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HOMOGENEOUS ISOPARAMETRIC HYPERSURFACES IN SPHERES WITH FOUR DISTINCT PRINCIPAL CURVATURES AND MOMENT MAPS

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Abstract. We study relations between moment maps of Hamiltonian actions and isoparametric hypersurfaces in spheres with four distinct principal curvatures. In this paper, we deal with the isoparametric hypersurfaces given by the isotropy representations of compact irreducible Hermitian symmetric spaces of classical type and of rank two. We show that such isoparametric hypersurfaces can be obtained by moment maps. More precisely, certain squared-norms of moment maps coincide with Cartan-Münzner polynomials, which are defining-equations, of above isoparametric hypersurfaces.

1. Introduction. A hypersurface N in a Riemannian manifold $(M, \langle\ ,\ \rangle)$ is called an *isoparametric hypersurface* if N is a level set of an isoparametric function on M. Here, a smooth function φ on M is said to be *isoparametric* if there exist two smooth functions $A(t)$ and $B(t)$ on $\boldsymbol{R}$ which satisfy

$$(1.1)\qquad \begin{cases} \|\mathrm{grad}\,\varphi\|^2 = \langle \mathrm{grad}\,\varphi, \mathrm{grad}\,\varphi\rangle = A\circ\varphi\,, \\ \qquad \Delta\varphi = B\circ\varphi\,. \end{cases}$$

We refer to Cecil [2], Thorbergsson [13], and the references therein, for history and general theory of isoparametric hypersurfaces. Note that isoparametric hypersurfaces in Euclidean spaces $\boldsymbol{R}^n$ and hyperbolic spaces $\boldsymbol{H}^n$ have been classified completely.

In this paper, we consider isoparametric hypersurfaces in spheres S^n. It is known that a hypersurface in S^n is isoparametric if and only if it has constant principal curvatures. Hence, a homogeneous hypersurface in S^n is isoparametric since it has constant principal curvatures. They have been characterized as principal orbits of the isotropy representation of symmetric spaces of rank two, and are classified completely by Hsiang and Lawson [6]. Their results state that a homogeneous hypersurface in S^n coincides with a principal orbit of the isotropy representation of a symmetric space of rank two.

Let g denote the number of distinct principal curvatures of an isoparametric hypersurface in S^n. We denote its principal curvatures by $\lambda_1 < \cdots < \lambda_g$ whose multiplicities are $m_1, \dots, m_g$, respectively. Then, Münzner [8] showed that

$$m_i = m_{i+2} \quad \text{for } i \bmod g\,.$$

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Thus, it is clear that all multiplicities are determined by m_1 and m_2. Note that $m_1 = m_2$ if g is odd. Münzner described an isoparametric function which defines an isoparametric hypersurface in S^n with (g, m_1, m_2). Such a function is obtained by the restriction on S^n of a homogeneous polynomial function $\varphi : \boldsymbol{R}^{n+1} \to \boldsymbol{R}$ of degree g which satisfies

$$(1.2)\qquad \begin{cases} \|\mathrm{grad}\,\varphi(P)\|^2 = g^2\|P\|^{2g-2}\,, \\ \Delta\varphi(P) = \dfrac{m_2 - m_1}{2} g^2\|P\|^{g-2}\,. \end{cases}$$

A homogeneous polynomial satisfying (1.2) is called a *Cartan-Münzner polynomial*. Münzner [9] also showed that g must be 1, 2, 3, 4 or 6. Note that isoparametric hypersurfaces in S^n with $g = 1, 2, 3$ have been classified completely.

We are interested in isoparametric hypersurfaces in S^n with four distinct principal curvatures. There exist homogeneous ones and non-homogeneous ones. Homogeneous ones are obtained from the isotropy representations of the following symmetric spaces of rank two:

(1) $\mathrm{SO}(2+n)/\mathrm{SO}(2) \times \mathrm{SO}(n)$,
(2) $\mathrm{SU}(2+n)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(n))$,
(3) $\mathrm{SO}(10)/\mathrm{U}(5)$,
(4) $\mathrm{E}_6/\mathrm{U}(1) \times \mathrm{Spin}(10)$,
(5) $\mathrm{Sp}(2+n)/\mathrm{Sp}(2) \times \mathrm{Sp}(n)$,
(6) $\mathrm{SO}(5) \times \mathrm{SO}(5)/\mathrm{SO}(5)$.

Among these, (1), (2), (3) and (4) are Hermitian symmetric spaces. Non-homogeneous examples are found by Ozeki and Takeuchi [11] for the first time by using the representations of Clifford algebras. The method of Ozeki and Takeuchi was generalized by Ferus, Karcher and Münzner [4]. They systematically constructed isoparametric hypersurfaces in S^n with four distinct principal curvatures from representations of Clifford algebras. Such an isoparametric hypersurface is called *FKM-type*. Recently, they have been *almost* classified by Cecil, Chi and Jensen [3] and Immervoll [7], and a complete classification of non-homogeneous ones is still open.

Our expectation is that every isoparametric hypersurface in S^n with four distinct principal curvatures is related to a moment map for a certain group action. In this paper, we show that this is true for the isoparametric hypersurfaces obtained by the isotropy representations of the above Hermitian symmetric spaces (1), (2) and (3).

Let G/K be an irreducible Hermitian symmetric space of compact type and of rank two. We denote the Cartan decomposition by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then, we get two K-invariant functions on $\mathfrak{p}$. One comes from a Cartan-Münzner polynomial. Since the rank of G/K is two, the results of Hsiang-Lawson imply that a principal K-orbit of the isotropy representation of G/K is a homogeneous isoparametric hypersurface in the unit sphere in $\mathfrak{p}$. By the definition of isoparametric hypersurfaces, there exists an isoparametric function on the sphere. According to the result of Münzner, this function is the restriction of a Cartan-Münzner polynomial φ. This φ is a K-invariant function on $\mathfrak{p}$. Another K-invariant function on $\mathfrak{p}$ comes from a moment map. Since G/K is Hermitian, the isotropy representation of G/K is a Hamiltonian action (see Section 2). Thus, there exists a moment map $\mu : \mathfrak{p} \to \mathfrak{k}^*$ for this action, where $\mathfrak{k}^*$ is the dual vector space of $\mathfrak{k}$. By definition, μ is K-equivariant. Hence, a composition function of μ and a K-invariant norm on $\mathfrak{k}^*$ is K-invariant. We have two K-invariant functions on $\mathfrak{p}$ in these ways. Our main theorem in this paper describes a relation between these two K-invariant functions. One can construct the first K-invariant function (a Cartan-Münzner polynomial) from the second one (a squared-norm of a moment map). More precisely,

MAIN THEOREM. *Let G/K be one of the following Hermitian symmetric spaces*: $\mathrm{SO}(2+n)/\mathrm{SO}(2) \times \mathrm{SO}(n)$, $\mathrm{SU}(2+n)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(n))$ *and* $\mathrm{SO}(10)/\mathrm{U}(5)$. *Let φ be the corresponding Cartan-Münzner polynomial. Then, there exists a K-invariant norm on $\mathfrak{k}^*$ such that φ coincides with the squared-norm of the moment map μ for the isotropy representation of G/K.*

This article is organized as follows. Section 2 gives a definition and some properties of a moment map. In particular, we introduce a useful formula to compute a moment map for the isotropy representation of a Hermitian symmetric space. In Section 3, we describe our main theorems, and give proofs thereafter.

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2. Hermitian symmetric spaces and moment maps. In this section, we recall Hamiltonian actions and their moment maps (we refer to Audin [1]), and study the moment map for the isotropy representation of a Hermitian symmetric space.

Let M be a smooth manifold of dimension $2n$ with a closed 2-form ω on M satisfying $\omega^{\wedge n} \neq 0$. The pair (M, ω) is called a *symplectic manifold*, and ω is called a *symplectic form* of M.

DEFINITION 2.1. Let K be a Lie group which acts on a symplectic manifold (M, ω). Let $\mathfrak{k}$ denote the Lie algebra of K. A K-action is said to be *Hamiltonian* if

(1) the action preserves ω,

(2) there exists a moment map $\mu : M \to \mathfrak{k}^*$, where $\mathfrak{k}^*$ is the dual space of $\mathfrak{k}$ as a vector space.

Here, a map $\mu : M \to \mathfrak{k}^*$ is called a *moment map* if

(1) $(d\mu)_P(Q)(\xi) = \omega_P(\widetilde{\xi}_P, Q)$ for all $P \in M$, all $Q \in T_P M$ and all $\xi \in \mathfrak{k}$,

(2) $\mu(a.P) = \left(\mathrm{Ad}^*\right)_a (\mu(P))$ for all $P \in M$ and all $a \in K$,

where $\widetilde{\xi}$ denotes a vector field on M defined by

$$M \ni P \overset{\widetilde{\xi}}{\longmapsto} \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi).P \in T_P M\,,$$

and $\mathrm{Ad}^* : K \to \mathrm{GL}(\mathfrak{k}^*)$ is the coadjoint representation.

The second property means that it is K-equivariant. A moment map for a Hamiltonian action is determined uniquely up to constant functions on M. It is not so easy to compute a moment map for a Hamiltonian action in general.

