymbol{C}^\times$, *with* $q \leq r := \min\{\dim V^j;\ j \in \boldsymbol{Z}_m\}$.

(ii) *For any set of complex numbers of the form* (i), *there exists a semisimple element of* $\mathfrak{g}_1$ *whose set of eigenvalues coincides with it.*

(iii) *Write* $\mathfrak{g}_1^{\mathrm{ss}}$ *the set of semisimple elements of* $\mathfrak{g}_1$. *Then the map* $\mathfrak{g}_1^{\mathrm{ss}}/G_0 \to \mathfrak{g}/G = \mathfrak{gl}(V)/GL(V)$ *defined by* $\mathcal{O} \mapsto \mathrm{Ad}(G) \cdot \mathcal{O}$ *is injective.*

PROOF. Let $A \in \mathfrak{g}_1$ be a semisimple element. By the definition of the diagram $\Delta(A)$, we easily see that $\Delta(A)$ is of the form

$$\sum_{i=1}^{q} \Delta_1^m(\langle\zeta\rangle\alpha_i) + \sum_{j\in \boldsymbol{Z}_m}(n_j - q)\Delta_1^m(j, \{0\}) \tag{2.1}$$

for some $\alpha_1, \dots, \alpha_q \in \boldsymbol{C}^\times$. Since each number $j \in \boldsymbol{Z}_m$ appears once in each $\Delta_1^m(\langle\zeta\rangle\alpha_i)$, j appears q-times in $\sum_{i=1}^q \Delta_l^m(\langle\zeta\rangle\alpha_i)$. Thus we have $q \le r$ and the non-zero eigenvalues of A are $\bigcup_{i=1}^q \langle\zeta\rangle\alpha_i$. This proves the claim (i).

For a given set of complex numbers of the form in (i), by Theorem 2.7, (ii), there exists an element $A \in \mathfrak{g}_1$ whose $\boldsymbol{Z}_m$-YDE $\Delta(A)$ is of the form (2.1). Therefore the eigenvalues of A are of the form in (i) and the claim (ii) is proved.

Suppose that $A \in \mathfrak{g}_1^{\mathrm{ss}}$ and that the eigenvalues of the $GL(V)$-orbit of A is the complex numbers in (i). Then the $\boldsymbol{Z}_m$-YDE of A must coincide with the diagram (2.1). Hence the map in (iii) is injective. □

By Propositions 1.7, 1.9 and Theorem 2.7, we obtain the following.

COROLLARY 2.9. *Let (G, θ) be a Θ-group of order m of type* (BCD-I) (*resp. Θ-group of order $2m$ of type* (A-O)) *and $(\tilde{G}, \Theta) = (GL(V), \theta)$* (*resp.* $(\tilde{G}, \Theta) = (GL(V), \theta^2)$) *the associated Θ-group of order m of type* (A-I). *Then, for the corresponding Θ-representations $(G_0, \mathfrak{g}_1)$ and $(\tilde{G}_0, \tilde{\mathfrak{g}}_1)$, we have the following.*

(i) *The map $\mathfrak{g}_1/G_0 \to \mathrm{YDE}_m(n_0, n_1, n_2, \dots, n_{m-1})$ which maps $\mathcal{O} \in \mathfrak{g}_1/G_0$ to the $\boldsymbol{Z}_m$-YDE $\Delta(\mathrm{Ad}(\tilde{G}) \cdot \mathcal{O})$ of the orbit $\mathrm{Ad}(\tilde{G}) \cdot \mathcal{O} \in \tilde{\mathfrak{g}}_1/\tilde{G}_0$, is injective.*

(ii) *Write $\mathfrak{g}_1^{\mathrm{ss}}$ the set of semisimple elements of $\mathfrak{g}_1$. Then the map $\mathfrak{g}_1^{\mathrm{ss}}/G_0 \to \mathfrak{gl}(V)/GL(V)$ defined by $\mathcal{O} \mapsto \mathrm{Ad}(GL(V)) \cdot \mathcal{O}$ is injective.*

3. Classification of semisimple orbits of Θ-representation of type (BCD-I) and (A-O).

(3.1) Type (BCD-I). Let (G, θ) be a Θ-group of order m of type (BCD-I) defined by an (ε, ω)-space $(V, (\ ,\ ), S)$ with m-automorphism and $(\tilde{G}, \theta) = (GL(V), \mathrm{Ad}(S))$ the associated Θ-group of type (A-I). We write $V = \bigoplus_{j\in \boldsymbol{Z}_m} V^j$ the $\boldsymbol{Z}_m$-gradation of V defined by S.

Let A be a semisimple element of $\mathfrak{g}_1$. Let U (resp. W) be an A-stable and S-stable subspace of V with basis $\{u^j;\ j \in \boldsymbol{Z}_m\}$ (resp. $\{w^j;\ j \in \boldsymbol{Z}_m\}$) such that

$$\alpha^{-1}A : u^0 \to u^1 \to \cdots u^{m-1} \to u^0 \text{ and } u^j \in V^j$$

$$(\text{resp. } \beta^{-1}A : w^0 \to w^1 \to \cdots w^{m-1} \to w^0 \text{ and } w^j \in V^j)\,,$$

where α (resp. β) is a non-zero complex number.

LEMMA 3.1. *Suppose that $(U, W) \neq \{0\}$. Then we have the following.*

(i) $(u^i, w^j) \neq 0$ *if and only if $i + j = \omega$ in $\boldsymbol{Z}_m$*

(ii) $-\beta/\alpha \in \langle\zeta\rangle$.

PROOF. (i) Suppose that $(u^i, w^j) \neq 0$. By the definition of $(V, (\ ,\ ), S)$, it holds

$$\zeta^{\omega}(u^i, w^j) = (Su^i, Sw^j) = \zeta^{i+j}(u^i, w^j)$$

(cf. Lemma 1.2). Hence $i + j = \omega$ in $\boldsymbol{Z}_m$.

(ii) Since $(U, W) \neq \{0\}$, there exist $p, q \in \boldsymbol{Z}_m$ such that $(u^p, w^q) \neq 0$. Then $p + q = \omega$ in $\boldsymbol{Z}_m$. From this, we compute

$$\begin{aligned}(u^p, w^q) &= (u^p, w^{\omega-p}) = \alpha^{-p}(A^p u^0, w^{\omega-p}) = (-\alpha)^{-p}(u^0, A^p w^{\omega-p}) \\ &= \left(-\frac{\beta}{\alpha}\right)^p (u^0, (\beta^{-1}A)^p w^{\omega-p}) = \left(-\frac{\beta}{\alpha}\right)^p (u^0, w^{\omega}).\end{aligned}$$

Hence $(u^0, w^{\omega}) \neq 0$. Suppose $i + j = \omega$ in $\boldsymbol{Z}_m$. Then $(u^i, w^j) \neq 0$ follows from $(u^i, w^j) = (-\beta/\alpha)^i (u^0, w^{\omega})$. If we put $i = m$ in the last equation, we obtain

$$(u^0, w^{\omega}) = (u^m, w^{\omega-m}) = \left(-\frac{\beta}{\alpha}\right)^m (u^0, w^{\omega}).$$

Hence (ii) follows. □

LEMMA 3.2. *Suppose that $(U, U) \neq \{0\}$. Then m is even and $(\varepsilon, \omega) = (1, 0)$ or $(\varepsilon, \omega) = (-1, 1)$.*

PROOF. Suppose that $(U, U) \neq \{0\}$. Apply Lemma 3.1 by putting $U = W$, $\alpha = \beta$ and $u^j = w^j$. Then we see $(u^0, u^{\omega}) \neq 0$ and

$$\begin{aligned}\varepsilon(u^0, u^{\omega}) &= \varepsilon((\alpha^{-1}A)^m u^0, u^{\omega}) = \varepsilon(-1)^{m-\omega}((\alpha^{-1}A)^{\omega} u^0, (\alpha^{-1}A)^{m-\omega} u^{\omega}) \\ &= \varepsilon(-1)^{m-\omega}(u^{\omega}, u^0) = (-1)^{m-\omega}(u^0, u^{\omega}).\end{aligned}$$

Hence $(-1)^{m-\omega} = \varepsilon$. On the other hand, by Lemma 3.1, (ii), $-1 \in \langle \zeta \rangle$ and hence m is even. Therefore Lemma 3.2 follows. □

LEMMA 3.3. *Suppose that m is even and that $(\varepsilon, \omega) = (1, 0)$ or $(\varepsilon, \omega) = (-1, 1)$. Then there exists a $(\ ,\ )$-orthogonal direct sum decomposition $V = V_0 \perp V_1 \perp V_2 \perp \cdots \perp V_l$ into A-stable and S-stable subspaces V_i of V with the following properties*:

(a) $A|_{V_0} = 0$.

(b) *For each $1 \leq k \leq l$, there exist $\alpha_k \in \boldsymbol{C}^{\times}$ and a basis $v^0, v^1, \ldots, v^{m-1}$ of V_k with $v^j \in V^j$ $(j \in \boldsymbol{Z}_m)$ such that A maps this basis as ${\alpha_k}^{-1}A : v^0 \to v^1 \to \cdots \to v^{m-1} \to v^0$.*

PROOF. If $A = 0$, the statement is trivial. We suppose that $A \neq 0$. It is enough to show that there exists a subspace V_1 with the property (b) such that $(\ ,\ )|_{V_1}$ is non-degenerate. Then apply the same procedure to the orthogonal complement $V_1^{\perp}$, and we obtain Lemma 3.3.

Since $A \in \tilde{\mathfrak{g}}_1$, by Lemma 2.3, there exist a subspace U of V, $\alpha \in \boldsymbol{C}^{\times}$ and a basis $u^0, u^1, \ldots, u^{m-1}$ of U with $u^j \in V^j$ $(j \in \boldsymbol{Z}_m)$ such that A maps this basis as $\alpha^{-1}A : u^0 \to u^1 \to \cdots \to u^{m-1} \to u^0$. If $(U, U) \neq \{0\}$, it follows from Lemma 3.1 that $(\ ,\ )|_U$ is non-degenerate. Then $V_1 = U$ is a desired subspace.

Next suppose that $(U, U) = \{0\}$. Then there exists a direct summand W in Lemma 2.3 such that $(U, W) \neq \{0\}$. Since $(U, V_A(0)) = (AU, V_A(0)) = (U, AV_A(0)) = \{0\}$, we have

$W \neq V_A(0)$. If $(W, W) \neq \{0\}$, $(\ ,\ )|_W$ is non-degenerate as before and we get a desired subspace $V_1 = W$. Hence we assume $(W, W) = \{0\}$.

Take a basis $w^0, w^1, \dots, w^{m-1}$ of W with $w^j \in V^j$ $(j \in \mathbf{Z}_m)$ such that A maps this basis as $\beta^{-1}A : w^0 \to w^1 \to \cdots \to w^{m-1} \to w^0$. Since $(U, W) \neq \{0\}$ and m is even, we have $\beta/\alpha \in \langle\zeta\rangle$ by Lemma 3.1, (ii). By changing of basis of W, if necessary, we may assume that $\beta = \alpha$, i.e.,

$$\alpha^{-1}A : u^0 \to u^1 \to \cdots \to u^{m-1} \to u^0,\ \ \alpha^{-1}A : w^0 \to w^1 \to \cdots \to w^{m-1} \to w^0 .$$

By Lemma 3.1, (i), we have $(u^\omega, w^0) \neq 0$. Let us put $v^j = u^j + w^j \in V^j$ $(j \in \mathbf{Z}_m)$ and $V_1 := \langle v^0, v^1, \dots, v^{m-1}\rangle_{\mathbf{C}}$. Then we have $\alpha^{-1}A : v^0 \to v^1 \to \cdots \to v^{m-1} \to v^0$. If some $v^j = 0$, we conclude $U = W$ which contradicts the assumption $(U, W) \neq \{0\}$. Hence any $v^j \neq 0$ and $v^0, v^1, \dots, v^{m-1}$ are linearly independent. We easily compute $(v^\omega, v^0) = \{1 + \varepsilon(-1)^\omega\}(u^\omega, w^0)$. Since $(\varepsilon, \omega) = (1, 0)$ or $(\varepsilon, \omega) = (-1, 1)$, we have $(v^\omega, v^0) = 2(u^\omega, w^0) \neq 0$. Therefore, $(\ ,\ )|_{V_1}$ is non-degenerate by Lemma 3.1, (i). □

LEMMA 3.4. *Suppose that m is odd or $(\varepsilon, \omega) = (1, 1)$ or $(\varepsilon, \omega) = (-1, 0)$, i.e., the complementary cases of Lemma 3.3. Then there exists an $(\ ,\ )$-orthogonal direct sum decomposition $V = V_0 \perp (V_1 \oplus V_1') \perp (V_2 \oplus V_2') \perp \cdots \perp (V_l \oplus V_l')$ into A-stable and S-stable subspaces of V with the following properties*:

(a) $A|_{V_0} = 0$.