Here, we will see that the isotropy representation of a compact Hermitian symmetric space is a Hamiltonian action, and there is a useful formula of its moment map. Let G/K be a compact Hermitian symmetric space. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. The isotropy representation of G/K is the K-action on $\mathfrak{p}$ by Ad. Let $\langle\ ,\ \rangle$ be an inner product on $\mathfrak{g}$ defined by

$$\langle X, Y\rangle := -B(X, Y) \quad \text{for } X, Y \in \mathfrak{g}, \tag{2.1}$$

where B is the Killing form of $\mathfrak{g}$. We also denote by $\langle\ ,\ \rangle$ its restrictions to $\mathfrak{p}$ and to $\mathfrak{k}$, respectively. It is clear that $\langle\ ,\ \rangle$ is K-invariant. Moreover, $\mathfrak{p}$ has a K-invariant complex structure J, defined by

$$J := \mathrm{ad}_Z|_{\mathfrak{p}},$$

where Z be a fixed non-zero element of the center $C(\mathfrak{k})$ of $\mathfrak{k}$. We identify $T_P\mathfrak{p}$ with $\mathfrak{p}$ because $\mathfrak{p}$ is a vector space. Therefore, $\mathfrak{p}$ has a canonical symplectic form

$$\omega_P(X_P, Y_P) := \langle J(X_P), Y_P\rangle \quad \text{for } X, Y \in \mathfrak{X}(\mathfrak{p}) \text{ and } P \in \mathfrak{p},$$

where $\mathfrak{X}(\mathfrak{p})$ is a set of all vector fields on $\mathfrak{p}$. Thus, $(\mathfrak{p}, \omega)$ is a symplectic manifold. It is clear that K preserves the symplectic form ω on $\mathfrak{p}$, since K preserves $\langle\ ,\ \rangle$ and J. The following proposition shows that this K-action on $\mathfrak{p}$ is Hamiltonian. Note that this formula is essentially obtained by Ohnita [10].

PROPOSITION 2.2. *Let G/K be a compact Hermitian symmetric space. We define a map $\mu : \mathfrak{p} \to \mathfrak{k}^*$ by $P \mapsto \mu_P$, where*

$$\mu_P(\xi) = \frac{1}{2}\langle(\mathrm{ad}_P)^2(Z),\ \xi\rangle \in \boldsymbol{R} \quad \textit{for } \xi \in \mathfrak{k}. \tag{2.2}$$

Then, μ is a moment map for the isotropy representation of G/K.

PROOF. We have only to check that μ satisfies two conditions in the definition of moment maps.

We show that μ satisfies the first condition of moment maps, that is, $d\mu$ is compatible with ω. Our first claim is that $d\mu$ satisfies

$$(d\mu)_P(Q)(\xi) = \langle[P, [Q, Z]], \xi\rangle \quad \text{for } P, Q \in \mathfrak{p}. \tag{2.3}$$

By definition, we have

$$(d\mu)_P(Q)(\xi) = \frac{1}{2}\langle d((\mathrm{ad}_P)^2(Z))_P(Q),\ \xi\rangle. \tag{2.4}$$

Direct computations show that

$$
\begin{aligned}
(d((\mathrm{ad}_P)^2(Z)))_P(Q) &= (d\,([P,[P,Z]]))_P\,(Q) \\
&= \lim_{t\to 0}\frac{[P+tQ,[P+tQ,Z]]-[P,[P,Z]]}{t} \\
&= [Q,[P,Z]]+[P,[Q,Z]] \\
&= -[P,[Z,Q]]-[Z,[Q,P]]+[P,[Q,Z]] \\
&= 2[P,[Q,Z]]\,,
\end{aligned}
\tag{2.5}
$$

because $[Q, P] \in \mathfrak{k}$ and $Z \in C(\mathfrak{k})$. Thus, we have (2.3). On the other hand, it follows from the definition of ω that

$$
\begin{aligned}
\omega_P(\widetilde{\xi}_P, Q) &= \langle J(\widetilde{\xi}_P), Q\rangle \\
&= \langle Q, J([\xi, P])\rangle \\
&= \langle Q, [Z,[\xi, P]]\rangle \\
&= \langle [P,[Q,Z]], \xi\rangle\,,
\end{aligned}
\tag{2.6}
$$

for all $\xi \in \mathfrak{k}$. Therefore, the first condition has been completed.

Next, we show that μ is K-equivariant. We remark that $\mathrm{Ad}_a(Z) = Z$ for all $a \in K$ because $Z \in C(\mathfrak{k})$. It turns out that

$$
\begin{aligned}
2\mu_{a.P}(\xi) &= 2\mu_{\mathrm{Ad}_a(P)}(\xi) \\
&= \langle [\mathrm{Ad}_a(P), [\mathrm{Ad}_a(P), Z]], \xi\rangle \\
&= \langle \mathrm{Ad}_a\,([P,[P,Z]])\,, \xi\rangle \\
&= \langle [P,[P,Z]],\ (\mathrm{Ad}_a)^{-1}\,(\xi)\rangle \\
&= 2\mu_P \circ (\mathrm{Ad}_a)^{-1}(\xi) \\
&= 2\left(\mathrm{Ad}^*\right)_a(\mu_P)\,(\xi)\,,
\end{aligned}
\tag{2.7}
$$

for all $\xi \in \mathfrak{k}$, $P \in \mathfrak{p}$ and $a \in K$. This means that μ is K-equivariant. □

REMARK 2.3. We identify $\mathfrak{k}^*$ with $\mathfrak{k}$ as vector spaces via the isomorphism defined by the K-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{k}$. So, we consider that the moment map μ for the isotropy representation of a compact Hermitian symmetric space G/K is $\mathfrak{k}$-valued map defined by

$$
\mu(P) := \frac{1}{2}(\mathrm{ad}_P)^2(Z) \in \mathfrak{k}
\tag{2.8}
$$

for all $P \in \mathfrak{p}$.

We need the squared-norm of the moment map for our main theorem. Because G/K is a Hermitian symmetric space, there exists a Lie subalgebra $\mathfrak{k}'$ of $\mathfrak{k}$ satisfying $\mathfrak{k} = \mathfrak{u}(1) \oplus \mathfrak{k}'$ as a Lie algebra, and $\mathfrak{u}(1) = \boldsymbol{R}Z$. Let μ_1 and μ_2 denote the compositions of μ with the canonical projections of $\mathfrak{k}$ onto $\mathfrak{u}(1)$ and $\mathfrak{k}'$, respectively. Hence,

$$
\mu_1 : \mathfrak{p} \to \mathfrak{u}(1)\,, \quad \mu_2 : \mathfrak{p} \to \mathfrak{k}'\,.
$$

It is remarkable that both μ_1 and μ_2 are K-equivariant. Let $\|\cdot\|$ be a K-invariant norm on $\mathfrak{k}$ defined by the inner product $\langle\ ,\ \rangle$ on $\mathfrak{k}$. Then, the squared-norms of μ_1 and μ_2 can be computed easily.

PROPOSITION 2.4. *The squared-norms of μ_1 and μ_2 are given by*
(1) $\|\mu_1(P)\|^2 = \|P\|^4/(4\|Z\|^2)$,
(2) $\|\mu_2(P)\|^2 = \|\mu(P)\|^2 - \|\mu_1(P)\|^2$
for $P \in \mathfrak{p}$, respectively.

PROOF. First, we compute $\|\mu_1(P)\|^2$. Because $\mathfrak{u}(1) = \boldsymbol{R}Z$, we can write

$$\mu_1(P) = (\langle \mu(P), Z\rangle/\|Z\|^2)Z\,. \tag{2.9}$$

From (2.8), we have

$$\begin{aligned} 2\langle \mu(P), Z\rangle &= \langle [P,[P,Z]], Z\rangle = \langle P, [[P,Z],Z]\rangle \\ &= \langle P, J^2(P)\rangle = \langle P, -P\rangle = -\|P\|^2\,. \end{aligned} \tag{2.10}$$

Therefore, we obtain

$$\mu_1(P) = -(\|P\|^2/(2\|Z\|^2))Z\,. \tag{2.11}$$

This shows our claim (1).

Next, we show our claim (2). Since $\mu = \mu_1 + \mu_2$, we have

$$\|\mu_2(P)\|^2 = \|\mu(P) - \mu_1(P)\|^2 = \|\mu(P)\|^2 + \|\mu_1(P)\|^2 - 2\langle \mu(P), \mu_1(P)\rangle\,. \tag{2.12}$$

By the equations (2.10) and (2.11), we have

$$\langle \mu(P), \mu_1(P)\rangle = -\frac{\|P\|^2}{2\|Z\|^2}\langle \mu(P), Z\rangle = \frac{\|P\|^4}{4\|Z\|^2} = \|\mu_1(P)\|^2\,.$$

This completes our claim (2). □

It is remarkable that both $\|\mu_1(P)\|^2$ and $\|\mu_2(P)\|^2$ are K-invariant functions on $\mathfrak{p}$, which is essential to our main theorem. In our main theorem we are concerned with the case of rank two, but Propositions 2.2 and 2.4 hold independently of the rank of G/K.

3. Main theorem. Let G/K be an irreducible Hermitian symmetric space of classical type and of rank two, that is, one of the following:

$$\mathrm{SO}(2+n)/\mathrm{SO}(2)\times\mathrm{SO}(n)\,,\quad \mathrm{SU}(2+n)/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(n))\,,\quad \mathrm{SO}(10)/\mathrm{U}(5)\,.$$

The symbols $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ and Z are same as above.

Now we define *a weighted squared-norm of the moment map* $f_{a,b} : \mathfrak{p} \to \boldsymbol{R}$ by

$$f_{a,b}(P) = a\|\mu_1(P)\|^2 + b\|\mu_2(P)\|^2 \quad \text{for } P \in \mathfrak{p}\,, \tag{3.1}$$

where a and b are real numbers, and $\|\cdot\|$ is a norm on $\mathfrak{k}$ defined by the Killing form B of $\mathfrak{g}$. Proposition 2.4 yields that

$$\begin{aligned} f_{a,b}(P) &= b\|\mu(P)\|^2 + (a-b)\|\mu_1(P)\|^2 \\ &= b\|\mu(P)\|^2 + (a-b)\frac{\|P\|^4}{4\|Z\|^2}. \end{aligned} \tag{3.2}$$

Our main theorem is the following:

THEOREM 3.1. *For each irreducible Hermitian symmetric space G/K of classical type and of rank two, there exists a pair of non-zero real numbers (a, b) such that the function $f_{a,b}$ is a Cartan-Münzner polynomial. A level hypersurface of $f_{a,b}$ is a homogeneous isoparametric hypersurface in S^n with four distinct principal curvatures obtained by a K-orbit of the isotropy representation of G/K.*

REMARK 3.2. (1) To prove the theorem, we have only to show that the function $f_{a,b}$ is a Cartan-Münzner polynomial. Since $f_{a,b}$ is K-invariant, every regular K-orbit is contained in a level hypersurface of $f_{a,b}$. In fact, they coincide each other, since they have the same dimension and a K-orbit is complete.

(2) Each of the functions we got is essentially the same as the ones computed by Ozeki and Takeuchi [12], although they do not use "moment maps".

We will give the proof of our main theorem for each G/K after the next sections by carrying out the following steps sequentially:

(a) compute μ, the moment map for the isotropy representation of G/K,

(b) compute $f_{a,b}$, the weighted squared-norm of μ,

(c) compute $\|\mathrm{grad}\, f_{a,b}\|^2$, the squared-norm of the gradient of $f_{a,b}$,

(d) compute $\Delta f_{a,b}$, the Laplacian of $f_{a,b}$,

(e) find (a, b) so that $f_{a,b}$ is a Cartan-Münzner polynomial, that is, satisfies the equations (1.2).

4. Case of $\mathrm{SO}(2+n)/\mathrm{SO}(2)\times\mathrm{SO}(n)$**.** In this section, we consider the case of $(G, K) = (\mathrm{SO}(2+n), \mathrm{SO}(2)\times\mathrm{SO}(n))$. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. The Cartan decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \left(\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{0} & A' \end{array} \right) ;\ A \in \mathfrak{so}(2),\, A' \in \mathfrak{so}(n) \right\}, \tag{4.1}$$

and

$$\mathfrak{p} = \left\{ P(X) = \left(\begin{array}{c|c} \mathbf{0} & -{}^tX \\ \hline X & \mathbf{0} \end{array} \right) ;\ X \in M_{n,2}(\boldsymbol{R}) \right\}. \tag{4.2}$$

In order to simplify the notation, we write P instead of $P(X)$ through this section. The Killing form B of $\mathfrak{g} = \mathfrak{so}(2+n)$ is given by

$$B(P, Q) = n\,\mathrm{Tr}(PQ) \quad \text{for } P, Q \in \mathfrak{g}.$$

Therefore, the G-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ is written as

$$\langle P, Q\rangle := -B(P, Q) = -n\,\mathrm{Tr}(PQ) \quad \text{for } P, Q \in \mathfrak{g}\,.$$

First, we compute the moment map $\mu : \mathfrak{p} \to \mathfrak{k}^* \simeq \mathfrak{k}$.

PROPOSITION 4.1. *In this case, the moment map μ is given by*

$$\mu(P) = \frac{1}{2}\left(\begin{array}{c|c} -J_1\,{}^tXX - {}^tXXJ_1 & \mathbf{0} \\ \hline \mathbf{0} & 2XJ_1\,{}^tX \end{array}\right) \in \mathfrak{k} \tag{4.3}$$

for $P = P(X) \in \mathfrak{p}$, where

$$J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\boldsymbol{R})\,.$$

PROOF. In this case,

$$Z = \left(\begin{array}{c|c} J_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \in \mathfrak{k} \tag{4.4}$$

defines a complex structure on $\mathfrak{p}$. So, Proposition 2.2 yields that

$$\mu(P) = \frac{1}{2}[P, [P, Z]] = \frac{1}{2}\left(\begin{array}{c|c} -J_1\,{}^tXX - {}^tXXJ_1 & \mathbf{0} \\ \hline \mathbf{0} & 2XJ_1\,{}^tX \end{array}\right) \in \mathfrak{k} \tag{4.5}$$

for all $P \in \mathfrak{p}$. □

The second step of the proof is to compute $f_{a,b}$.

PROPOSITION 4.2. *The weighted squared-norm $f_{a,b}$ of μ is written as*

$$f_{a,b}(P) = -\frac{1}{2}nb\,\mathrm{Tr}(P^4) + \frac{n}{8}(a + 2b)(\mathrm{Tr}(P^2))^2 \quad \textit{for } P \in \mathfrak{p}\,. \tag{4.6}$$

PROOF. We recall that $f_{a,b}(P)$ is written as

$$f_{a,b}(P) = b\|\mu(P)\|^2 + (a - b)\|\mu_1(P)\|^2\,. \tag{4.7}$$

by the equation (3.2). So, it is sufficient to compute $\|\mu(P)\|^2$ and $\|\mu_1(P)\|^2$.

First, we compute $\|\mu_1(P)\|^2$. We claim that

$$\|\mu_1(P)\|^2 = \frac{n}{8}(\mathrm{Tr}(P^2))^2\,. \tag{4.8}$$

From Proposition 2.4, it follows that $\|\mu_1(P)\|^2 = \|P\|^4/(4\|Z\|^2)$. The definition of the inner product $\langle\ ,\ \rangle$ on $\mathfrak{p}$ for this case shows that $\|Z\|^2 = 2n$ and $\|P\|^4 = n^2\left(\mathrm{Tr}(P^2)\right)^2$, respectively. These complete the claim.

Next, we show

$$\|\mu(P)\|^2 = -\frac{n}{2}\,\mathrm{Tr}(P^4) + \frac{3n}{8}(\mathrm{Tr}(P))^2\,. \tag{4.9}$$

Recall $\|\mu(P)\|^2 = -n\,\mathrm{Tr}(\mu(P)^2)$, by the definition of the inner product $\langle\ ,\ \rangle$. We obtain

$$\mathrm{Tr}(\mu(P)^2) = -(1/2)\,\mathrm{Tr}(({}^tXX)^2) + (3/2)\,\mathrm{Tr}((J_1{}^tXX)^2) \tag{4.10}$$

from

$$\mu(P)^2 = \frac{1}{4}\left(\begin{array}{c|c} (J_1{}^tXX)^2 + ({}^tXXJ_1)^2 + J_1({}^tXX)^2J_1 - ({}^tXX)^2 & \mathbf{0} \\ \hline \mathbf{0} & 4(XJ_1{}^tX)^2 \end{array}\right). \tag{4.11}$$

Next, to prove

$$\mathrm{Tr}(\mu(P)^2) = \mathrm{Tr}(({}^tXX)^2) - \frac{3}{2}(\mathrm{Tr}({}^tXX))^2 \quad \text{for } P \in \mathfrak{p}, \tag{4.12}$$

it is sufficient to show that

$$\mathrm{Tr}((J_1{}^tXX)^2) = \mathrm{Tr}(({}^tXX)^2) - (\mathrm{Tr}({}^tXX))^2. \tag{4.13}$$

In fact, ${}^tXX$ can be written as

$${}^tXX = \begin{pmatrix} p & r \\ r & q \end{pmatrix}$$

by some real numbers p, q and r, since ${}^tXX$ is a symmetric matrix over $\boldsymbol{R}$ of degree two. By an easy calculation, we obtain

$$({}^tXX)^2 = \begin{pmatrix} p^2+r^2 & (p+q)r \\ (p+q)r & q^2+r^2 \end{pmatrix} \quad \text{and} \quad (J_1{}^tXX)^2 = \begin{pmatrix} r^2-pq & 0 \\ 0 & r^2-pq \end{pmatrix},$$

and so $\mathrm{Tr}(({}^tXX)^2) = p^2+q^2+2r^2$ and $\mathrm{Tr}((J_1{}^tXX)^2) = 2r^2-2pq$ follow. Thus, we get

$$\mathrm{Tr}((J_1{}^tXX)^2) - \mathrm{Tr}(({}^tXX)^2) = -(p+q)^2 = -(\mathrm{Tr}({}^tXX))^2.$$

Thus we obtain (4.12). Finally,

$$\mathrm{Tr}(\mu(P)^2) = \frac{1}{2}\mathrm{Tr}(P^4) - \frac{3}{8}(\mathrm{Tr}(P^2))^2 \quad \text{for } P \in \mathfrak{p} \tag{4.14}$$

follows from

$$P^2 = \left(\begin{array}{c|c} -{}^tXX & \mathbf{0} \\ \hline \mathbf{0} & -X\,{}^tX \end{array}\right) \quad \text{and} \quad P^4 = \left(\begin{array}{c|c} ({}^tXX)^2 & \mathbf{0} \\ \hline \mathbf{0} & (X\,{}^tX)^2 \end{array}\right), \tag{4.15}$$

and we have

$$\mathrm{Tr}({}^tXX) = -(1/2)\,\mathrm{Tr}(P^2) \quad \text{and} \quad \mathrm{Tr}(({}^tXX)^2) = (1/2)\,\mathrm{Tr}(P^4). \tag{4.16}$$

Thus we obtain the proposition. □

The next step of the proof is to compute $\|\mathrm{grad}\, f_{a,b}\|^2$.