(b) *For each $1 \leq k \leq l$, there exist $\alpha_k \in \mathbf{C}^\times$, bases $u^0, u^1, \dots, u^{m-1}$ of V_k and $w^0, w^1, \dots, w^{m-1}$ of V_k' with $u^j, w^j \in V^j$ $(j \in \mathbf{Z}_m)$ such that A maps these bases as*

$$\alpha_k{}^{-1}A : u^0 \to u^1 \to \cdots \to u^{m-1} \to u^0,\ \ -\alpha_k{}^{-1}A : w^0 \to w^1 \to \cdots \to w^{m-1} \to w^0 .$$

(c) *For each $1 \leq k \leq l$, $(V_k, V_k) = (V_k', V_k') = \{0\}$ and $(\ ,\ )|_{V_k \oplus V_k'}$ is non-degenerate.*

PROOF. As before, it is enough to show that there exists a subspace $V_1 \oplus V_1'$ with the properties (b) and (c). Let $V = V_A(0) \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_p$ be the direct sum decomposition of V for $A \in \tilde{\mathfrak{g}}_1$ (cf. Lemma 2.3). As before, $(V_k, V_A(0)) = \{0\}$ $(1 \leq k \leq p)$. By Lemma 3.2, we see $(V_k, V_k) = \{0\}$ $(1 \leq k \leq p)$. Since $(\ ,\ )$ is non-degenerate, we may assume that $(V_1, V_2) \neq \{0\}$. Take bases $u^0, u^1, \dots, u^{m-1}$ of V_1 and $w^0, w^1, \dots, w^{m-1}$ of V_2 with $u^j, w^j \in V^j$ $(j \in \mathbf{Z}_m)$ so that

$$\alpha^{-1}A : u^0 \to u^1 \to \cdots \to u^{m-1} \to u^0,\ \ \beta^{-1}A : w^0 \to w^1 \to \cdots \to w^{m-1} \to w^0 .$$

Since $\beta \in \langle\zeta\rangle(-\alpha)$ by Lemma 3.1, (ii), by changing of basis of V_2, if necessary, we may assume that $\beta = -\alpha$, i.e., $(-\alpha)^{-1}A : w^0 \to w^1 \to \cdots \to w^{m-1} \to w^0$. Then by putting $V_1' := V_2$, we obtain the desired $V_1 \oplus V_1'$. □

THEOREM 3.5. *Let (G, θ) be a Θ-group of order m of type (BCD-I) defined by an (ε, ω)-space $(V, (\ ,\ ), S)$ with m-automorphism and $V = V^0 \oplus V^1 \oplus \cdots \oplus V^{m-1}$ the $\mathbf{Z}_m$-gradation of V defined by S.*

(i) *Suppose that $(\varepsilon, \omega) = (1, 0)$ or $(-1, 1)$ and m is even. We write $r := \min\{\dim V^j;\ j \in \mathbf{Z}_m\}$.*

(1) *The eigenvalues of any semisimple element of* $\mathfrak{g}_1$ *can be written as*

$$\alpha_1, \zeta\alpha_1, \dots, \zeta^{m-1}\alpha_1,\ \alpha_2, \zeta\alpha_2, \dots, \zeta^{m-1}\alpha_2, \dots,\ \alpha_q, \zeta\alpha_q, \dots, \zeta^{m-1}\alpha_q,\ \overbrace{0, \dots, 0}^{\dim V - mq}$$

for some $\alpha_1, \alpha_2, \dots, \alpha_q \in \boldsymbol{C}^\times$ *with* $q \le r$.

(2) *Conversely, for any set of complex numbers of the form* (1), *there exists a semisimple element of* $\mathfrak{g}_1$ *whose set of eigenvalues coincides with it.*

(ii) *For any triple* (ε, ω, m) *except for the case* (i), *we write* $r := \min\{[\dim V^j/2];\ j \in \boldsymbol{Z}_m\}$.

(1) *The eigenvalues of any semisimple element of* $\mathfrak{g}_1$ *can be written as*

$$\alpha_1, \zeta\alpha_1, \dots, \zeta^{m-1}\alpha_1, -\alpha_1, -\zeta\alpha_1, \dots, -\zeta^{m-1}\alpha_1\,,$$

$$\alpha_2, \zeta\alpha_2, \dots, \zeta^{m-1}\alpha_2, -\alpha_2, -\zeta\alpha_2, \dots, -\zeta^{m-1}\alpha_2, \dots\,,$$

$$\alpha_q, \zeta\alpha_q, \dots, \zeta^{m-1}\alpha_q, -\alpha_q, -\zeta\alpha_q, \dots, -\zeta^{m-1}\alpha_q,\ \overbrace{0, \dots, 0}^{\dim V - 2mq}$$

for some $\alpha_1, \alpha_2, \dots, \alpha_q \in \boldsymbol{C}^\times$ *with* $q \le r$.

(2) *Conversely, for any set of complex numbers of the form* (1), *there exists a semisimple element of* $\mathfrak{g}_1$ *whose set of eigenvalues coincides with it.*

PROOF. The claims (i, 1) and (ii, 1) follow from Lemmas 3.3 and 3.4.

The claims (i, 2) and (ii, 2) follow from the construction of Cartan subspaces in Section 5 (Lemma 5.12). □

(3.2) Type (A-O). Let (G, θ) be a Θ-group of order $2m$ of type (A-O) defined by a vector space $(V, \langle\ ,\ \rangle)$ with (ω, m)-bilinear form and S the (ω, m)-automorphism of V corresponding to $(V, \langle\ ,\ \rangle)$. Write $(\tilde{G}, \mathrm{Ad}(S)) = (G, \theta^2)$ the associated Θ-group of order m of type (A-I). We put $\xi = e^{\pi\sqrt{-1}/m}$ and $\zeta = e^{2\pi\sqrt{-1}/m} = \xi^2$.

Let A be a semisimple element of $\mathfrak{g}_1$. Let U (resp. W) be an A-stable and S-stable subspace of V with basis $\{u^j;\ j \in \boldsymbol{Z}_m\}$ (resp. $\{w^j;\ j \in \boldsymbol{Z}_m\}$) such that

$$\alpha^{-1}A : u^0 \to u^1 \to \cdots u^{m-1} \to u^0 \text{ with } u^j \in V^j$$

$$(\text{resp. } \beta^{-1}A : w^0 \to w^1 \to \cdots w^{m-1} \to w^0 \text{ with } w^j \in V^j)\,,$$

where α (resp. β) is a non-zero complex number (cf. Lemma 2.3). We notice that $\langle w^j, u^i\rangle = \xi^{-\omega}\langle u^i, Sw^j\rangle = \xi^{-\omega}\zeta^j\langle u^i, w^j\rangle$.

The proofs of the following two lemmas are similar to those of Lemmas 3.1 and 3.2, and we omit them.

LEMMA 3.6. *Suppose that* $\langle U, W\rangle \neq \{0\}$. *Then we have the following.*

(i) $\langle u^i, w^j\rangle \neq 0$ *if and only if* $i + j = \omega$ *in* $\boldsymbol{Z}_m$.

(ii) $\xi(-\beta/\alpha) \in \langle\zeta\rangle$.

LEMMA 3.7. *Suppose that* m *is even or* $\omega = 1$. *Then* $\langle U, U\rangle = \{0\}$.

LEMMA 3.8. *Suppose that m is odd and that $\omega = 0$. Then there exists an orthogonal direct sum decomposition $V = V_0 \perp V_1 \perp V_2 \perp \cdots \perp V_l$ into A-stable and S-stable subspaces of V with the following properties*:

(a) $A|_{V_0} = 0$.

(b) $\langle V_i, V_j \rangle = \langle V_j, V_i \rangle = \{0\}$ *if* $i \neq j$.

(c) *For each* $1 \leq k \leq l$, *there exist* $\alpha_k \in \boldsymbol{C}^\times$ *and a basis* $v^0, v^1, \dots, v^{m-1}$ *of* V_k *with* $v^j \in V^j$ $(j \in \boldsymbol{Z}_m)$ *such that* ${\alpha_k}^{-1} A : v^0 \to v^1 \to \cdots \to v^{m-1} \to v^0$.

PROOF. If $A = 0$, the statement is trivial. Thus we suppose that $A \neq 0$. We will show that there exists a subspace U such that $\langle\ ,\ \rangle|_U$ is non-degenerate and (c) is satisfied for $V_k = U$. Then for $v \in V$, we see $\langle U, v \rangle = \xi^{-\omega} \langle v, SU \rangle = \langle v, U \rangle$. By applying the same procedure to the orthogonal complement $U^\perp := \{v \in V; \langle U, v \rangle = \langle v, U \rangle = 0\}$ (which is A-stable and S-stable subspace of V), we obtain Lemma 3.8.

Let $V = V_A(0) \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_p$ be the direct sum decomposition of V for $A \in \tilde{\mathfrak{g}}_1$ (cf. Lemma 2.3). If $\langle V_k, V_k \rangle \neq 0$ for some $1 \leq k \leq p$, $U = V_k$ is the desired subspace.

Suppose that $\langle V_k, V_k \rangle = \{0\}$ for any $1 \leq k \leq p$. Since $\langle V_k, V_A(0) \rangle = \langle V_A(0), V_k \rangle = 0$ $(1 \leq k \leq p)$ as before, we may assume that $\langle V_1, V_2 \rangle \neq \{0\}$. Take bases $u^0, u^1, \dots, u^{m-1}$ of V_1 and $w^0, w^1, \dots, w^{m-1}$ of V_2 with $u^j, w^j \in V^j$ $(j \in \boldsymbol{Z}_m)$ so that

$$\alpha^{-1} A : u^0 \to u^1 \to \cdots \to u^{m-1} \to u^0 \quad \text{and} \quad \beta^{-1} A : w^0 \to w^1 \to \cdots \to w^{m-1} \to w^0 .$$

Then $\xi(-\beta/\alpha) \in \langle \zeta \rangle$ by Lemma 3.6. Since m is odd and $\xi^m = -1$, we easily see $\beta/\alpha \in \langle \zeta \rangle$. By changing of a basis of W, if necessary, we may assume that $\beta = \alpha$, i.e.,

$$\alpha^{-1} A : u^0 \to u^1 \to \cdots \to u^{m-1} \to u^0, \ \ \alpha^{-1} A : w^0 \to w^1 \to \cdots \to w^{m-1} \to w^0 .$$

By Lemma 3.6, (i), we have $\langle u^0, w^0 \rangle \neq 0$. Let us put $v^j = u^j + w^j \in V^j$ $(j \in \boldsymbol{Z}_m)$ and $U := \langle v^0, v^1, \dots, v^{m-1} \rangle_{\boldsymbol{C}}$. Then $\alpha^{-1} A : v^0 \to v^1 \to \cdots \to v^{m-1} \to v^0$. Clearly $v^0, v^1, \dots, v^{m-1}$ are linearly independent and we compute

$$\langle v^0, v^0 \rangle = \langle u^0, w^0 \rangle + \langle w^0, u^0 \rangle = \langle u^0, w^0 \rangle + \xi^{-\omega} \langle u^0, S w^0 \rangle = 2 \langle u^0, w^0 \rangle \neq 0 .$$

Therefore, $\langle\ ,\ \rangle|_{V_1}$ is non-degenerate by Lemma 3.6, (i). □

LEMMA 3.9. *Suppose that m is even or $\omega = 1$, i.e., the complementary cases of Lemma 3.8. Then there exists an orthogonal direct sum decomposition* $V = V_0 \perp (V_1 \oplus V_1') \perp (V_2 \oplus V_2') \perp \cdots \perp (V_l \oplus V_l')$ *into A-stable and S-stable subspaces of V with the following properties*:

(a) $A|_{V_0} = 0$.