PROPOSITION 4.3. *The squared-norm of the gradient* grad $f_{a,b}$ *is given by*

$$\|\mathrm{grad}\, f_{a,b}(P)\|^2 = -\frac{1}{4}(a^2+4ab+2b^2)n(\mathrm{Tr}(P^2))^3 + b(2a+b)n\,\mathrm{Tr}(P^2)\,\mathrm{Tr}(P^4).$$

PROOF. The gradient of $f_{a,b}$ is given by

$$\operatorname{grad} f_{a,b}(P) = \sum_{i=1}^{2n} \frac{\partial f_{a,b}(P)}{\partial x_i} x_i \in \mathfrak{p} \tag{4.17}$$

with respect to an orthonormal basis $\{x_1, \dots, x_{2n}\}$ of $\mathfrak{p}$. Since the expression on the right-hand side is independent of the choice of an orthonormal basis of $\mathfrak{p}$, we take the following orthonormal basis $\{P_{i,j}\,;\ 1 \leq i \leq n,\ j = 1, 2\}$:

$$P_{i,j} := \frac{1}{\sqrt{2n}} \left(\begin{array}{c|c} \mathbf{0} & -E_{j,i}^{(2,n)} \\ \hline E_{i,j}^{(n,2)} & \mathbf{0} \end{array} \right) \in \mathfrak{p}\,, \tag{4.18}$$

where $E_{i,j}^{(n,2)}$ is an $n \times 2$ matrix whose (i, j)-entry is 1 and the others are 0.

First, we compute $\operatorname{grad} f_{a,b}$. We show that

$$\operatorname{grad} f_{a,b}(P) = 2bP^3 - \frac{1}{2}(a + 2b)\operatorname{Tr}(P^2)P\,. \tag{4.19}$$

Recall that

$$f_{a,b}(P) = -\frac{1}{2} nb \operatorname{Tr}(P^4) + \frac{n}{8}(a + 2b)(\operatorname{Tr}(P^2))^2 \quad \text{for } P \in \mathfrak{p}\,. \tag{4.20}$$

The differential of $\operatorname{Tr}(P^4)$ of order one is given by

$$\frac{\partial}{\partial P_{i,j}} \operatorname{Tr}(P^4) = \lim_{t \to 0} \frac{\operatorname{Tr}((P + tP_{i,j})^4) - \operatorname{Tr}(P^4)}{t} = 4\operatorname{Tr}(P^3 P_{i,j})\,. \tag{4.21}$$

Similarly, the differential of $(\operatorname{Tr}(P^2))^2$ of order one is given by

$$\frac{\partial}{\partial P_{i,j}} (\operatorname{Tr}(P^2))^2 = 4\operatorname{Tr}(P^2)\operatorname{Tr}(PP_{i,j})\,. \tag{4.22}$$

Therefore, the gradient of $f_{a,b}$ is written as

$$\begin{aligned} \operatorname{grad} f_{a,b}(P) &= \sum \Big(-2nb \operatorname{Tr}(P^3 P_{i,j}) + \frac{n}{2}(a + 2b)\operatorname{Tr}(P^2)\operatorname{Tr}(PP_{i,j}) \Big) P_{i,j} \\ &= \sum \Big(2b\langle P^3, P_{i,j}\rangle - \frac{1}{2}(a + 2b)\operatorname{Tr}(P^2)\langle P, P_{i,j}\rangle \Big) P_{i,j} \\ &= 2bP^3 - \frac{1}{2}(a + 2b)\operatorname{Tr}(P^2)P\,, \end{aligned}$$

because $P^3 \in \mathfrak{p}$ and $P = \sum \langle P, P_{i,j}\rangle P_{i,j}$ for all $P \in \mathfrak{p}$.

Next, we compute $\|\operatorname{grad} f_{a,b}\|^2$. From (4.19), we have

$$\begin{aligned} \|\operatorname{grad} f_{a,b}(P)\|^2 = -4nb^2 \operatorname{Tr}(P^6) &+ 2nb(a + 2b)\operatorname{Tr}(P^2)\operatorname{Tr}(P^4) \\ &- \frac{n}{4}(a + 2b)^2(\operatorname{Tr}(P^2))^3\,. \end{aligned} \tag{4.23}$$

To complete the proof of the proposition, we have only to show that

$$\operatorname{Tr}(P^6) = -\frac{1}{8}(\operatorname{Tr}(P^2))^3 + \frac{3}{4}\operatorname{Tr}(P^2)\operatorname{Tr}(P^4)\,. \tag{4.24}$$

In fact, simple computations show that $\mathrm{Tr}(P^2) = -2\,\mathrm{Tr}({}^tXX)$, $\mathrm{Tr}(P^4) = 2\,\mathrm{Tr}(({}^tXX)^2)$ and $\mathrm{Tr}(P^6) = -2\,\mathrm{Tr}(({}^tXX)^3)$. Since ${}^tXX$ is a symmetric matrix over $\boldsymbol{R}$ of degree two, let us write

$$
{}^tXX = \begin{pmatrix} p & r \\ r & q \end{pmatrix}, \tag{4.25}
$$

where p, q and r are real numbers. Thus, the above traces can be written as follows respectively:

$$
\begin{aligned}
\mathrm{Tr}(P^2) &= -2\,\mathrm{Tr}({}^tXX) = -2(p+q)\,, \\
\mathrm{Tr}(P^4) &= 2\,\mathrm{Tr}(({}^tXX)^2) = 2(p^2+q^2+2r^2) = 2((p+q)^2 - 2(pq-r^2))\,, \\
\mathrm{Tr}(P^6) &= -2\,\mathrm{Tr}(({}^tXX)^3) = -2(p^3+q^3+3r^2(p+q)) \\
&= -2((p+q)^3 - 3(p+q)(pq-r^2))\,.
\end{aligned} \tag{4.26}
$$

We find that

$$
\begin{aligned}
-\frac{1}{8}&(\mathrm{Tr}(P^2))^3 + \frac{3}{4}\,\mathrm{Tr}(P^2)\,\mathrm{Tr}(P^4) \\
&= -\frac{1}{8}(-2(p+q))^3 + \frac{3}{4}(-2(p+q))2((p+q)^2 - 2(pq-r^2)) \\
&= (p+q)^3 - 3(p+q)((p+q)^2 - 2(pq-r^2)) \\
&= \mathrm{Tr}(P^6)\,.
\end{aligned} \tag{4.27}
$$

This proves the equation (4.24). By substituting the equation (4.24) for (4.23), we finish the proof of the proposition. □

PROPOSITION 4.4. *The Laplacian $\Delta f_{a,b}$ of $f_{a,b}$ are written as*

$$
\Delta f_{a,b}(P) = -(n(a+b) + (a-b))\,\mathrm{Tr}(P^2)\,.
$$

PROOF. The Laplacian of $f_{a,b}(P)$ is given by

$$
\Delta f_{a,b}(P) = \sum_{i=1}^{2n} \frac{\partial^2}{\partial {x_i}^2} f_{a,b}(P)\,.
$$

Since the expression on the right-hand side is independent of the choice of an orthonormal basis of $\mathfrak{p}$, we compute $\Delta f_{a,b}$ with respect to the basis $\{P_{i,j}\}$ defined by the matrices (4.18).

It follows from a linearity of Laplacian that

$$
\Delta f_{a,b}(P) = -\frac{n}{2} b \Delta\,\mathrm{Tr}(P^4) + \frac{n(a+2b)}{8}\Delta(\mathrm{Tr}(P^2))^2\,. \tag{4.28}
$$

From (4.21) and (4.22), the differentials of $\mathrm{Tr}(P^4)$ and $(\mathrm{Tr}(P^2))^2$ of order two are given by

$$\begin{aligned}\frac{\partial^2}{\partial {P_{i,j}}^2}\,\mathrm{Tr}(P^4) &= 8\,\mathrm{Tr}(P^2 {P_{i,j}}^2) + 4\,\mathrm{Tr}((P P_{i,j})^2)\,,\\ \frac{\partial^2}{\partial {P_{i,j}}^2}(\mathrm{Tr}(P^2))^2 &= 8(\mathrm{Tr}(P P_{i,j}))^2 + 4\,\mathrm{Tr}(P^2)\,\mathrm{Tr}(P_{i,j}^2)\\ &= 8(\mathrm{Tr}(P P_{i,j}))^2 - \frac{4}{n}\,\mathrm{Tr}(P^2)\,.\end{aligned} \tag{4.29}$$

Thus, we have

$$\begin{aligned}\Delta f_{a,b}(P) = &-4nb\sum \mathrm{Tr}(P^2 {P_{i,j}}^2) - 2nb\sum \mathrm{Tr}((P P_{i,j})^2)\\ &+ n(a+2b)\sum(\mathrm{Tr}(P P_{i,j}))^2 - (a+2b)n\,\mathrm{Tr}(P^2)\,.\end{aligned} \tag{4.30}$$

We will compute each sum in this equation. We show

$$\sum \mathrm{Tr}(P^2 {P_{i,j}}^2) = -\frac{n+2}{4n}\,\mathrm{Tr}(P^2)\,. \tag{4.31}$$

Since

$${P_{i,j}}^2 = \frac{1}{2n}\left(\begin{array}{c|c} -E^{(2,2)}_{j,j} & \mathbf{0} \\ \hline \mathbf{0} & -E^{(n,n)}_{i,i}\end{array}\right),$$

we have

$$\begin{aligned}\sum \mathrm{Tr}(P^2 {P_{i,j}}^2) &= \mathrm{Tr}\left(P^2 \sum {P_{i,j}}^2\right)\\ &= \mathrm{Tr}\left(\left(\begin{array}{c|c} -{}^tXX & \mathbf{0} \\ \hline \mathbf{0} & -X\,{}^tX\end{array}\right)\cdot\frac{1}{2n}\left(\begin{array}{c|c} -nI_2 & \mathbf{0} \\ \hline \mathbf{0} & -2I_n\end{array}\right)\right)\\ &= \frac{1}{2n}(n\,\mathrm{Tr}({}^tXX) + 2\,\mathrm{Tr}(X{}^tX))\\ &= \frac{n+2}{2n}\,\mathrm{Tr}({}^tXX)\,.\end{aligned} \tag{4.32}$$

Since $\mathrm{Tr}(P^2) = -2\,\mathrm{Tr}({}^tXX)$, we get the equation (4.31). Next we show

$$\sum \mathrm{Tr}((P P_{i,j})^2) = -\frac{1}{2n}\,\mathrm{Tr}(P^2)\,. \tag{4.33}$$

Since

$$(P P_{i,j})^2 = \frac{1}{2n}\left(\begin{array}{c|c} \left({}^tXE^{(n,2)}_{i,j}\right)^2 & \mathbf{0} \\ \hline \mathbf{0} & \left(E^{(2,n)}_{j,i}X\right)^2\end{array}\right), \tag{4.34}$$

we find that

$$\mathrm{Tr}((P P_{i,j})^2) = \frac{1}{n}\,\mathrm{Tr}((E^{(2,n)}_{j,i}X)^2)\,.$$

Here, we write $X = (x_{i,j})$. Then, a simple computation shows $E^{(2,n)}_{j,i} X E^{(2,n)}_{j,i} = x_{i,j} E^{(2,n)}_{j,i}$. Hence, it turns out that

$$\sum \mathrm{Tr}((PP_{i,j})^2) = \frac{1}{n}\mathrm{Tr}\Big(\Big(\sum E^{(2,n)}_{j,i} X E^{(2,n)}_{j,i}\Big)X\Big) = \frac{1}{n}\mathrm{Tr}({}^tXX)\,. \tag{4.35}$$

Since $\mathrm{Tr}(P^2) = -2\,\mathrm{Tr}({}^tXX) = -2\,\mathrm{Tr}(X\,{}^tX)$, we obtain the equation (4.33). Next we show

$$\sum (\mathrm{Tr}(PP_{i,j}))^2 = -\frac{1}{n}\mathrm{Tr}(P^2)\,. \tag{4.36}$$

This follows from

$$\sum (\mathrm{Tr}(PP_{i,j}))^2 = \sum \Big(-\frac{1}{n}\langle P, P_{i,j}\rangle\Big)^2 = \frac{1}{n^2}\|P\|^2 = -\frac{1}{n}\mathrm{Tr}(P^2)\,. \tag{4.37}$$

By substituting (4.31), (4.33) and (4.36) for the equation (4.30), we finish the proof. □

Now, we find a pair (a, b) of real numbers so that $f_{a,b}$ is a Cartan-Münzner polynomial.

THEOREM 4.5. *When* $G/K = \mathrm{SO}(2+n)/\mathrm{SO}(2)\times\mathrm{SO}(n)$, *the weighted squared-norm* $f_{a,b}$ *of* μ *is a Cartan-Münzner polynomial if* $(a, b) = (-8n, 16n)$.

PROOF. If $(a, b) = (-8n, 16n)$, then $\|\mathrm{grad}\, f_{a,b}(P)\|^2$ and $\Delta f_{a,b}(P)$ are written as

$$\begin{cases} \|\mathrm{grad}\, f_{a,b}(P)\|^2 = -16n^3(\mathrm{Tr}(P^2))^3 = 16\|P\|^6\,, \\ \Delta f_{a,b}(P) = -8n(n-3)\,\mathrm{Tr}(P^2) = 8(n-3)\|P\|^2\,, \end{cases} \tag{4.38}$$

respectively. Thus, from (1.2), $f_{a,b}(P)$ is a Cartan-Münzner polynomial for $g = 4$ and $(m_1, m_2) = (1, n-2)$. □

5. Case of $\mathrm{SU}(2+n)/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(n))$**.** In this section, we give a proof of the main theorem in the case where $G = \mathrm{SU}(2+n)$ and $K = \mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(n))$. The Lie algebras corresponding G and K are $\mathfrak{g} = \mathfrak{su}(2+n)$ and $\mathfrak{k} = \mathfrak{u}(1)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(n)$, respectively. The Cartan decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$, where

$$\mathfrak{k} = \left\{\left(\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{0} & A' \end{array}\right)\,;\ A\in\mathfrak{u}(2),\ A'\in\mathfrak{u}(n),\ \mathrm{Tr}(A)+\mathrm{Tr}(A') = 0\right\}, \tag{5.1}$$

and

$$\mathfrak{p} = \left\{P(X) = \left(\begin{array}{c|c} \mathbf{0} & -{}^t\overline{X} \\ \hline X & \mathbf{0} \end{array}\right)\,;\ X\in M_{n,2}(\boldsymbol{C})\right\}. \tag{5.2}$$

In order to simplify the notation, we write P instead of $P(X)$ through this section. In this case, the Killing form B of $\mathfrak{g} = \mathfrak{su}(2+n)$ is given by

$$B(P, Q) = 2(n+2)\,\mathrm{Re}\,\mathrm{Tr}(PQ) \quad \text{for } P, Q\in\mathfrak{g}\,, \tag{5.3}$$

where Re means the real part of a complex number. Thus, the G-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ is written as

$$\langle P, Q\rangle := -B(P, Q) = -2(n+2)\,\mathrm{Re}\,\mathrm{Tr}(PQ) \quad \text{for } P, Q\in\mathfrak{g}\,. \tag{5.4}$$

We remark that $\mathrm{Tr}(P^2), \mathrm{Tr}(P^4), \mathrm{Tr}(P^6)\in\boldsymbol{R}$ for all $P\in\mathfrak{p}$.

First, we compute the moment map μ.

PROPOSITION 5.1. *In this case, the moment map μ is given by*

$$\mu(P) = \sqrt{-1}\left(\begin{array}{c|c} -{}^t\overline{X}X & \mathbf{0} \\ \hline \mathbf{0} & X{}^t\overline{X} \end{array}\right) \in \mathfrak{k}\,. \tag{5.5}$$

PROOF. In this case,

$$Z = \frac{\sqrt{-1}}{n+2}\left(\begin{array}{c|c} nI_2 & \mathbf{0} \\ \hline \mathbf{0} & -2I_n \end{array}\right) \in \mathfrak{k} \tag{5.6}$$

defines a complex structure on $\mathfrak{p}$. Thus, Proposition 2.2 yields that

$$\mu(P) = \frac{1}{2}[P,[P,Z]] = \sqrt{-1}\left(\begin{array}{c|c} -{}^t\overline{X}X & \mathbf{0} \\ \hline \mathbf{0} & X{}^t\overline{X} \end{array}\right) \tag{5.7}$$

for each $P \in \mathfrak{p}$. □

Next, we compute the weighted squared-norm $f_{a,b}(P)$ of μ.

PROPOSITION 5.2. *The weighted squared-norm $f_{a,b}(P)$ of μ can be written as*

$$f_{a,b}(P) = 2(n+2)b\,\mathrm{Tr}(P^4) + \frac{(n+2)^2}{4n}(a-b)(\mathrm{Tr}(P^2))^2\,. \tag{5.8}$$

PROOF. Recall that $f_{a,b}(P) = b\|\mu(P)\|^2 + (a-b)\|\mu_1(P)\|^2$ by Proposition 2.4. Thus, it is sufficient to compute $\|\mu(P)\|^2$ and $\|\mu_1(P)\|^2$.

First, we calculate $\|\mu(P)\|^2$. We show

$$\|\mu(P)\|^2 = 2(n+2)\,\mathrm{Tr}(P^4)\,. \tag{5.9}$$

From the definition of the inner product $\langle\ ,\ \rangle$, the squared-norm of μ is given by

$$\|\mu(P)\|^2 = -2(n+2)\,\mathrm{Re}\,\mathrm{Tr}(\mu(P)^2)\,. \tag{5.10}$$

We obtain

$$\mathrm{Re}\,\mathrm{Tr}(\mu(P)^2) = -\,\mathrm{Tr}(P^4) \tag{5.11}$$

from

$$\mu(P)^2 = -\left(\begin{array}{c|c} ({}^t\overline{X}X)^2 & \mathbf{0} \\ \hline \mathbf{0} & (X{}^t\overline{X})^2 \end{array}\right) = -P^4\,, \tag{5.12}$$

and $\mathrm{Tr}(P^4) \in \boldsymbol{R}$.

Next, we compute $\|\mu_1(P)\|^2$. We claim that

$$\|\mu_1(P)\|^2 = \frac{(n+2)^2}{4n}(\mathrm{Tr}(P^2))^2\,. \tag{5.13}$$

From Proposition 2.4, it follows that $\|\mu_1(P)\|^2 = \|P\|^4/(4\|Z\|^2)$. The definition of the inner product $\langle\ ,\ \rangle$ on $\mathfrak{p}$ for this case shows that $\|Z\|^2 = 4n$ and $\|P\|^2 = -2(n+2)\,\mathrm{Re}\,\mathrm{Tr}(P^2)$, respectively. Because $\mathrm{Tr}(P^2) \in \boldsymbol{R}$, these complete the claim.

Thus, (5.9) and (5.13) show the proposition. □

The third step of the proof is to compute $\|\mathrm{grad}\, f_{a,b}\|^2$.