(b) *For each* $1 \leq k \leq l$, *there exist* $\alpha_k \in \boldsymbol{C}^\times$, *bases* $u^0, u^1, \dots, u^{m-1}$ *of* V_k *and* $w^0, w^1, \dots, w^{m-1}$ *of* V_k' *with* $u^j, w^j \in V^j$ $(j \in \boldsymbol{Z}_m)$ *such that*

$$\begin{aligned} {\alpha_k}^{-1} A &: u^0 \to u^1 \to \cdots \to u^{m-1} \to u^0 \quad \textit{and} \\ (-\xi^{-1}\alpha_k)^{-1} A &: w^0 \to w^1 \to \cdots \to w^{m-1} \to w^0 . \end{aligned}$$

(c) *For each* $1 \leq k \leq l$, $\langle V_k, V_k \rangle = \langle V_k', V_k' \rangle = \{0\}$ *and* $\langle\ ,\ \rangle|_{V_k \oplus V_k'}$ *is non-degenerate.*

PROOF. We assume that $A \neq 0$. It is enough to show that there exists a subspace $U \oplus U'$ with the properties (b) and (c). Let $V = V_A(0) \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_p$ be the direct sum decomposition in Lemma 2.3 for $A \in \tilde{\mathfrak{g}}_1$. As before, $\langle V_k, V_A(0)\rangle = \{0\}$ $(1 \leq k \leq p)$. By Lemma 3.6, $\langle V_k, V_k\rangle = \{0\}$ $(1 \leq k \leq p)$. Since $\langle\ ,\ \rangle$ is non-degenerate, we may assume that $\langle V_1, V_2\rangle \neq \{0\}$. Take bases $u^0, u^1, \dots, u^{m-1}$ of V_1 and $w^0, w^1, \dots, w^{m-1}$ of V_2 with $u^j, w^j \in V^j$ $(j \in \boldsymbol{Z}_m)$ as

$$\alpha^{-1}A : u^0 \to u^1 \to \cdots \to u^{m-1} \to u^0 \ \text{ and } \ \beta^{-1}A : w^0 \to w^1 \to \cdots \to w^{m-1} \to w^0 .$$

Since $\beta \in \langle\zeta\rangle(-\xi^{-1}\alpha)$ by Lemma 3.6, (ii), there exists integer i such that $\beta = \zeta^i(-\xi^{-1}\alpha)$. Then $(\zeta^i)^j w^j$ $(j \in \boldsymbol{Z}_m)$ is a basis of V_2 such that

$$(-\xi^{-1}\alpha)^{-1}A : w^0 \to (\zeta^i)w^1 \to (\zeta^i)^2 w^2 \to \cdots \to (\zeta^i)^{m-1}w^{m-1} \to w^0 .$$

Put $U = V_1$ and $U' = V_2$. Then $\langle\ ,\ \rangle|_{U\oplus U'}$ is non-degenerate by Lemma 3.6, (i). □

THEOREM 3.10. *Let (G, θ) be a Θ-group of order $2m$ of type* (A-O) *defined by a vector space $(V, \langle\ ,\ \rangle)$ with (ω, m)-bilinear form and S the (ω, m)-automorphism of V corresponding to $(V, \langle\ ,\ \rangle)$. Let $V = V^0 \oplus V^1 \oplus \cdots \oplus V^{m-1}$ be the $\boldsymbol{Z}_m$-gradation of V defined by S.*

(i) *Suppose that $\omega = 0$ and m is odd. We write $r := \min\{\dim V^j;\ j \in \boldsymbol{Z}_m\}$.*

(1) *The eigenvalues of any semisimple element of $\mathfrak{g}_1$ can be written as*

$$\alpha_1, \zeta\alpha_1, \dots, \zeta^{m-1}\alpha_1,\ \alpha_2, \zeta\alpha_2, \dots, \zeta^{m-1}\alpha_2, \dots,\ \alpha_q, \zeta\alpha_q, \dots, \zeta^{m-1}\alpha_q,\ \overbrace{0, \dots, 0}^{\dim V - mq}$$

for some $\alpha_1, \alpha_2, \dots, \alpha_q \in \boldsymbol{C}^\times$ with $q \leq r$.

(2) *Conversely, for any set of complex numbers of the form* (1), *there exists a semisimple element of $\mathfrak{g}_1$ whose set of eigenvalues coincides with it.*

(ii) *For any pair (ω, m) except for the case (i), we write $r := \min\{[\dim V^j/2];\ j \in \boldsymbol{Z}_m\}$.*

(1) *The eigenvalues of any semisimple element of $\mathfrak{g}_1$ can be written as*

$$\alpha_1, \zeta\alpha_1, \dots, \zeta^{m-1}\alpha_1, -\xi^{-1}\alpha_1, -\xi^{-1}\zeta\alpha_1, \dots, -\xi^{-1}\zeta^{m-1}\alpha_1 ,$$

$$\alpha_2, \zeta\alpha_2, \dots, \zeta^{m-1}\alpha_2, -\xi^{-1}\alpha_2, -\xi^{-1}\zeta\alpha_2, \dots, -\xi^{-1}\zeta^{m-1}\alpha_2, \dots ,$$

$$\alpha_q, \zeta\alpha_q, \dots, \zeta^{m-1}\alpha_q, -\xi^{-1}\alpha_q, -\xi^{-1}\zeta\alpha_q, \dots, -\xi^{-1}\zeta^{m-1}\alpha_q,\ \overbrace{0, \dots, 0}^{\dim V - 2mq}$$

for some $\alpha_1, \alpha_2, \dots, \alpha_q \in \boldsymbol{C}^\times$ with $q \leq r$.

(2) *Conversely, for any set of complex numbers of the form* (1), *there exists a semisimple element of $\mathfrak{g}_1$ whose set of eigenvalues coincides with it.*

PROOF. The claims (i, 1) and (ii, 1) follow from Lemmas 3.8 and 3.9.

The claims (i, 2) and (ii, 2) follow from the construction of Cartan subspaces in Section 5 (Lemma 5.23). □

4. Rings of invariants of classical Θ-representations.

(4.1) Surjectivity of the restriction maps. The main theorem of this section is the following.

THEOREM 4.1. *Let* (G, θ) *be a classical* Θ*-group of types* (A-I), (BCD-I) *or* (A-O). *Then for the inclusion* $(G_0, \mathfrak{g}_1) \hookrightarrow (GL(V), \mathfrak{gl}(V))$, *the restriction map*

$$\mathrm{rest} : \boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)} \to \boldsymbol{C}[\mathfrak{g}_1]^{G_0}, \ f \mapsto f|_{\mathfrak{g}_1}$$

is surjective.

Since $\boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)} \to \boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ decomposes as

$$\boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)} \to \boldsymbol{C}[\mathfrak{g}]^{G} \to \boldsymbol{C}[\mathfrak{g}_1]^{G_0},$$

we know that the restriction map $\boldsymbol{C}[\mathfrak{g}]^{G} \to \boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ is also surjective and obtain the following.

COROLLARY 4.2. *For a classical* Θ*-group* (G, θ) *of types* (A-I), (BCD-I) *or* (A-O), *the restriction map* rest : $\boldsymbol{C}[\mathfrak{g}]^{G} \to \boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ *is surjective.*

The proof of Theorem 4.1 will be given in (4.2) and (4.3). Before giving a proof of Theorem 4.1, we recall some facts on the affine quotients by reductive groups. Suppose a reductive algebraic group G acts on an affine variety X. Since the invariant ring $\boldsymbol{C}[X]^G$ is finitely generated by Hilbert's theorem, we can consider the affine variety $X//G := \mathrm{Spec}(\boldsymbol{C}[X]^G)$. It is known that $X//G$ is the categorical quotient of X under the action of G. The morphism $\pi_{(G,X)} : X \to X//G$ defined by the inclusion $\boldsymbol{C}[X]^G \hookrightarrow \boldsymbol{C}[X]$ is called the affine quotient map under G. Clearly $\pi_{(G,X)}$ maps any G-orbit to a point of $X//G$. Moreover, any fibre of $\pi_{(G,X)}$ contains exactly one closed G-orbit (see for example [PoV, Section 4]). Therefore, we obtain a natural identification

$$\{\text{closed } G\text{-orbits in } X\} \simeq X//G, \quad \mathcal{O} \mapsto \pi_{(G,X)}(\mathcal{O}).$$

For a Θ-representation $(G_0, \mathfrak{g}_1)$ defined by (G, θ), we denote by $\mathfrak{g}_1^{\mathrm{ss}}$ the set of semisimple elements in $\mathfrak{g}_1$. Then it is known by [V, Proposition 3] that the set $\mathfrak{g}_1^{\mathrm{ss}}/G_0$ of semisimple G_0-orbits coincides with that of closed G_0-orbits in $\mathfrak{g}_1$. Thus we have a natural identification $\mathfrak{g}_1^{\mathrm{ss}}/G_0 = \mathfrak{g}_1//G_0$.

(4.2) Type (A-I). Let (G, θ) be a Θ-group of order m of type (A-I) defined by a vector space (V, S) with m-automorphism. We also consider the group $G_0^Z := \{g \in G\,;\ \mathrm{Ad}(\theta(g)) = \mathrm{Ad}(g)\}$ which contains G_0.

LEMMA 4.3. (i) G_0 *is a normal subgroup of* G_0^Z *and the quotient* G_0^Z/G_0 *is a finite group.*

(ii) *For* $x \in \mathfrak{g}_1$, *the orbit* $\mathrm{Ad}(G_0) \cdot x$ *is closed in* $\mathfrak{g}_1$ *if and only if* x *is semisimple.*

(iii) *The map* $\mathfrak{g}_1^{\mathrm{ss}}/G_0 \to \mathfrak{g}_1^{\mathrm{ss}}/G_0^Z$, $\mathcal{O} \mapsto \mathrm{Ad}(G_0^Z) \cdot \mathcal{O}$ *is bijective. In particular, it holds* $\mathrm{Ad}(G_0^Z) \cdot \mathcal{O} = \mathcal{O}$ *for any* $\mathcal{O} \in \mathfrak{g}_1^{\mathrm{ss}}/G_0$.

(iv) $\boldsymbol{C}[\mathfrak{g}_1]^{G_0} = \boldsymbol{C}[\mathfrak{g}_1]^{G_0^Z}$.

PROOF. (i) For $g \in G_0^Z$, we put $\alpha(g) := \theta(g)g^{-1}$. Clearly $\alpha(g) \in Z(G)$ and we obtain a homomorphism $\alpha : G_0^Z \to Z(G)$. It is easily verified that $\mathrm{Im}\,\alpha \subset \{c\ \mathrm{id}_V ; c \in \langle\zeta\rangle\}$. Since $\mathrm{Ker}\,\alpha = G_0$, the claim (i) follows. The claim (ii) follows from [V, Proposition 3]. (iii) By Corollary 2.8, (iii), the map $\mathfrak{g}_1^{\mathrm{ss}}/G_0 \to \mathfrak{g}/G$ ($\mathcal{O} \mapsto \mathrm{Ad}(G)\cdot\mathcal{O}$) is injective. Since this map is decomposed as $\mathfrak{g}_1^{\mathrm{ss}}/G_0 \to \mathfrak{g}_1^{\mathrm{ss}}/G_0^Z \to \mathfrak{g}/G$, $\mathfrak{g}_1^{\mathrm{ss}}/G_0 \to \mathfrak{g}_1^{\mathrm{ss}}/G_0^Z$ is also injective. (iv) Take an invariant $f \in \boldsymbol{C}[\mathfrak{g}_1]^{G_0}$. To show that $f \in \boldsymbol{C}[\mathfrak{g}_1]^{G_0^Z}$, it is enough to show that $f(g\cdot x) = f(x)$ for any $x \in \mathfrak{g}_1$ and any $g \in G_0^Z$. Take $y \in \overline{G_0\cdot x}$ so that $G_0\cdot y$ is the unique closed orbit in $\overline{G_0\cdot x}$. Then $f(y) = f(x)$. Since G_0 is a normal subgroup of G_0^Z, we have

$$g\cdot y \in g\cdot\overline{G_0\cdot x} = \overline{(gG_0g^{-1})\cdot(g\cdot x)} = \overline{G_0\cdot(g\cdot x)}\,.$$

Hence $f(g\cdot y) = f(g\cdot x)$. Since y is semisimple by (ii), we have $g\cdot y \in G_0^Z\cdot y = G_0\cdot y$. Hence $f(g\cdot y) = f(y)$. Thus we obtain $f(g\cdot x) = f(g\cdot y) = f(y) = f(x)$. □

THEOREM 4.4 ([O3, Theorem 8]). *Let $\theta : \mathcal{G} \to \mathcal{G}$ be an automorphism of a reductive algebraic group $\mathcal{G}$ over $\boldsymbol{C}$. We denote by $\theta : \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{G})$ the corresponding automorphism of the Lie algebra. Let $\tilde{G}$ be a θ-stable reductive subgroup of $\mathcal{G}$ and $\tilde{L}$ a θ-stable and $\mathrm{Ad}(\tilde{G})$-stable subspace of $\mathrm{Lie}(\mathcal{G})$. Define a closed subgroup G' of $\tilde{G}$ by $G' = \{g \in \tilde{G} ; \mathrm{Ad}_{\tilde{L}}(g) = \mathrm{Ad}_{\tilde{L}}(\theta(g))\}$. Let α be an element of $GL(\tilde{L})$ such that $\alpha(\mathrm{Ad}(g)X) = \mathrm{Ad}(g)\alpha(X)$ for any $g \in \tilde{G}$ and $X \in \tilde{L}$. Define an element $\varphi \in GL(\tilde{L})$ by $\varphi(X) = \alpha^{-1}(\theta(X))$ $(X \in \tilde{L})$. Put $L := \{X \in \tilde{L}; \varphi(X) = X\ (\Leftrightarrow\ \theta(X) = \alpha(X))\}$. Suppose that φ has a finite order. Then, for the inclusion $(G', L) \hookrightarrow (\tilde{G}, \tilde{L})$, we have the following*:

(i) *For the correspondence*

$$L/G' \to \tilde{L}/\tilde{G},\ \mathcal{O} \mapsto \tilde{\mathcal{O}} := \mathrm{Ad}(\tilde{G})\cdot\mathcal{O}\,,$$

$\tilde{\mathcal{O}}$ is closed in $\tilde{L}$ if and only if $\mathcal{O}$ is closed in L.

(ii) *The morphism $L//G' \to \tilde{L}//\tilde{G}$ corresponding to the restriction map* rest : $\boldsymbol{C}[\tilde{L}]^{\tilde{G}} \to \boldsymbol{C}[L]^{G'}$ *is finite, that is, $\boldsymbol{C}[L]^{G'}$ is integral over the image $\boldsymbol{C}[\tilde{L}]^{\tilde{G}}|_L$.*

(iii) *Suppose that the morphism $L//G' \to \tilde{L}//\tilde{G}$ of (ii) is injective. Then the morphism $L//G' = \mathrm{Spec}(\boldsymbol{C}[L]^{G'}) \to \mathrm{Spec}(\boldsymbol{C}[\tilde{L}]^{\tilde{G}}|_L)$ corresponding to $\boldsymbol{C}[\tilde{L}]^{\tilde{G}}|_L \hookrightarrow \boldsymbol{C}[L]^{G'}$ is bijective and birational (i.e., the quotient fields of $\boldsymbol{C}[\tilde{L}]^{\tilde{G}}|_L$ and $\boldsymbol{C}[L]^{G'}$ coincide). In particular, since $\boldsymbol{C}[L]^{G'}$ is normal (i.e., integrally closed in the quotient field), $\boldsymbol{C}[L]^{G'}$ is the integral closure of $\boldsymbol{C}[\tilde{L}]^{\tilde{G}}|_L$ in the quotient field.*

By putting $\mathcal{G} = \tilde{G} = G = GL(V)$, $\tilde{L} = \mathfrak{g} = \mathfrak{gl}(V)$, $\theta = \mathrm{Ad}(S)$ and $\alpha(X) = \zeta X (X \in \mathfrak{g})$, G' and L become $G' = G_0^Z$ and $L = \mathfrak{g}_1$. By Corollary 2.8 and Lemma 4.3, (iii), the map $\mathfrak{g}_1//G_0^Z \to \mathfrak{g}//G$ is injective. Then by Theorem 4.4, (iii), $\boldsymbol{C}[\mathfrak{g}_1]^{G_0^Z}$ is the integral closure of $\boldsymbol{C}[\mathfrak{g}]^G|_{\mathfrak{g}_1}$ in its quotient field. Since $\boldsymbol{C}[\mathfrak{g}_1]^{G_0} = \boldsymbol{C}[\mathfrak{g}_1]^{G_0^Z}$, we have the following.

LEMMA 4.5. *$\boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ is the integral closure of $\boldsymbol{C}[\mathfrak{g}]^G|_{\mathfrak{g}_1}$ in its quotient field.*

NOTATION 4.6. (i) For an n-dimensional vector space V, define functions $P_1, P_2, \ldots, P_n \in \boldsymbol{C}[\mathfrak{gl}(V)]$ by

$$\det(t\ \mathrm{id}_V - X) = t^n + P_1(X)t^{n-1} + \cdots + P_n(X)\,.$$

It is well-known that $\boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)} = \boldsymbol{C}[P_1, P_2, \dots, P_n]$.

(ii) For r-variables $t_1, t_2, \dots, t_r$, we define elementary symmetric polynomials $F_1, F_2, \dots, F_r \in \boldsymbol{C}[t_1, t_2, \dots, t_r]$ by

$$(t - t_1)(t - t_2)\cdots(t - t_r) = t^r + F_1(t_1, t_2, \dots, t_r)t^{r-1} + \cdots + F_r(t_1, t_2, \dots, t_r)\,.$$

PROOF OF THEOREM 4.1 FOR THE TYPE (A-I). We put $n = \dim V$. It is enough to show that $\boldsymbol{C}[\mathfrak{g}]^G|_{\mathfrak{g}_1} = \boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\mathfrak{g}_1}$ is a polynomial ring. For any $X \in \mathfrak{g}_1$, by Corollary 2.8, the eigenvalues of X are of the form

$$\alpha_1, \zeta\alpha_1, \dots, \zeta^{m-1}\alpha_1, \alpha_2, \zeta\alpha_2, \dots, \zeta^{m-1}\alpha_2, \dots, \alpha_r, \zeta\alpha_r, \dots, \zeta^{m-1}\alpha_r, \overbrace{0, \dots, 0}^{n-mr}\,.$$

Since

$$\det(t\,\mathrm{id}_V - X) = \left(\prod_{k=1}^{r}(t-\alpha_k)(t-\zeta\alpha_k)\cdots(t-\zeta^{m-1}\alpha_k)\right)t^{n-mr} = \left(\prod_{k=1}^{r}(t^m-\alpha_k^m)\right)t^{n-mr}\,,$$

we have $P_{mj}(X) = F_j(\alpha_1^m, \alpha_2^m, \dots, \alpha_r^m)$ and $P_k(X) = 0$ $(k \neq mj,\ 1 \leq j \leq r)$. Since $\alpha_1, \alpha_2, \dots, \alpha_r$ can take any values, $P_{mj}|_{\mathfrak{g}_1} (1 \leq j \leq r)$ are algebraically independent. Therefore $\boldsymbol{C}[\mathfrak{g}]^G|_{\mathfrak{g}_1} = \boldsymbol{C}[P_{mj}|_{\mathfrak{g}_1}]_{1\leq j\leq r}$ is a polynomial ring. □

Cosequently, we have

$$\boldsymbol{C}[\mathfrak{g}_1]^{G_0} = \boldsymbol{C}[P_{mj}|_{\mathfrak{g}_1}]_{1\leq j\leq r}\,. \tag{4.1}$$

(4.3) Types (BCD-I) and (A-O). Let (G, θ) be a Θ-group of order m of type (BCD-I) or a Θ-group of order $2m$ of type (A-O). We put $\tilde{G} = GL(V)$ and consider the associated Θ-group $(\tilde{G}, \mathrm{Ad}(S))$ of order m of type (A-I). We notice that $\theta = \mathrm{Ad}(S)$ in the case of type (BCD-I) and that $\theta^2 = \mathrm{Ad}(S)$ in the case of type (A-O). As before, we put $\zeta = e^{2\pi\sqrt{-1}/m}$ and $\xi = e^{\pi\sqrt{-1}/m}$. We also notice that

$$G_0 = G^\theta = \{g \in \tilde{G}_0;\, g^* = g^{-1}\}\,, \quad \mathfrak{g}_1 = \begin{cases}\{X \in \tilde{\mathfrak{g}}_1;\, X^* = -X\} & ((\text{BCD-I}))\,,\\ \{X \in \tilde{\mathfrak{g}}_1;\, X^* = -\xi X\} & ((\text{A-O}))\,.\end{cases}$$

In both cases, we write $V = V^0 \oplus V^1 \oplus \cdots \oplus V^{m-1}$ the $\boldsymbol{Z}_m$-gradation of V defined by S. To give a proof of Theorem 4.1 for these cases, we need the following.

THEOREM 4.7 ([O3, Theorem 12]). *In the setting of Proposition* 1.8, *we assume furthermore the following.*

(c) *The element $\varphi \in GL(\tilde{L})$, defined by $\varphi(X) = \alpha^{-1}(\sigma(X))$ $(X \in \tilde{L})$, has a finite order.*

Then we have the following:

(i) *For the correspondence*

$$L/H \to \tilde{L}/\tilde{H},\ \mathcal{O} \mapsto \tilde{\mathcal{O}} := \mathrm{Ad}(\tilde{H})\cdot\mathcal{O}\,,$$

$\tilde{\mathcal{O}}$ is closed in $\tilde{L}$ if and only if $\mathcal{O}$ is closed in L.

(ii) *The morphism $L//H \to$ $\mathrm{Spec}(\boldsymbol{C}[\tilde{L}]^{\tilde{H}}|_L)$, defined by $\boldsymbol{C}[\tilde{L}]^{\tilde{H}}|_L \hookrightarrow \boldsymbol{C}[L]^H$, is bijective and gives a normalization of the variety* $\mathrm{Spec}(\boldsymbol{C}[\tilde{L}]^{\tilde{H}}|_L)$ *(i.e., $L//H$ is normal and* *the morphism is finite, birational). In particular,* $\boldsymbol{C}[L]^H$ *is the integral closure of* $\boldsymbol{C}[\tilde{L}]^{\tilde{H}}|_L$ *in its quotient field.*

By applying Theorem 4.7 to the inclusion $(G_0, \mathfrak{g}_1) \hookrightarrow (\tilde{G}_0, \tilde{\mathfrak{g}}_1)$, we obtain the following.

LEMMA 4.8. $\boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ *is the integral closure of* $\boldsymbol{C}[\tilde{\mathfrak{g}}_1]^{\tilde{G}_0}|_{\mathfrak{g}_1}$ *in its quotient field.*

By the case of (A-I) of Theorem 4.1, we have $\boldsymbol{C}[\tilde{\mathfrak{g}}_1]^{\tilde{G}_0} = \boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\tilde{\mathfrak{g}}_1}$. Hence $\boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ is the integral closure of $\boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\mathfrak{g}_1}$. Therefore, to prove Theorem 4.1 for the cases (BCD-I) and (A-O), it is enough to show that $\boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\mathfrak{g}_1}$ is a polynomial ring.

Before giving a proof for types (BCD-I) and (A-O), we prepare the polynomials $Q_1, Q_2, \dots, Q_r$ defined as follows.

LEMMA 4.9. *Let* $x_1, x_2, \dots, x_r$ *be variables. Define* $a_1, a_2, \dots, a_{2r} \in \boldsymbol{C}[x_1, x_2, \dots, x_r]$ *by*

$$(t^r + x_1 t^{r-1} + \cdots + x_{r-1} t + x_r)^2 = t^{2r} + a_1 t^{2r-1} + \cdots + a_{2r-1} t + a_{2r} .$$

Then it holds $\boldsymbol{C}[a_1, a_2, \dots, a_r] = \boldsymbol{C}[x_1, x_2, \dots, x_r]$. *In other words, there exist polynomials* $Q_1, Q_2, \dots, Q_r \in \boldsymbol{C}[t_1, t_2, \dots, t_r]$ *in variables* $t_1, t_2, \dots, t_r$ *such that*

$$x_i = Q_i(a_1, a_2, \dots, a_r) \ (1 \le i \le r) .$$

To give a proof of Theorem 4.1 for Types (BCD-I) and (A-O), we separate the Θ-groups of type (BCD-I) and (A-O) into the following three cases.

Case I. (a) (G, θ) is of type (BCD-I), $(\varepsilon, \omega) = (1, 0)$ or $(-1, 1)$ and m is even.

(b) (G, θ) is of type (A-O), $\omega = 0$ and m is odd.

For Case I, we put $r = \min\{\dim V^j;\ j \in \boldsymbol{Z}_m\}$ (cf. Theorem 3.5, (i) and Theorem 3.10, (i)).

Case II. (a) (G, θ) is of type (BCD-I) and m is odd.

(b) (G, θ) is of type (A-O) and m is even.

Case III. (a) (G, θ) is of type (BCD-I), $(\varepsilon, \omega) = (1, 1)$ or $(-1, 0)$ and m is even.

(b) (G, θ) is of type (A-O), $\omega = 1$ and m is odd.

For Cases II and III, we put $r = \min\{[\dim V^j/2];\ j \in \boldsymbol{Z}_m\}$ (cf. Theorem 3.5, (ii) and Theorem 3.10, (ii)).

PEROOF OF THEOREM 4.1 FOR CASE I. Let (G, θ) be a Θ-group in Case I. As in the proof for type (A-I), we can show that $\boldsymbol{C}[\mathfrak{g}]^G|_{\mathfrak{g}_1} = \boldsymbol{C}[P_{mj}|_{\mathfrak{g}_1}]_{1 \le j \le r}$ and that $P_{mj}|_{\mathfrak{g}_1} (1 \le j \le r)$ are algebraically independent, by using Theorem 3.5, (i) and Theorem 3.10. Thus, $\boldsymbol{C}[\mathfrak{g}]^G|_{\mathfrak{g}_1}$ is a polynomial ring and hence Theorem 4.1 is proved for Case I. □

Consequently, we have

$$\boldsymbol{C}[\mathfrak{g}_1]^{G_0} = \boldsymbol{C}[P_{mj}|_{\mathfrak{g}_1}]_{1 \le j \le r} . \tag{4.2}$$

From now on, we assume that (G, θ) is a Θ-group in Case II or Case III. Then the following lemma is an easy consequence of Theorem 3.5, (ii) and Theorem 3.10, (ii).