PROPOSITION 5.3. *In this case, the squared-norm of the gradient* grad $f_{a,b}$ *are written as*

$$\begin{aligned}\|\mathrm{grad}\, f_{a,b}(P)\|^2 = {} & \frac{n+2}{2}\left(8b^2 - \frac{(n+2)^2}{n^2}(a-b)^2\right)(\mathrm{Tr}(P^2))^3 \\ & - 8(n+2)\left(3b^2 + \frac{n+2}{n}b(a-b)\right)\mathrm{Tr}(P^2)\,\mathrm{Tr}(P^4)\,. \end{aligned} \tag{5.14}$$

PROOF. It is sufficient to compute $\|\mathrm{grad}\, f_{a,b}(P)\|^2$ with respect to an orthonormal basis $\{P_{i,j}, Q_{i,j}\,;\ 1 \le i \le n,\ j = 1, 2\}$ of $\mathfrak{p}$ defined by

$$P_{i,j} = \frac{1}{2\sqrt{n+2}}\left(\begin{array}{c|c} \mathbf{0} & -E^{(2,n)}_{j,i} \\ \hline E^{(n,2)}_{i,j} & \mathbf{0} \end{array}\right), \tag{5.15}$$

$$Q_{i,j} = \frac{\sqrt{-1}}{2\sqrt{n+2}}\left(\begin{array}{c|c} \mathbf{0} & E^{(2,n)}_{j,i} \\ \hline E^{(n,2)}_{i,j} & \mathbf{0} \end{array}\right). \tag{5.16}$$

First, we show

$$\mathrm{grad}\, f_{a,b}(P) = -4bP^3 - \frac{n+2}{2n}(a-b)\,\mathrm{Tr}(P^2)P\,. \tag{5.17}$$

Recall (5.8). The differentials of $\mathrm{Tr}(P^4)$ and $\left(\mathrm{Tr}(P^2)\right)^2$ of order one are given by

$$\begin{aligned} & \frac{\partial}{\partial P_{i,j}}\mathrm{Tr}(P^4) = 4\,\mathrm{Tr}(P^3P_{i,j})\,, \qquad \frac{\partial}{\partial Q_{i,j}}\mathrm{Tr}(P^4) = 4\,\mathrm{Tr}(P^3Q_{i,j})\,, \\ & \frac{\partial}{\partial P_{i,j}}(\mathrm{Tr}(P^2))^2 = 4\,\mathrm{Tr}(P^2)\,\mathrm{Tr}(PP_{i,j})\,, \qquad \frac{\partial}{\partial Q_{i,j}}(\mathrm{Tr}(P^2))^2 = 4\,\mathrm{Tr}(P^2)\,\mathrm{Tr}(PQ_{i,j})\,, \end{aligned} \tag{5.18}$$

respectively. Hence, we find that

$$\begin{aligned} & \mathrm{grad}\, f_{a,b}(P) \\ & = \sum\left(\frac{\partial f_{a,b}(P)}{\partial P_{i,j}}P_{i,j} + \frac{\partial f_{a,b}(P)}{\partial Q_{i,j}}Q_{i,j}\right) \\ & = \sum\left(8(n+2)b\,\mathrm{Tr}(P^3P_{i,j}) + \frac{(n+2)^2}{n}(a-b)\,\mathrm{Tr}(P^2)\,\mathrm{Tr}(PP_{i,j})\right)P_{i,j} \\ & \quad + \sum\left(8(n+2)b\,\mathrm{Tr}(P^3Q_{i,j}) + \frac{(n+2)^2}{n}(a-b)\,\mathrm{Tr}(P^2)\,\mathrm{Tr}(PQ_{i,j})\right)Q_{i,j} \\ & = \sum\left(-4b\langle P^3, P_{i,j}\rangle - \frac{n+2}{2n}(a-b)\,\mathrm{Tr}(P^2)\langle P, P_{i,j}\rangle\right)P_{i,j} \\ & \quad + \sum\left(-4b\langle P^3, Q_{i,j}\rangle - \frac{n+2}{2n}(a-b)\,\mathrm{Tr}(P^2)\langle P, Q_{i,j}\rangle\right)Q_{i,j} \end{aligned}$$

$$= -4bP^3 - \frac{n+2}{2n}(a-b)\operatorname{Tr}(P^2)P\,,$$

because $P^3 \in \mathfrak{p}$ and $P = \sum \big(\langle P, P_{i,j}\rangle P_{i,j} + \langle P, Q_{i,j}\rangle Q_{i,j}\big)$ for any $P \in \mathfrak{p}$.

From (5.17), we have

$$\begin{aligned}\|\mathrm{grad}\, f_{a,b}\|^2 = {}& -32b^2(n+2)\operatorname{Tr}(P^6) - \frac{8(n+2)^2}{n} b(a-b)\operatorname{Tr}(P^2)\operatorname{Tr}(P^4) \\ & - \frac{(n+2)^3}{2n^2}(a-b)^2(\operatorname{Tr}(P^2))^3\,,\end{aligned} \tag{5.19}$$

since $\operatorname{Tr}(P^2)$, $\operatorname{Tr}(P^4)$, $\operatorname{Tr}(P^6) \in \boldsymbol{R}$ for all $P \in \mathfrak{p}$. To complete the proof, we have only to show that

$$\operatorname{Tr}(P^6) = -\frac{1}{8}(\operatorname{Tr}(P^2))^3 + \frac{3}{4}\operatorname{Tr}(P^2)\operatorname{Tr}(P^4) \quad \text{for } P \in \mathfrak{p}\,. \tag{5.20}$$

First of all, we have

$$\operatorname{Tr}(P^2) = -2\operatorname{Tr}({}^t\overline{X}X),\ \operatorname{Tr}(P^4) = 2\operatorname{Tr}(({}^t\overline{X}X)^2),\ \operatorname{Tr}(P^6) = -2\operatorname{Tr}(({}^t\overline{X}X)^3)\,.$$

Since ${}^t\overline{X}X$ is a Hermitian matrix of degree two, write ${}^t\overline{X}X$ as follows:

$${}^t\overline{X}X = \begin{pmatrix} p & \bar{r} \\ r & q \end{pmatrix}, \tag{5.21}$$

where p and q are real numbers and r is a complex number. Then, the above traces can be written as

$$\begin{aligned}&\operatorname{Tr}(P^2) = -2(p+q)\,,\\ &\operatorname{Tr}(P^4) = 2(p^2+q^2+2|r|^2) = 2((p+q)^2 - 2(pq-|r|^2))\,,\\ &\operatorname{Tr}(P^6) = -2(p^3+q^3+3|r|^2(p+q)) = -2((p+q)^3 - 3(p+q)(pq-|r|^2))\,,\end{aligned}$$

where $|\cdot|$ means the absolute value of a complex number. Thus, we obtain that

$$\begin{aligned}& -\frac{1}{8}(\operatorname{Tr}(P^2))^3 + \frac{3}{4}\operatorname{Tr}(P^2)\operatorname{Tr}(P^4) \\ &\quad = -\frac{1}{8}(-2(p+q))^3 + \frac{3}{4}(-2(p+q))2((p+q)^2 - 2(pq-|r|^2)) \\ &\quad = (p+q)^3 - 3(p+q)((p+q)^2 - 2(pq-|r|^2)) \\ &\quad = \operatorname{Tr}(P^6)\,.\end{aligned} \tag{5.22}$$

This proves the equation (5.20). By substituting (5.20) for (5.19), we finish the proof. □

PROPOSITION 5.4. *In this case, the Laplacian $\Delta f_{a,b}$ of $f_{a,b}$ is written as*

$$\Delta f_{a,b}(P) = -\frac{n+2}{n}(2n(a+b) + (a-b))\operatorname{Tr}(P^2)\,. \tag{5.23}$$

PROOF. Since the Laplacian is a linear operator, we can write $\Delta f_{a,b}$ as

$$\Delta f_{a,b} = 2(n+2)b\Delta\operatorname{Tr}(P^4) + \frac{(n+2)^2}{4n}(a-b)\Delta(\operatorname{Tr}(P^2))^2\,. \tag{5.24}$$

From (5.18), the differentials of $\mathrm{Tr}(P^4)$ and $(\mathrm{Tr}(P^2))^2$ of degree two are given by

$$\begin{aligned}\frac{\partial^2}{\partial {P_{i,j}}^2}\,\mathrm{Tr}(P^4) &= 8\,\mathrm{Tr}(P^2 {P_{i,j}}^2) + 4\,\mathrm{Tr}((P P_{i,j})^2)\,,\\ \frac{\partial^2}{\partial {Q_{i,j}}^2}\,\mathrm{Tr}(P^4) &= 8\,\mathrm{Tr}(P^2 {Q_{i,j}}^2) + 4\,\mathrm{Tr}((P Q_{i,j})^2)\,,\\ \frac{\partial^2}{\partial {P_{i,j}}^2}(\mathrm{Tr}(P^2))^2 &= 8(\mathrm{Tr}(P P_{i,j}))^2 - \frac{2}{n+2}\,\mathrm{Tr}(P^2),\\ \frac{\partial^2}{\partial {Q_{i,j}}^2}(\mathrm{Tr}(P^2))^2 &= 8(\mathrm{Tr}(P Q_{i,j}))^2 - \frac{2}{n+2}\,\mathrm{Tr}(P^2)\,.\end{aligned} \tag{5.25}$$

Thus, we have

$$\begin{aligned}\Delta f_{a,b}(P) = {}& 16(n+2)b\sum(\mathrm{Tr}(P^2 {P_{i,j}}^2) + \mathrm{Tr}(P^2 {Q_{i,j}}^2))\\ &+ 8(n+2)b\sum(\mathrm{Tr}((P P_{i,j})^2) + \mathrm{Tr}((P {Q_{i,j}}^2)))\\ &+ \frac{2(n+2)^2}{n}(a-b)\sum((\mathrm{Tr}(P P_{i,j}))^2 + (\mathrm{Tr}(P Q_{i,j}))^2)\\ &- 2(n+2)(a-b)\,\mathrm{Tr}(P^2)\,.\end{aligned} \tag{5.26}$$