LEMMA 4.10. *For a* Θ*-group* (G, θ) *contained in Case II or Case III, we have the following.*

(i) *For any* $X \in \mathfrak{g}_1$, *there exist complex numbers* $\alpha_1, \alpha_2, \dots, \alpha_r \in \boldsymbol{C}$ *such that*

$$\det(t\ \mathrm{id}_V - X) = \begin{cases} (\prod_{k=1}^{r}(t^{2m} - \alpha_k^{2m}))t^{n-2mr} & \text{(Case II)} \\ (\prod_{k=1}^{r}(t^{m} - \alpha_k^{m}))^2 t^{n-2mr} & \text{(Case III).} \end{cases}$$

(ii) *For any* $\alpha_1, \alpha_2, \dots, \alpha_r \in \boldsymbol{C}$, *there exists an element* $X \in \mathfrak{g}_1$ *which satisfies* (i).

PROOF OF THEOREM 4.1 FOR CASES II AND III. Let us give a proof of Theorem 4.1 for Cases II and III.

Let (G, θ) be a Θ-group in Case II. For any $X \in \mathfrak{g}_1$, there exist complex numbers $\alpha_1, \alpha_2, \dots, \alpha_r \in \boldsymbol{C}$ such that

$$\det(t\ \mathrm{id}_V - X) = ((t^{2m})^r + F_1(\alpha_1^{2m}, \dots, \alpha_r^{2m})(t^{2m})^{r-1} + \cdots + F_r(\alpha_1^{2m}, \dots, \alpha_r^{2m}))t^{n-2mr}$$

by Lemma 4.10. Therefore $P_{2im}(X) = F_i(\alpha_1^{2m}, \dots, \alpha_r^{2m})$ $(1 \le i \le r)$ and $P_k(X)$ for other k's are zero. Hence we have $\boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\mathfrak{g}_1} = \boldsymbol{C}[P_{2im}|_{\mathfrak{g}_1}]_{1 \le i \le r}$.

On the other hand, for given $\alpha_1, \alpha_2, \dots, \alpha_r \in \boldsymbol{C}$, there exists $X \in \mathfrak{g}_1$ such that $P_{2im}(X) = F_i(\alpha_1^{2m}, \dots, \alpha_r^{2m})$ $(1 \le i \le r)$. Hence $P_{2im}|_{\mathfrak{g}_1}$ $(1 \le i \le r)$ are algebraically independent and $\boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\mathfrak{g}_1}$ is a polynomial ring. Consequently, we have

$$\boldsymbol{C}[\mathfrak{g}_1]^{G_0} = \boldsymbol{C}[P_{2mj}|_{\mathfrak{g}_1}]_{1 \le j \le r}\,. \tag{4.3}$$

Next consider Case III. Let (G, θ) be a Θ-group in Case III. For any $X \in \mathfrak{g}_1$, there exist complex numbers $\alpha_1, \alpha_2, \dots, \alpha_r \in \boldsymbol{C}$ such that

$$\det(t\ \mathrm{id}_V - X) = ((t^{m})^r + F_1(\alpha_1^{m}, \dots, \alpha_r^{m})(t^{m})^{r-1} + \cdots + F_r(\alpha_1^{m}, \dots, \alpha_r^{m}))^2 t^{n-2mr}$$

by Lemma 4.10. Therefore $P_k(X) = 0$ for $k \ne im$ $(1 \le i \le 2r)$. Let $Q_1, Q_2, \dots, Q_r$ be the polynomials obtained in Lemma 4.9. Define functions $f_1, f_2, \dots, f_r \in \boldsymbol{C}[\mathfrak{gl}(V)]$ by

$$f_i = Q_i(P_m, P_{2m}, \dots, P_{rm})\,.$$

Then we have $f_i(X) = F_i(\alpha_1^m, \dots, \alpha_r^m)$ $(1 \le i \le r)$. Since

$$((t^m)^r + f_1(X)(t^m)^{r-1} + \cdots + f_r(X))^2 = (t^m)^{2r} + P_m(X)(t^m)^{2r-1} + \cdots + P_{2rm}(X)\,,$$

we have

$$\begin{aligned} \boldsymbol{C}[P_m|_{\mathfrak{g}_1}, P_{2m}|_{\mathfrak{g}_1}, \dots, P_{rm}|_{\mathfrak{g}_1}] &\supset \boldsymbol{C}[f_1|_{\mathfrak{g}_1}, f_2|_{\mathfrak{g}_1}, \dots, f_r|_{\mathfrak{g}_1}] \\ &\supset \boldsymbol{C}[P_m|_{\mathfrak{g}_1}, P_{2m}|_{\mathfrak{g}_1}, \dots, P_{rm}|_{\mathfrak{g}_1}] = \boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\mathfrak{g}_1}\,. \end{aligned}$$

Hence we have

$$\boldsymbol{C}[P_m|_{\mathfrak{g}_1}, P_{2m}|_{\mathfrak{g}_1}, \dots, P_{rm}|_{\mathfrak{g}_1}] = \boldsymbol{C}[f_1|_{\mathfrak{g}_1}, f_2|_{\mathfrak{g}_1}, \dots, f_r|_{\mathfrak{g}_1}] = \boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\mathfrak{g}_1}\,.$$

By Lemma 4.10, (ii), $f_1|_{\mathfrak{g}_1}, f_2|_{\mathfrak{g}_1}, \dots, f_r|_{\mathfrak{g}_1}$ are algebraically independent and hence $\boldsymbol{C}[\mathfrak{gl}(V)]^{GL(V)}|_{\mathfrak{g}_1}$ is a polynomial ring. Consequently, we have

$$\boldsymbol{C}[\mathfrak{g}_1]^{G_0} = \boldsymbol{C}[f_j|_{\mathfrak{g}_1}]_{1 \le j \le r} = \boldsymbol{C}[P_{mj}|_{\mathfrak{g}_1}]_{1 \le j \le r}\,. \tag{4.4}$$

Therefore the proof of Theorem 4.1 is completed. □

COROLLARY 4.11. *For a classical Θ-group (G,θ), $\boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ is isomorphic to a polynomial ring of r variables. Moreover, algebraically independent generators of the ring $\boldsymbol{C}[\mathfrak{g}_1]^{G_0}$ are given in the equations* (4.1) *through* (4.4).

5. Cartan subspaces and Weyl groups.

(5.1) Inclusion theorem for orbits and Weyl groups. Let (G,θ) be a general reductive Θ-group. A maximal abelian subspace $\mathfrak{c}$ of $\mathfrak{g}_1$ which consists of semisimple elements is called a Cartan subspace of the Θ-representation $(G_0,\mathfrak{g}_1)$ ([V]). It is known by [V, Theorem 3.1] that any two Cartan subspaces are conjugate by an element of the identity component $(G_0)^0$ of G_0.

Let $\mathfrak{c}$ be a Cartan subspace of (G,θ). Although Vinberg studied the Weyl group $W((G_0)^0,\mathfrak{c}) = N_{(G_0)^0}(\mathfrak{c})/Z_{(G_0)^0}(\mathfrak{c})$, in this paper, we study

$$W(G_0,\mathfrak{c}) := N_{G_0}(\mathfrak{c})/Z_{G_0}(\mathfrak{c})$$

which we call the Weyl group of the Θ-representation $(G_0,\mathfrak{g}_1)$.

THEOREM 5.1 (cf. [V, Theorem 3.2]). *The correspondence of orbits*

$$\mathfrak{c}/W(G_0,\mathfrak{c}) \to \mathfrak{g}_1/G_0,\ \mathcal{O} \mapsto \mathrm{Ad}(G_0)\cdot\mathcal{O}$$

is injective.

PROOF. It was shown in [V, Theorem 3.2] that the map $\mathfrak{c}/W((G_0)^0,\mathfrak{c}) \to \mathfrak{g}_1/(G_0)^0$ is injective. But the proof can be applied to our setting and we obtain Theorem 5.1. □

THEOREM 5.2. *Let H be a complex reductive algebraic group and K a reductive closed subgroup. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{c}$ a subspace of $\mathfrak{t}$. Let us consider the following groups*:

$$W = W(H,\mathfrak{t}) := N_H(\mathfrak{t})/Z_H(\mathfrak{t}) \subset GL(\mathfrak{t}),\quad W(K,\mathfrak{c}) := N_K(\mathfrak{c})/Z_K(\mathfrak{c}) \subset GL(\mathfrak{c}),$$

$$N_W(\mathfrak{c})|_{\mathfrak{c}} := \{w|_{\mathfrak{c}};\ w \in W,\ w\cdot\mathfrak{c} = \mathfrak{c}\}.$$

Then we have the following.

(i) *As subgroups of $GL(\mathfrak{c})$, it holds $W(K,\mathfrak{c}) \subset N_W(\mathfrak{c})|_{\mathfrak{c}}$.*

(ii) *If the map $\mathfrak{c}/W(K,\mathfrak{c}) \to \mathfrak{h}/H$ defined by $\mathcal{O} \mapsto \mathrm{Ad}(H)\cdot\mathcal{O}$ is injective, it holds that $W(K,\mathfrak{c}) = N_W(\mathfrak{c})|_{\mathfrak{c}}$.*

PROOF. (i) Let us put

$$\mathfrak{a} := \mathfrak{z}(\mathfrak{z}_{\mathfrak{h}}(\mathfrak{c})),\quad \mathfrak{s} := [\mathfrak{z}_{\mathfrak{h}}(\mathfrak{c}),\mathfrak{z}_{\mathfrak{h}}(\mathfrak{c})].$$

Since $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{c})$ is reductive, we have $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{c}) = \mathfrak{a}\oplus\mathfrak{s}$. Since $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{c})$, there exists a Cartan subalgebra $\mathfrak{t}'$ of $\mathfrak{s}$ such that $\mathfrak{t} = \mathfrak{a}\oplus\mathfrak{t}'$. Then for any $g \in N_H(\mathfrak{c})$, since $\mathrm{Ad}(g)\cdot\mathfrak{z}_{\mathfrak{h}}(\mathfrak{c}) = \mathfrak{z}_{\mathfrak{h}}(\mathfrak{c})$, we have $\mathrm{Ad}(g)\cdot\mathfrak{a} = \mathfrak{a}$ and $\mathrm{Ad}(g)\cdot\mathfrak{s} = \mathfrak{s}$.

Let us take $w \in W(K,\mathfrak{c})$ and $g \in N_K(\mathfrak{c})$ such that $w = \mathrm{Ad}(g)|_{\mathfrak{c}}$. Since $\mathrm{Ad}(g)\cdot\mathfrak{a} = \mathfrak{a}$, $\mathrm{Ad}(g)\cdot\mathfrak{s} = \mathfrak{s}$ and $\mathrm{Ad}(g)\cdot\mathfrak{t}'$ is a Cartan subalgebra of $\mathfrak{s}$, there exists an element h of the connected subgroup of H corresponding to $\mathfrak{s}$ such that $\mathfrak{t}' = \mathrm{Ad}(h)\mathrm{Ad}(g)\cdot\mathfrak{t}' = \mathrm{Ad}(hg)\cdot\mathfrak{t}'$. Since $h \in Z_H(\mathfrak{c})$ and $\mathfrak{a}$ is the center of $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{c})$, $\mathrm{Ad}(h)$ acts trivially on $\mathfrak{a}$. Thus we see $\mathrm{Ad}(hg)\cdot$

$\mathfrak{a} = \mathrm{Ad}(g) \cdot \mathfrak{a} = \mathfrak{a}$ and $\mathrm{Ad}(hg) \cdot \mathfrak{t} = \mathfrak{t}$. Hence $hg \in N_H(\mathfrak{t})$. Let us put $w' := \mathrm{Ad}(hg)|_{\mathfrak{t}} \in W$. Then, since $\mathrm{Ad}(h)$ acts trivially on $\mathfrak{c}$, we have $w' \in N_W(\mathfrak{c})$ and $w = \mathrm{Ad}(g)|_{\mathfrak{c}} = \mathrm{Ad}(hg)|_{\mathfrak{c}} = w'|_{\mathfrak{c}} \in N_W(\mathfrak{c})|_{\mathfrak{c}}$.

(ii) Since $W(K, \mathfrak{c}) \subset N_W(\mathfrak{c})|_{\mathfrak{c}}$, it holds that $W(K, \mathfrak{c}) \cdot x \subset W \cdot x$ for any $x \in \mathfrak{c}$. Hence the injection $\mathfrak{c}/W(K, \mathfrak{c}) \to \mathfrak{h}/H$ is decomposed as

$$\mathfrak{c}/W(K, \mathfrak{c}) \to \mathfrak{t}/W \to \mathfrak{h}/H .$$

Therefore, $\mathfrak{c}/W(K, \mathfrak{c}) \to \mathfrak{t}/W$ is also injective. Since this map is decomposed as

$$\mathfrak{c}/W(K, \mathfrak{c}) \to \mathfrak{c}/N_W(\mathfrak{c})|_{\mathfrak{c}} \to \mathfrak{t}/W ,$$

the map $\mathfrak{c}/W(K, \mathfrak{c}) \to \mathfrak{c}/N_W(\mathfrak{c})|_{\mathfrak{c}}$ is bijective. Thus, for any $x \in \mathfrak{c}$, we have $W(K, \mathfrak{c}) \cdot x = N_W(\mathfrak{c})|_{\mathfrak{c}} \cdot x$.

Let us show that there exists $x \in \mathfrak{c}$ such that $Z_H(\mathfrak{c}) = Z_H(x)$. We may assume that $H \subset GL(V)$. It is clear that we can take $x \in \mathfrak{c}$ so that $Z_{GL(V)}(\mathfrak{c}) = Z_{GL(V)}(x)$. Then, by taking the intersections with H, we obtain the above equality. Thus we take $x \in \mathfrak{c}$ as above. For any $w \in N_W(\mathfrak{c})|_{\mathfrak{c}}$, take $g \in N_H(\mathfrak{c}) \cap N_H(\mathfrak{t})$ such that $w = \mathrm{Ad}(g)|_{\mathfrak{c}}$. Since $w \cdot x \in N_W(\mathfrak{c})|_{\mathfrak{c}} \cdot x = W(K, \mathfrak{c}) \cdot x$, there exists $w_1 \in W(K, \mathfrak{c})$ such that $w \cdot x = w_1 \cdot x$. Take $g_1 \in N_K(\mathfrak{c})$ so that $w_1 = \mathrm{Ad}(g_1)|_{\mathfrak{c}}$. Then clearly $\mathrm{Ad}(g_1^{-1} g) \cdot x = x$ and hence $g_1^{-1} g \in Z_H(x) = Z_H(\mathfrak{c})$. Therefore, for any $y \in \mathfrak{c}$, we have $w \cdot y = \mathrm{Ad}(g) \cdot y = \mathrm{Ad}(g_1) \cdot y = w_1 \cdot y$. Hence $w = w_1 \in W(K, \mathfrak{c})$ and we obtain $N_W(\mathfrak{c})|_{\mathfrak{c}} \subset W(K, \mathfrak{c})$. □

THEOREM 5.3. *Let (G, θ) be one of the classical Θ-groups in* (1.1); $(G_0, \mathfrak{g}_1) \hookrightarrow (GL(V), \mathfrak{gl}(V))$. *Let $\mathfrak{c} \subset \mathfrak{g}_1$ be a Cartan subspace and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{gl}(V)$ which contains $\mathfrak{c}$. We consider the following groups*:

$$W(G_0, \mathfrak{c}) = N_{G_0}(\mathfrak{c})/Z_{G_0}(\mathfrak{c}), \quad W(GL(V), \mathfrak{t}) = N_{GL(V)}(\mathfrak{t})/Z_{GL(V)}(\mathfrak{t}) ,$$

$$N_{W(GL(V),\mathfrak{t})}(\mathfrak{c})|_{\mathfrak{c}} := \{w|_{\mathfrak{c}};\, w \in N_{W(GL(V),\mathfrak{t})}(\mathfrak{c})\} .$$

Notice that $W(GL(V), \mathfrak{t})$ is isomorphic to the symmetric group of degree $\dim V$. *Then we have $W(G_0, \mathfrak{c}) = N_{W(GL(V),\mathfrak{t})}(\mathfrak{c})|_{\mathfrak{c}}$.*

PROOF. By Corollary 2.8, (iii) or Corollary 2.9, (iii), the map $\mathfrak{g}_1^{\mathrm{ss}}/G_0 \to \mathfrak{gl}(V)/GL(V)$ defined by $\mathcal{O} \mapsto \mathrm{Ad}(GL(V)) \cdot \mathcal{O}$ is injective. On the other hand, $\mathfrak{c}/W(G_0, \mathfrak{c}) \to \mathfrak{g}_1^{\mathrm{ss}}/G_0$ is also injective by Theorem 5.1. Hence $\mathfrak{c}/W(G_0, \mathfrak{c}) \to \mathfrak{gl}(V)/GL(V)$ is injective. By applying Theorem 5.2, (ii) to $H = GL(V)$ and $K = G_0$, we obtain the equality. □

(5.2) Cartan subspaces and Weyl groups of Θ-representations of type (A-I). Let $(G, \theta) = (GL(V), \mathrm{Ad}(S))$ be a Θ-group of order m of type (A-I) defined by a vector space (V, S) with an m-automorphism. Let $V = \bigoplus_{j \in \mathbf{Z}_m} V^j$ be the $\mathbf{Z}_m$-gradation of V defined by S. We put

$$\zeta = e^{2\pi\sqrt{-1}/m},\ n = \dim V,\ n_j = \dim V^j\ (j \in \mathbf{Z}_m),\ r = \min\{n_j\,;\ j \in \mathbf{Z}_m\} .$$

For each $j \in \mathbf{Z}_m$, take a basis $v_1^j, v_2^j, \dots, v_r^j, v_{r+1}^j, \dots, v_{n_j}^j$ of V^j and define $X_k \in \mathfrak{g}$ $(1 \le k \le r)$ by $X_k v_l^j = \delta_{k,l} v_k^{j+1}$. It is clear that $X_k \in \mathfrak{g}_1$. We define a subspace $\mathfrak{c}$ of $\mathfrak{g}_1$ by

$\mathfrak{c} = \langle X_1, X_2, \dots, X_r \rangle_{\boldsymbol{C}}$. Next we put

$$u_k^j = \sum_{i \in \mathbf{Z}_m} (\zeta^{-j})^i v_k^i \ (1 \le k \le r,\ j \in \mathbf{Z}_m)\,,$$

$$\mathcal{B}_{\mathfrak{c}} = \{u_k^j\,;\ 1 \le k \le r,\ j \in \mathbf{Z}_m\}, \quad \mathcal{B}_0 = \{v_l^j\,;\ l > r,\ j \in \mathbf{Z}_m\}\,.$$

Thus we obtain a basis $\mathcal{B} := \mathcal{B}_{\mathfrak{c}} \cup \mathcal{B}_0$ of V. Then we have the following.

LEMMA 5.4. (i) *$X_k u_l^j = \delta_{k,l} \zeta^j u_k^j$ and $X_k v = 0$ for $v \in \mathcal{B}_0$.*
(ii) *$\mathfrak{c}$ is a Cartan subspace of (G, θ).*

PROOF. (i) is obtained by easy computation.

(ii) We easily see that (a) $\mathfrak{c}$ is abelian, and (b) $\mathfrak{c}$ consists of semisimple elements. By Corollary 4.11, $\dim(\mathfrak{g}_1//G_0) = \dim \operatorname{Spec}(\boldsymbol{C}[\mathfrak{g}_1]^{G_0}) = r = \dim \mathfrak{c}$. By [V, Theorem 4.5], $\dim(\mathfrak{g}_1//G_0)$ coincides with the dimension of a Cartan subspace of (G, θ). Hence $\mathfrak{c}$ is maximal in the sense of (a) and (b). Therefore $\mathfrak{c}$ is a Cartan subspace of (G, θ). □

Let us show that the Weyl group of $(G_0, \mathfrak{g}_1)$ is essentially the normalizer of $\mathfrak{c}$ in the Weyl group of $G = GL(V)$.

By using the basis $\mathcal{B}$, we define a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{gl}(V)$ and the Weyl group $W_{GL(V)}$ of $G = GL(V)$ by

(5.1)
$$\mathfrak{t} = \{X \in \mathfrak{gl}(V);\ Xu \in \boldsymbol{C}u \text{ for any } u \in \mathcal{B}\}, \quad W_{GL(V)} = N_{GL(V)}(\mathfrak{t})/Z_{GL(V)}(\mathfrak{t}) \simeq S_n\,,$$

where S_n denotes the symmetric group of degree n. The permutation group $P(\mathcal{B})$ of the set $\mathcal{B}$ is naturally identified with a subgroup of $GL(V)$ and we have a natural identification

$$W_{GL(V)} = \operatorname{Ad}(P(\mathcal{B}))|_{\mathfrak{t}}\,. \tag{5.2}$$

DEFINITION 5.5. (i) For $\sigma \in S_r$ and $(p_1, p_2, \dots, p_r) \in (\mathbf{Z}_m)^r$ $(1 \le k \le r)$, define $g = g(p_1, p_2, \dots, p_r; \sigma) \in P(\mathcal{B})$ so that it satisfies (a) $gu_k^j = u_{\sigma(k)}^{j-p_k}$, and (b) $gv = v$ for any $v \in \mathcal{B}_0$.

(ii) Define a subgroup $W_{\mathfrak{c}}$ of $W_{GL(V)}$ by

$$W_{\mathfrak{c}} := \{\operatorname{Ad}(g(p_1, p_2, \dots, p_r; \sigma))|_{\mathfrak{t}}\,;\ \sigma \in S_r,\ (p_1, p_2, \dots, p_r) \in (\mathbf{Z}_m)^r\}\,.$$

LEMMA 5.6. *The equalities* $\operatorname{Ad}(g(p_1, p_2, \dots, p_r; \sigma))X_k = \zeta^{p_k} X_{\sigma(k)}$ $(1 \le k \le r)$ *hold.*

PROOF. By the definition of $g = g(p_1, p_2, \dots, p_r; \sigma)$, it is easy to see that $g^{-1}u_l^i = u_{\sigma^{-1}(l)}^{i+p_{\sigma^{-1}(l)}}$. Then we have $(gX_k g^{-1})u_l^i = gX_k u_{\sigma^{-1}(l)}^{i+p_{\sigma^{-1}(l)}} = 0$ if $l \neq \sigma(k)$ and

$$\begin{aligned}(gX_k g^{-1})u_{\sigma(k)}^i &= gX_k u_k^{i+p_k} = g(\zeta^{i+p_k} u_k^{i+p_k}) = \zeta^{i+p_k} u_{\sigma(k)}^{i+p_k-p_k} \\ &= \zeta^{p_k}(\zeta^i u_{\sigma(k)}^i) = \zeta^{p_k} X_{\sigma(k)} u_{\sigma(k)}^i\,.\end{aligned}$$

Hence we obtain Lemma 5.6. □

LEMMA 5.7. *For any* $w \in N_{W_{GL(V)}}(\mathfrak{c})$, *there exist* $\sigma \in S_r$ *and* $(p_1, p_2, \dots, p_r) \in (\mathbf{Z}_m)^r$ *such that* $w|_{\mathfrak{c}} = \mathrm{Ad}(g(p_1, p_2, \dots, p_r; \sigma))|_{\mathfrak{c}}$.

PROOF. Take $g \in P(\mathcal{B})$ so that $\mathrm{Ad}(g)|_{\mathfrak{t}} = w$. Then it is easy to see that $g \cdot \mathcal{B}_{\mathfrak{c}} = \mathcal{B}_{\mathfrak{c}}$ and $g \cdot \mathcal{B}_0 = \mathcal{B}_0$. Since w normalizes $\mathfrak{c}$, for each $1 \leq k \leq r$, there exist $c_i \in \boldsymbol{C}$ $(1 \leq i \leq r)$ such that $w \cdot X_k = \mathrm{Ad}(g)X_k = \sum_{i=1}^r c_i X_i$. The matrix expression of $w \cdot X_k$ with respect to the basis $\mathcal{B}$ is diagonal and the number of non-zero entries is just m. We know that only one c_i is non-zero. Thus there exist $\sigma \in S_r$ and $a_k \in \boldsymbol{C}^\times$ such that $\mathrm{Ad}(g)X_k = a_k X_{\sigma(k)}$. By comparing the eigenvalues, we have $a_k \in \langle \zeta \rangle$. Therefore, there exists $p_k \in \mathbf{Z}_m$ such that $\mathrm{Ad}(g) \cdot X_k = \zeta^{p_k} X_{\sigma(k)} = \mathrm{Ad}(g(p_1, p_2, \dots, p_r; \sigma)) \cdot X_k$. Hence $w|_{\mathfrak{c}} = \mathrm{Ad}(g(p_1, p_2, \dots, p_r; \sigma))|_{\mathfrak{c}}$. □

Following Shephard and Todd [ST], let us denote by $G(m, 1, r)$ the group of the monomial matrices of size $r \times r$ whose non-zero entries are contained in $\langle \zeta \rangle$.