Since

$$ {P_{i,j}}^2 = {Q_{i,j}}^2 = -\frac{1}{4(n+2)}\left(\begin{array}{c|c} E^{(2,2)}_{j,j} & \mathbf{0} \\ \hline \mathbf{0} & E^{(n,n)}_{i,i}\end{array}\right) \tag{5.27}$$

and $\mathrm{Tr}(P^2) = -2\,\mathrm{Tr}({}^t\overline{X} X)$, we have

$$\begin{aligned}&\sum(\mathrm{Tr}(P^2 {P_{i,j}}^2) + \mathrm{Tr}(P^2 {Q_{i,j}}^2))\\ &\quad= \mathrm{Tr}\left(P^2 \sum({P_{i,j}}^2 + {Q_{i,j}}^2)\right)\\ &\quad= 2\,\mathrm{Tr}\left(P^2 \sum\left({P_{i,j}}^2\right)\right)\\ &\quad= 2\,\mathrm{Tr}\left(\left(\begin{array}{c|c} -{}^t\overline{X}X & \mathbf{0} \\ \hline \mathbf{0} & -X\,{}^t\overline{X}\end{array}\right)\cdot\frac{-1}{4(n+2)}\left(\begin{array}{c|c} nI_2 & \mathbf{0} \\ \hline \mathbf{0} & 2I_n\end{array}\right)\right)\\ &\quad= \frac{1}{2(n+2)}(n\,\mathrm{Tr}({}^t\overline{X}X) + 2\,\mathrm{Tr}(X\,{}^t\overline{X}))\\ &\quad= \frac{1}{2}\,\mathrm{Tr}({}^t\overline{X}X)\\ &\quad= -\frac{1}{4}\,\mathrm{Tr}(P^2)\,.\end{aligned} \tag{5.28}$$

From

$$(P P_{i,j})^2 = \frac{1}{4(n+2)}\left(\begin{array}{c|c} \left({}^t\overline{X} E^{(n,2)}_{i,j}\right)^2 & \mathbf{0} \\ \hline \mathbf{0} & \left(X E^{(2,n)}_{j,i}\right)^2\end{array}\right) = -(P Q_{i,j})^2\,,$$

we obtain

$$\sum(\mathrm{Tr}((PP_{i,j})^2) + \mathrm{Tr}((PQ_{i,j}{}^2))) = 0\,. \tag{5.29}$$

Next, we obtain

$$\begin{aligned}
&\sum_{i,j}((\mathrm{Tr}(PP_{i,j}))^2 + (\mathrm{Tr}(PQ_{i,j}))^2)\\
&\quad = \sum_{i,j}\left(\left(-\frac{1}{2(n+2)}\langle P, P_{i,j}\rangle\right)^2 + \left(-\frac{1}{2(n+2)}\langle P, Q_{i,j}\rangle\right)^2\right)\\
&\quad = \frac{1}{4(n+2)^2}\sum_{i,j}(\langle P, P_{i,j}\rangle^2 + \langle P, Q_{i,j}\rangle^2)\\
&\quad = \frac{1}{4(n+2)^2}\|P\|^2\\
&\quad = -\frac{1}{2(n+2)}\,\mathrm{Tr}(P^2)\,.
\end{aligned} \tag{5.30}$$

By substituting (5.28), (5.29) and (5.30) for (5.26), we finish the proof. □

Now, we find a pair (a, b) of real numbers so that $f_{a,b}$ is a Cartan-Münzner polynomial.

THEOREM 5.5. *When* $G/K = \mathrm{SU}(2+n)/\mathrm{S}\,(\mathrm{U}(2)\times\mathrm{U}(n))$ *and* $(a, b) = (32(n-1), -16(n+2))$, *the weighted squared-norm* $f_{a,b}$ *of* μ *is a Cartan-Münzner polynomial.*

PROOF. If $(a, b) = (32(n-1), -16(n+2))$, then the above propositions state that

$$\begin{cases} \|\mathrm{grad}\, f_{a,b}\|^2 = -128(n+2)^3(\mathrm{Tr}(P^2))^3 = 16\|P\|^6\,,\\ \Delta f_{a,b} = -16(n+2)(2n-5)\,\mathrm{Tr}(P^2) = 4(2n-5)\|P\|^2\,. \end{cases} \tag{5.31}$$

Thus, from (1.2), $f_{a,b}(P)$ is a Cartan-Münzner polynomial for $g = 4$ and $(m_1, m_2) = (2, 2n-3)$. □

6. Case of $\mathrm{SO}(10)/\mathrm{U}(5)$**.** Finally, we consider the case of $G/K = \mathrm{SO}(10)/\mathrm{U}(5)$. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. We put

$$\mathfrak{k} = \left\{\left(\begin{array}{c|c} A & A' \\ \hline -A' & A \end{array}\right)\ ;\ A \in \mathrm{Alt}_5(\boldsymbol{R}),\ A' \in \mathrm{Sym}_5(\boldsymbol{R})\right\}, \tag{6.1}$$

and

$$\mathfrak{p} = \left\{P(X, Y) = \left(\begin{array}{c|c} X & Y \\ \hline Y & -X \end{array}\right)\ ;\ X, Y \in \mathrm{Alt}_5(\boldsymbol{R})\right\}. \tag{6.2}$$

It is clear that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In order to simplify the notation, we write P instead of $P(X, Y)$ through this section. The Killing form B of $\mathfrak{g} = \mathfrak{so}(10)$ is given by

$$B(P, Q) = 8\,\mathrm{Tr}(PQ) \quad \text{for } P, Q \in \mathfrak{g}\,. \tag{6.3}$$

Thus, the G-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ is written as

$$\langle P, Q\rangle := -B(P, Q) = -8\,\mathrm{Tr}(PQ) \quad \text{for } P, Q \in \mathfrak{g}\,. \tag{6.4}$$

First, we compute the moment map μ for the isotropy representation of SO(10)/ U(5).

PROPOSITION 6.1. *In this case, the moment map μ is given by*

$$\mu(P) = \left(\begin{array}{c|c} -[X,Y] & X^2+Y^2 \\ \hline -X^2-Y^2 & -[X,Y] \end{array} \right) \in \mathfrak{k} \tag{6.5}$$

for all $P \in \mathfrak{p}$, where $[\ ,\ ]$ is a canonical bracket product on a Lie algebra $\mathfrak{gl}_5(\boldsymbol{R})$.

PROOF. First, we put

$$Z = \frac{1}{2} \left(\begin{array}{c|c} \mathbf{0} & I_5 \\ \hline -I_5 & \mathbf{0} \end{array} \right).$$

It is clear that $Z \in C(\mathfrak{k})$ and $J := \mathrm{ad}_Z|_{\mathfrak{p}}$ gives a complex structure on $\mathfrak{p}$. Thus, Proposition 2.2 yields that

$$\mu(P) = \frac{1}{2}[P,[P,Z]] = \left(\begin{array}{c|c} -[X,Y] & X^2+Y^2 \\ \hline -X^2-Y^2 & -[X,Y] \end{array} \right) \tag{6.6}$$

for $P \in \mathfrak{p}$. □

The next step is to compute $f_{a,b}$.

PROPOSITION 6.2. *In this case, $f_{a,b}$ is given by*

$$f_{a,b}(P) = 8b\,\mathrm{Tr}(P^4) + \frac{4(a-b)}{5}(\mathrm{Tr}(P^2))^2 . \tag{6.7}$$

PROOF. Recall that $f_{a,b}(P) = b\|\mu(P)\|^2 + (a-b)\|\mu_1(P)\|^2$.
First, we show

$$\|\mu(P)\|^2 = 8\,\mathrm{Tr}(P^4) . \tag{6.8}$$

From the definition of $\langle\ ,\ \rangle$, we have $\|\mu(P)\|^2 = -8\,\mathrm{Tr}(\mu(P)^2)$, and

$$\mathrm{Tr}(\mu(P)^2) = -2\,\mathrm{Tr}((X^2+Y^2)^2 - [X,Y]^2) \tag{6.9}$$

follows from

$$\mu(P)^2 = \left(\begin{array}{c|c} -(X^2+Y^2)^2 + [X,Y]^2 & * \\ \hline * & -(X^2+Y^2)^2 + [X,Y]^2 \end{array} \right).$$

Since $\mathrm{Tr}(P^4) = 2\,\mathrm{Tr}\left((X^2+Y^2)^2 - [X,Y]^2\right)$, we get (6.8).
Next, we show

$$\|\mu_1(P)\|^2 = \frac{4}{5}(\mathrm{Tr}(P^2))^2 . \tag{6.10}$$

By Proposition 2.4, we have $\|\mu_1(P)\|^2 = \|P\|^4/(4\|Z\|^2)$. The definition of the K-invariant inner product $\langle\ ,\ \rangle$ shows that $\|Z\|^2 = 20$ and $\|P\|^2 = -8\,\mathrm{Tr}(P^2)$. These completes the claim.