PROPOSITION 5.8. *The homomorphism* $\rho : W_{\mathfrak{c}} \to GL(\mathfrak{c})$ *defined by* $w \mapsto w|_{\mathfrak{c}}$ *is injective and the image coincides with the Weyl group* $W(G_0, \mathfrak{c})$. *As a consequence*, *we have* $W(G_0, \mathfrak{c}) \simeq W_{\mathfrak{c}} \simeq G(m, 1, r)$.

PROOF. The injectivity of ρ is trivial. By Theorem 5.3 and Lemma 5.7, we have

$$W(G_0, \mathfrak{c}) = \rho(N_{W_{GL(V)}}(\mathfrak{c})) = \rho(W_{\mathfrak{c}}) \simeq W_{\mathfrak{c}} \,.$$

By Lemma 5.6, $W_{\mathfrak{c}}$ is isomorphic to $G(m, 1, r)$. □

(5.3) Cartan subspaces and Weyl groups of Θ-representations of type (BCD-I). Let (G, θ) a Θ-group of order m of type (BCD-I) defined by an (ε, ω)-space $(V, (\ ,\ ), S)$ with m-automorphism. We use the notations of (4.3). To construct Cartan subspaces, we first give the following lemma, the proof of which is similar to that of Lemma 3.1.

LEMMA 5.9. *For* $i, j \in \mathbf{Z}_m$, *it holds* $(V^i, V^j) \neq \{0\}$ *if and only if* $i + j = \omega$ *in* $\mathbf{Z}_m$. *For such* i *and* j, $(\ ,\ )|_{V^i + V^j}$ *is non-degenerate.*

REMARK 5.10. The cases for which there exists $i \in \mathbf{Z}_m$ such that $i = \omega - i$ (i.e., $(\ ,\ )|_{V^i}$ is non-degenerate) are just the following:

(i) $(\varepsilon, \omega) = (1, 0)$ and $i = 0$ or $i = m/2$ in Case I (m: even).

(ii) $\omega = 0$ and $i = 0$ or $\omega = 1$ and $i = (m + 1)/2$ in Case II (m: odd).

(iii) $(\varepsilon, \omega) = (-1, 0)$ and $i = 0$ or $i = m/2$ in Case III (m: even).

Applying the normalization algorithms of symmetric or alternating bilinear forms to $(\ ,\ )|_{V^i + V^{\omega - i}}$, we have the following.

LEMMA 5.11. (i) *In Case I, for each* $j \in \mathbf{Z}_m$, *there exist linearly independent vectors* $v_1^j, v_2^j, \dots, v_r^j$ *in* V^j *such that*

$$(v_p^i, v_q^j) = \delta_{p,q} \delta_{j,\omega-i} (-1)^i \ (i, j \in \mathbf{Z}_m,\ 1 \leq p, q \leq r) \,.$$

In this case, *we put* $U^j := \langle v_1^j, v_2^j, \dots, v_r^j \rangle_{\boldsymbol{C}}$ *and* $U := \bigoplus_{j \in \mathbf{Z}_m} U^j$.

(ii) *In Cases II and III, for each* $j \in \boldsymbol{Z}_m$, *there exist linearly independent vectors* $v_1^j, v_2^j, \dots, v_r^j, w_1^j, w_2^j, \dots, w_r^j$ *in* V^j *such that*

$$(v_p^i, w_q^j) = \delta_{p,q}\delta_{j,\omega-i} \ (i, j \in \boldsymbol{Z}_m,\ 1 \le p, q \le r)\,,$$

$$(v_p^i, v_q^j) = (w_p^i, w_q^j) = 0 \ (i, j \in \boldsymbol{Z}_m,\ 1 \le p, q \le r)\,.$$

In this case, we put $U^j := \langle v_1^j, v_2^j, \dots, v_r^j, w_1^j, w_2^j, \dots, w_r^j\rangle_{\boldsymbol{C}}$ *and* $U := \bigoplus_{j\in\boldsymbol{Z}_m} U^j$.

Let U be the subspace of V defined in Lemma 5.11. Then clearly $(\ ,\ )|_U$ is non-degenerate and we have the orthogonal decomposition $V = U \perp U^{\perp}$. Based on the above basis of U, we define $X_k \in \mathfrak{gl}(V)$ by

$$X_k v_p^j = \delta_{k,p} v_k^{j+1} \ (1 \le k, p \le r,\ j \in \boldsymbol{Z}_m), \quad X_k|_{U^\perp} = 0$$

for Case I, and

$$X_k v_p^j = \delta_{k,p} v_k^{j+1},\ -X_k w_p^j = \delta_{k,p} w_k^{j+1} \ (1 \le k, p \le r,\ j \in \boldsymbol{Z}_m), \quad X_k|_{U^\perp} = 0$$

for Cases II and III.

As in (5.2), X_k is contained in $\tilde{\mathfrak{g}}_1$ and semisimple. We define a subspace $\mathfrak{c}$ of $\tilde{\mathfrak{g}}_1$ by $\mathfrak{c} = \langle X_1, X_2, \dots, X_r\rangle_{\boldsymbol{C}}$. Then we can verify the following.

LEMMA 5.12. (i) $X_k \in \mathfrak{g}_1$ *and* $\mathfrak{c} \subset \mathfrak{g}_1$.

(ii) *In Case I, for* $\alpha_k \in \boldsymbol{C}$ $(1 \le k \le r)$, *the set of eigenvalues of* $\sum_{k=1}^r \alpha_k X_k \in \mathfrak{c}$ *is the same as that in Theorem* 3.5, (i, 1) *with* $q = r$.

(iii) *In Cases II and III, for* $\alpha_k \in \boldsymbol{C}$ $(1 \le k \le r)$, *the set of eigenvalues of* $\sum_{k=1}^r \alpha_k X_k \in \mathfrak{c}$ *is the same as that in Theorem* 3.5, (ii, 1) *with* $q = r$.

By Lemma 5.12, Theorem 3.5, (i, 2) and (ii, 2) are proved.

As in the proof of Lemma 5.4, (ii), we can show the following proposition by using Corollary 4.11.

PROPOSITION 5.13. $\mathfrak{c}$ *is a Cartan subspace of the* Θ*-representation* $(G_0, \mathfrak{g}_1)$ *of type* (BCD-I).

We give a basis $\mathcal{B}$ of V as below. By using the basis $\mathcal{B}$, we define a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{gl}(V)$ and the Weyl group $W_{GL(V)}$ of $GL(V)$ as in (5.1). We use the identification (5.2).

Case I. We put

$$u_k^j = \sum_{i\in\boldsymbol{Z}_m} (\zeta^{-j})^i v_k^i \ (1 \le k \le r,\ j \in \boldsymbol{Z}_m), \quad \mathcal{B}_{\mathfrak{c}} = \{u_k^j\,;\ 1 \le k \le r,\ j \in \boldsymbol{Z}_m\}\,.$$

Then $\mathcal{B}_{\mathfrak{c}}$ is a basis of U. By taking any basis $\mathcal{B}_0$ of $U^\perp$, we obtain a basis $\mathcal{B} = \mathcal{B}_{\mathfrak{c}} \cup \mathcal{B}_0$ of V.

For $\sigma \in S_r$ and $(p_1, p_2, \dots, p_r) \in (\boldsymbol{Z}_m)^r$ $(1 \le k \le r)$, we define $g(p_1, p_2, \dots, p_r; \sigma) \in P(\mathcal{B})$ and a subgroup $W_{\mathfrak{c}}$ of $W_{GL(V)}$ as in Definition 5.5.

Cases II and III. We put

$$u_k^j = \sum_{i\in\boldsymbol{Z}_m} (\zeta^{-j})^i v_k^i, \quad \overline{u}_k^j = \sum_{i\in\boldsymbol{Z}_m} (\zeta^{-j})^i w_k^i \quad (1 \le k \le r,\ j \in \boldsymbol{Z}_m)\,,$$

$$\mathcal{B}_{\mathfrak{c}} = \{u_k^j,\ \overline{u}_k^j\ ;\ 1 \le k \le r,\ j \in \mathbf{Z}_m\}.$$

Then $\mathcal{B}_{\mathfrak{c}}$ is a basis of U. By taking any basis $\mathcal{B}_0$ of $U^{\perp}$, we obtain a basis $\mathcal{B} = \mathcal{B}_{\mathfrak{c}} \cup \mathcal{B}_0$ of V. We easily see the following.

LEMMA 5.14. *$X_k u_p^j = \delta_{k,p}\zeta^j u_k^j$ and $-X_k\overline{u}_p^j = \delta_{k,p}\zeta^j\overline{u}_k^j$.*

REMARK 5.15. (i) In Case II, since m is odd, we have $\langle\zeta\rangle \cup (-\langle\zeta\rangle) = \langle\xi\rangle$. Hence the non-zero eigenvalues of X_k are $1, \xi, \xi^2, \ldots, \xi^{2m-1}$ each of which appears with multiplicity one.

(ii) In Case III, since m is even, we have $\langle\zeta\rangle \cup (-\langle\zeta\rangle) = \langle\zeta\rangle$. Hence the non-zero eigenvalues of X_k are $1, \zeta, \zeta^2, \ldots, \zeta^{m-1}$ each of which appears with multiplicity two.

DEFINITION 5.16. In Case III, for $\sigma \in S_r$ and $(p_1, p_2, \ldots, p_r) \in (\mathbf{Z}_m)^r$, define $g = g(p_1, p_2, \ldots, p_r; \sigma) \in P(\mathcal{B})$ by (a) $gu_k^j = u_{\sigma(k)}^{j-p_k}$, $g\overline{u}_k^j = \overline{u}_{\sigma(k)}^{j-p_k}$, and (b) $gv = v$ for any $v \in \mathcal{B}_0$.

Define a subgroup $W_{\mathfrak{c}}$ of $W_{GL(V)}$ by

$$W_{\mathfrak{c}} := \{\mathrm{Ad}(g(p_1, p_2, \ldots, p_r; \sigma))|_{\mathfrak{t}}\ ; \sigma \in S_r,\ (p_1, p_2, \ldots, p_r) \in (\mathbf{Z}_m)^r\}.$$

In Case II, by Remark 5.15, the non-zero eigenvalues of X_k are $1, \xi, \xi^2, \ldots, \xi^{2m-1}$ each of which appears with multiplicity one. Let y_k^i $(i \in \mathbf{Z}_{2m})$ be the unique eigenvector of X_k contained in $\mathcal{B}_{\mathfrak{c}}$ having eigenvalue ξ^i. Clearly we have $\mathcal{B}_{\mathfrak{c}} = \{y_k^i\ ;\ 1 \le k \le r,\ i \in \mathbf{Z}_{2m}\}$.

DEFINITION 5.17. In Case II, for $\sigma \in S_r$ and $(p_1, p_2, \ldots, p_r) \in (\mathbf{Z}_{2m})^r$, define $g = g(p_1, p_2, \ldots, p_r; \sigma) \in P(\mathcal{B})$ by (a) $gy_k^j = y_{\sigma(k)}^{j-p_k}$, and (b) $gv = v$ for any $v \in \mathcal{B}_0$.

Define a subgroup $W_{\mathfrak{c}}$ of $W_{GL(V)}$ by

$$W_{\mathfrak{c}} := \{\mathrm{Ad}(g(p_1, p_2, \ldots, p_r; \sigma))|_{\mathfrak{t}}\ ; \sigma \in S_r,\ (p_1, p_2, \ldots, p_r) \in (\mathbf{Z}_{2m})^r\}.$$

For these three cases, statements similar to Lemma 5.6 also hold as follows.

LEMMA 5.18. (i) *In Cases I and III, we have* $\mathrm{Ad}(g(p_1, p_2, \ldots, p_r; \sigma))X_k = \zeta^{p_k}X_{\sigma(k)}$ $(1 \le k \le r)$ *for* $\sigma \in S_r$ *and* $(p_1, p_2, \ldots, p_r) \in (\mathbf{Z}_m)^r$.

(ii) *In Case II, we have* $\mathrm{Ad}(g(p_1, p_2, \ldots, p_r; \sigma))X_k = \xi^{p_k}X_{\sigma(k)}$ $(1 \le k \le r)$ *for* $\sigma \in S_r$ *and* $(p_1, p_2, \ldots, p_r) \in (\mathbf{Z}_{2m})^r$.

Then statements similar to Lemma 5.7 also hold for these cases and Theorem 5.3 implies the following.

PROPOSITION 5.19. *The homomorphism $\rho : W_{\mathfrak{c}} \to GL(\mathfrak{c})$, $\rho(w) = w|_{\mathfrak{c}}$ $(w \in W_{\mathfrak{c}})$ is injective and the image coincides with the Weyl group $W(G_0, \mathfrak{c})$. As a consequence, we have*

$W(G_0, \mathfrak{c}) \simeq G(m, 1, r)$ $(r = \min\{\dim V^j;\ j \in \mathbf{Z}_m\})$ *in* Case I,

$W(G_0, \mathfrak{c}) \simeq G(2m, 1, r)$ $(r = \min\{[\dim V^j/2];\ j \in \mathbf{Z}_m\})$ *in* Case II,

$W(G_0, \mathfrak{c}) \simeq G(m, 1, r)$ $(r = \min\{[\dim V^j/2];\ j \in \mathbf{Z}_m\})$ *in* Case III.

(5.4) Cartan subspaces and Weyl groups of Θ-representations of type (A-O). Let (G, θ) be a Θ-group of order $2m$ of type (A-O) defined by a vector space $(V, \langle\ ,\ \rangle)$ with

(ω, m)-bilinear form. We use the notations of (4.3). The proof of the following lemma is similar to that of Lemma 3.6 and we omit it.

LEMMA 5.20. *For $i, j \in \boldsymbol{Z}_m$, it holds $\langle V^i, V^j \rangle \neq \{0\}$ if and only if $i + j = \omega$ in $\boldsymbol{Z}_m$. For such i and j, $\langle\ ,\ \rangle|_{V^i+V^j}$ is non-degenerate.*

REMARK 5.21. The cases for which there exists $i \in \boldsymbol{Z}_m$ such that $i = \omega - i$ (i.e., $\langle\ ,\ \rangle|_{V^i}$ is non-degenerate) are just the following:

(i) $i = 0$ in Case I. In this case, $\langle\ ,\ \rangle|_{V^0}$ is symmetric.

(ii) $\omega = 0$ and $i = 0$ or $i = m/2$ in Case II (m is even). In this case, $\langle\ ,\ \rangle|_{V^0}$ is symmetric and $\langle\ ,\ \rangle|_{V^{m/2}}$ is alternating.

(iii) $i = (m+1)/2$ in Case III (m is odd). In this case, $\langle\ ,\ \rangle|_{V^{(m+1)/2}}$ is alternating.

We easily see the following:

(a) For $u \in V^i$ and $v \in V^{\omega-i}$ ($i \in \boldsymbol{Z}_m$), $\langle u, v\rangle = \xi^{-\omega}\zeta^i\langle v, u\rangle$.

(b) In Case I, $-\xi = \zeta^{(m+1)/2}$, $(-\xi)^{-1} = \zeta^{(m-1)/2}$ and $\zeta^{-i}(-\xi)^i = (-\xi)^{-i}$ ($i \in \boldsymbol{Z}_m$).

Then normalization algorithms of non-degenerate bilinear forms $\langle\ ,\ \rangle|_{V^i+V^{\omega-i}}$ imply the following.

LEMMA 5.22. (i) *In Case I, for each $j \in \boldsymbol{Z}_m$, there exist linearly independent vectors $v_1^j, v_2^j, \dots, v_r^j$ in V^j such that*

$$\langle v_p^i, v_q^j\rangle = \delta_{p,q}\delta_{-i,j}(-\xi)^i \quad (i, j \in \boldsymbol{Z}_m,\ 1 \le p, q \le r)\,.$$

In this case, we put $U^j := \langle v_1^j, v_2^j, \dots, v_r^j\rangle_{\boldsymbol{C}}$ and $U := \bigoplus_{j\in\boldsymbol{Z}_m} U^j$.

(ii) *In Cases II and III, for each $j \in \boldsymbol{Z}_m$, there exist linearly independent vectors $v_1^j, v_2^j, \dots, v_r^j, w_1^j, w_2^j, \dots, w_r^j$ in V^j such that*

$$\langle v_p^i, w_q^j\rangle = \delta_{p,q}\delta_{j,\omega-i} \quad (i, j \in \boldsymbol{Z}_m,\ 1 \le p, q \le r)\,,$$

$$\langle v_p^i, v_q^j\rangle = \langle w_p^i, w_q^j\rangle = 0 \quad (i, j \in \boldsymbol{Z}_m,\ 1 \le p, q \le r)\,.$$

In these cases, we put $U^j := \langle v_1^j, v_2^j, \dots, v_r^j, w_1^j, w_2^j, \dots, w_r^j\rangle_{\boldsymbol{C}}$ and $U := \bigoplus_{j\in\boldsymbol{Z}_m} U^j$.

Let U be the subspace of V defined in Lemma 5.22. Then clearly $\langle\ ,\ \rangle|_U$ is non-degenerate and we have the orthogonal decomposition $V = U \perp U^\perp$, where $U^\perp = \{v \in V; \langle U, v\rangle = \{0\}\}$. Here we easily see that $\langle U, v\rangle = \langle v, U\rangle$ since U is S-stable.

Based on the above basis of U, we define $X_k \in \mathfrak{gl}(V)$ by

$X_k v_p^j = \delta_{k,p}v_k^{j+1}$ ($1 \le k, p \le r$, $j \in \boldsymbol{Z}_m$), $X_k|_{U^\perp} = 0$ in Case I, and

$X_k v_p^j = \delta_{k,p}v_k^{j+1}$, $-\xi X_k w_p^j = \delta_{k,p}w_k^{j+1}$ ($1 \le k, p \le r$, $j \in \boldsymbol{Z}_m$), $X_k|_{U^\perp} = 0$ in Cases II and III.

As in (5.2), X_k is contained in $\tilde{\mathfrak{g}}_1$ and semisimple. We define a subspace $\mathfrak{c}$ of $\tilde{\mathfrak{g}}_1$ by $\mathfrak{c} = \langle X_1, X_2, \dots, X_r\rangle_{\boldsymbol{C}}$. Then we can verify the following.

LEMMA 5.23. (i) *$X_k \in \mathfrak{g}_1$ and $\mathfrak{c} \subset \mathfrak{g}_1$.*

(ii) *In Case I, for $\alpha_k \in \boldsymbol{C}$ ($1 \le k \le r$), the set of eigenvalues of $\sum_{k=1}^r \alpha_k X_k \in \mathfrak{c}$ is the same as that in Theorem* 3.10, (i, 1) *with $q = r$.*

(iii) *In Cases II and III, for* $\alpha_k \in \boldsymbol{C}$ $(1 \le k \le r)$, *the set of eigenvalues of* $\sum_{k=1}^{r} \alpha_k X_k \in \mathfrak{c}$ *is the same as that in Theorem* 3.10, (ii, 1) *with* $q = r$.

By Lemma 5.23, Theorem 3.10, (i, 2) and (ii, 2) are proved.

As in the proof of Lemma 5.4, (ii), we can show the following proposition by using Corollary 4.11.

PROPOSITION 5.24. $\mathfrak{c}$ *is a Cartan subspace of the* Θ*-representation* $(G_0, \mathfrak{g}_1)$ *of type* (A-O).

Now let us determine the Weyl group $W(G_0, \mathfrak{c})$. We give a basis $\mathcal{B}$ of V as below. By using the basis $\mathcal{B}$, we define a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{gl}(V)$ and the Weyl group $W_{GL(V)}$ of $GL(V)$ as in (5.1). We use the identification (5.2).

Case I. As in the case of (A-I), we put

$$u_k^j = \sum_{i \in \boldsymbol{Z}_m} (\zeta^{-j})^i v_k^i \ \ (1 \le k \le r,\ j \in \boldsymbol{Z}_m),\ \ \mathcal{B}_{\mathfrak{c}} = \{u_k^j\,;\ 1 \le k \le r,\ j \in \boldsymbol{Z}_m\}\,.$$

Then $\mathcal{B}_{\mathfrak{c}}$ is a basis of U. By taking any basis of $\mathcal{B}_0$ of $U^{\perp}$, we obtain a basis $\mathcal{B} = \mathcal{B}_{\mathfrak{c}} \cup \mathcal{B}_0$ of V.

For $\sigma \in S_r$ and $(p_1, p_2, \dots, p_r) \in (\boldsymbol{Z}_m)^r$ $(1 \le k \le r)$, define $g(p_1, p_2, \dots, p_r; \sigma) \in P(\mathcal{B})$ and a subgroup $W_{\mathfrak{c}}$ of $W_{GL(V)}$ as in Definition 5.5.

Cases II and III. We put

$$u_k^j = \sum_{i \in \boldsymbol{Z}_m} (\zeta^{-j})^i v_k^i,\quad \overline{u}_k^j = \sum_{i \in \boldsymbol{Z}_m} (\zeta^{-j})^i w_k^i \ \ (1 \le k \le r,\ j \in \boldsymbol{Z}_m)\,,$$

$$\mathcal{B}_{\mathfrak{c}} = \{u_k^j,\ \overline{u}_k^j;\ 1 \le k \le r,\ j \in \boldsymbol{Z}_m\}\,.$$

Then $\mathcal{B}_{\mathfrak{c}}$ is a basis of U. By taking any basis of $\mathcal{B}_0$ of $U^{\perp}$, we obtain a basis $\mathcal{B} = \mathcal{B}_{\mathfrak{c}} \cup \mathcal{B}_0$ of V. We easily see the following.

LEMMA 5.25. $X_k u_p^j = \delta_{k,p} \zeta^j u_k^j$ *and* $-\xi X_k \overline{u}_p^j = \delta_{k,p} \zeta^j \overline{u}_k^j$.

REMARK 5.26. (i) In Case II, since m is even, we have $\langle \zeta \rangle \cup (-\xi^{-1} \langle \zeta \rangle) = \langle \xi \rangle$. Hence the non-zero eigenvalues of X_k are $1, \xi, \xi^2, \dots, \xi^{2m-1}$ each of which appears with multiplicity one.

(ii) In Case III, since m is odd, we have $\langle \zeta \rangle \cup (-\xi^{-1} \langle \zeta \rangle) = \langle \zeta \rangle$. Hence the non-zero eigenvalues of X_k are $1, \zeta, \zeta^2, \dots, \zeta^{m-1}$ each of which appears with multiplicity two.

In Case III, for $\sigma \in S_r$ and $(p_1, p_2, \dots, p_r) \in (\boldsymbol{Z}_m)^r$, define $g(p_1, p_2, \dots, p_r; \sigma) \in P(\mathcal{B})$ and a subgroup $W_{\mathfrak{c}}$ of $W_{GL(V)}$ as in Definition 5.16.

In Case II, by Remark 5.26, the non-zero eigenvalues of X_k are $1, \xi, \xi^2, \dots, \xi^{2m-1}$ each of which appears with multiplicity one. Let y_k^i $(i \in \boldsymbol{Z}_{2m})$ be the unique eigenvector of X_k contained in $\mathcal{B}_{\mathfrak{c}}$ having eigenvalue ξ^i. Clearly we have $\mathcal{B}_{\mathfrak{c}} = \{y_k^i;\ 1 \le k \le r,\ i \in \boldsymbol{Z}_{2m}\}$. In this case, for $\sigma \in S_r$ and $(p_1, p_2, \dots, p_r) \in (\boldsymbol{Z}_{2m})^r$, define $g(p_1, p_2, \dots, p_r; \sigma) \in P(\mathcal{B})$ and a subgroup $W_{\mathfrak{c}}$ of $W_{GL(V)}$ as in Definition 5.17.

For these three cases, statements similar to Lemma 5.6 also hold as follows.

LEMMA 5.27. (i) *In Cases I and III, we have* $\mathrm{Ad}(g(p_1, p_2, \dots, p_r; \sigma))X_k = \zeta^{p_k} X_{\sigma(k)}$ $(1 \leq k \leq r)$ *for* $\sigma \in S_r$ *and* $(p_1, p_2, \dots, p_r) \in (\mathbf{Z}_m)^r$.

(ii) *In Case II, we have* $\mathrm{Ad}(g(p_1, p_2, \dots, p_r; \sigma))X_k = \xi^{p_k} X_{\sigma(k)}$ $(1 \leq k \leq r)$ *for* $\sigma \in S_r$ *and* $(p_1, p_2, \dots, p_r) \in (\mathbf{Z}_{2m})^r$.

Then statements similar to Lemma 5.7 also hold for these cases and Theorem 5.3 implies the following.

PROPOSITION 5.28. *The homomorphism* $\rho : W_{\mathfrak{c}} \to GL(\mathfrak{c})$, $\rho(w) = w|_{\mathfrak{c}}$ $(w \in W_{\mathfrak{c}})$ *is injective and the image coincides with the Weyl group* $W(G_0, \mathfrak{c})$. *As a consequence, we have*
$W(G_0, \mathfrak{c}) \simeq G(m, 1, r)$ $(r = \min\{\dim V^j;\ j \in \mathbf{Z}_m\})$ *in* Case I,
$W(G_0, \mathfrak{c}) \simeq G(2m, 1, r)$ $(r = \min\{[\dim V^j/2];\ j \in \mathbf{Z}_m\})$ *in* Case II,
$W(G_0, \mathfrak{c}) \simeq G(m, 1, r)$ $(r = \min\{[\dim V^j/2];\ j \in \mathbf{Z}_m\})$ *in* Case III.

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