Thus, (6.8) and (6.10) show the proposition. □

PROPOSITION 6.3. *In this case*, $\|\mathrm{grad}\, f_{a,b}\|^2$ *is given by*

$$\|\mathrm{grad}\, f_{a,b}\|^2 = -\frac{4}{25}(8a^2 - 16ab - 17b^2)(\mathrm{Tr}(P^2))^3 - \frac{16}{5}b(8a+7b)\,\mathrm{Tr}(P^2)\,\mathrm{Tr}(P^4)\,. \tag{6.11}$$

PROOF. Now, we take the following orthonormal basis $\{P_{i,j}, Q_{i,j}\,;\ 1 \le i < j \le 5\}$:

$$P_{i,j} = \frac{1}{4\sqrt{2}}\left(\begin{array}{c|c} E_{i,j} - E_{j,i} & \mathbf{0} \\ \hline \mathbf{0} & -E_{i,j} + E_{j,i} \end{array}\right), \tag{6.12}$$

$$Q_{i,j} = \frac{1}{4\sqrt{2}}\left(\begin{array}{c|c} \mathbf{0} & E_{i,j} - E_{j,i} \\ \hline E_{i,j} - E_{j,i} & \mathbf{0} \end{array}\right), \tag{6.13}$$

where $E_{i,j}$ is a square matrix of degree five whose (i, j)-entry is 1 and the others are 0.

First, we compute $\mathrm{grad}\, f_{a,b}$. We show

$$\mathrm{grad}\, f_{a,b} = -4bP^3 - \frac{2}{5}(a-b)\,\mathrm{Tr}(P^2)P\,. \tag{6.14}$$

Recall (6.7). The differentials of $\mathrm{Tr}(P^4)$ and $\big(\mathrm{Tr}(P^2)\big)^2$ of order one are given by

$$\begin{aligned} &\frac{\partial}{\partial P_{i,j}}\mathrm{Tr}(P^4) = 4\,\mathrm{Tr}(P^3P_{i,j})\,, \qquad \frac{\partial}{\partial Q_{i,j}}\mathrm{Tr}(P^4) = 4\,\mathrm{Tr}(P^3Q_{i,j})\,, \\ &\frac{\partial}{\partial P_{i,j}}(\mathrm{Tr}(P^2))^2 = 4\,\mathrm{Tr}(P^2)\,\mathrm{Tr}(PP_{i,j})\,, \qquad \frac{\partial}{\partial Q_{i,j}}(\mathrm{Tr}(P^2))^2 = 4\,\mathrm{Tr}(P^2)\,\mathrm{Tr}(PQ_{i,j})\,, \end{aligned} \tag{6.15}$$

respectively. Thus, $\mathrm{grad}\, f_{a,b}(P)$ is written as

$$\begin{aligned} \mathrm{grad}\, f_{a,b}(P) &= \sum_{1\le i<j\le 5}\left(32b\,\mathrm{Tr}(P^3P_{i,j}) + \frac{16}{5}(a-b)\,\mathrm{Tr}(P^2)\,\mathrm{Tr}(PP_{i,j})\right)P_{i,j} \\ &\quad + \sum_{1\le i<j\le 5}\left(32b\,\mathrm{Tr}(P^3Q_{i,j}) + \frac{16}{5}(a-b)\,\mathrm{Tr}(P^2)\,\mathrm{Tr}(PQ_{i,j})\right)Q_{i,j} \\ &= -4b\sum(\langle P^3, P_{i,j}\rangle P_{i,j} + \langle P^3, Q_{i,j}\rangle Q_{i,j}) \\ &\quad - \frac{2}{5}(a-b)\,\mathrm{Tr}(P^2)\sum(\langle P, P_{i,j}\rangle P_{i,j} + \langle P, Q_{i,j}\rangle Q_{i,j}) \\ &= -4bP^3 - \frac{2}{5}(a-b)\,\mathrm{Tr}(P^2)P\,, \end{aligned}$$

because $P^3 \in \mathfrak{p}$ and $P = \sum(\langle P, P_{i,j}\rangle P_{i,j} + \langle P, Q_{i,j}\rangle Q_{i,j})$ for all $P \in \mathfrak{p}$.

Next, we compute $\|\mathrm{grad}\, f_{a,b}\|^2$. From (6.14), we have

$$\|\mathrm{grad}\, f_{a,b}(P)\|^2 = -128b^2\,\mathrm{Tr}(P^6) - \frac{128}{5}b(a-b)\,\mathrm{Tr}(P^2)\,\mathrm{Tr}(P^4) - \frac{32}{25}(a-b)^2(\mathrm{Tr}(P^2))^3\,. \tag{6.16}$$

Here, we have

$$\operatorname{Tr}(P^6) = -\frac{1}{32}(\operatorname{Tr}(P^2))^3 + \frac{3}{8}\operatorname{Tr}(P^2)\operatorname{Tr}(P^4) \tag{6.17}$$

for every $P \in \mathfrak{p}$. This can be checked by direct computations. By substituting (6.17) for (6.16), we complete this proposition. □

PROPOSITION 6.4. *In this case, $\Delta f_{a,b}$ is given by*

$$\Delta f_{a,b} = -\frac{4}{5}(11a + 9b)\operatorname{Tr}(P^2). \tag{6.18}$$

PROOF. Because a Laplacian is a linear operator, it follows that

$$\Delta f_{a,b}(P) = 8b\Delta\operatorname{Tr}(P^4) + \frac{4(a-b)}{5}\Delta(\operatorname{Tr}(P^2))^2.$$

From (6.15), the differentials of $\operatorname{Tr}(P^4)$ and $\left(\operatorname{Tr}(P^2)\right)^2$ of order two are given by

$$\begin{aligned}
\frac{\partial^2}{\partial {P_{i,j}}^2}\operatorname{Tr}(P^4) &= 8\operatorname{Tr}(P^2{P_{i,j}}^2) + 4\operatorname{Tr}((PP_{i,j})^2),\\
\frac{\partial^2}{\partial {Q_{i,j}}^2}\operatorname{Tr}(P^4) &= 8\operatorname{Tr}(P^2{Q_{i,j}}^2) + 4\operatorname{Tr}((PQ_{i,j})^2),\\
\frac{\partial^2}{\partial {P_{i,j}}^2}(\operatorname{Tr}(P^2))^2 &= 8(\operatorname{Tr}(PP_{i,j}))^2 - \frac{1}{2}\operatorname{Tr}(P^2),\\
\frac{\partial^2}{\partial {Q_{i,j}}^2}(\operatorname{Tr}(P^2))^2 &= 8(\operatorname{Tr}(PQ_{i,j}))^2 - \frac{1}{2}\operatorname{Tr}(P^2).
\end{aligned} \tag{6.19}$$

Thus, we have

$$\begin{aligned}
\Delta f_{a,b}(P) = 64b&\sum(\operatorname{Tr}(P^2{P_{i,j}}^2) + \operatorname{Tr}(P^2{Q_{i,j}}^2))\\
&+ 32b\sum(\operatorname{Tr}((PP_{i,j})^2) + \operatorname{Tr}((PQ_{i,j})^2))\\
&+ \frac{32(a-b)}{5}\sum((\operatorname{Tr}(PP_{i,j}))^2 + (\operatorname{Tr}(PQ_{i,j}))^2)\\
&- 10\operatorname{Tr}(P^2).
\end{aligned} \tag{6.20}$$

We will compute each sum in this equation. We show

$$\sum_{1\le i<j\le 5}(\operatorname{Tr}(P^2{P_{i,j}}^2) + \operatorname{Tr}(P^2{Q_{i,j}}^2)) = -\frac{1}{4}\operatorname{Tr}(P^2). \tag{6.21}$$

Since

$${P_{i,j}}^2 = {Q_{i,j}}^2 = \frac{1}{32}\left(\begin{array}{c|c} -E_{i,i} - E_{j,j} & \mathbf{0} \\ \hline \mathbf{0} & -E_{i,i} - E_{j,j} \end{array}\right), \tag{6.22}$$

we have

$$\sum_{1\le i<j\le 5}(\mathrm{Tr}(P^2P_{i,j}{}^2)+\mathrm{Tr}(P^2Q_{i,j}{}^2))$$

$$(6.23)\qquad \begin{aligned}&=\mathrm{Tr}\left(P^2\sum(P_{i,j}{}^2+Q_{i,j}{}^2)\right)\\&=\mathrm{Tr}\left(\left(\begin{array}{c|c}X^2+Y^2&[X,Y]\\\hline -[X,Y]&X^2+Y^2\end{array}\right)\cdot\frac{1}{32}\left(\begin{array}{c|c}-8I_5&\mathbf{0}\\\hline \mathbf{0}&-8I_5\end{array}\right)\right)\\&=-\frac{1}{2}\mathrm{Tr}(X^2+Y^2).\end{aligned}$$

Since $\mathrm{Tr}(P^2)=-2\,\mathrm{Tr}(X^2+Y^2)$, we get (6.21). Next, we claim

$$(6.24)\qquad \sum_{1\le i<j\le 5}(\mathrm{Tr}((PP_{i,j})^2)+\mathrm{Tr}((PQ_{i,j})^2))=0\,.$$

This follows from

$$(6.25)\qquad \left(PP_{i,j}\right)^2=\frac{1}{32}\left(\begin{array}{c|c}(XF_{i,j})^2-(YF_{i,j})^2&-XF_{i,j}YF_{i,j}-YF_{i,j}XF_{i,j}\\\hline XF_{i,j}YF_{i,j}+YF_{i,j}XF_{i,j}&(XF_{i,j})^2-(YF_{i,j})^2\end{array}\right)$$

and

$$(6.26)\qquad (PQ_{i,j})^2=\frac{1}{32}\left(\begin{array}{c|c}-(XF_{i,j})^2+(YF_{i,j})^2&XF_{i,j}YF_{i,j}+YF_{i,j}XF_{i,j}\\\hline -XF_{i,j}YF_{i,j}-YF_{i,j}XF_{i,j}&-(XF_{i,j})^2+(YF_{i,j})^2\end{array}\right),$$

where $F_{i,j}=E_{i,j}-E_{j,i}$. Next, we show

$$(6.27)\qquad \sum_{1\le i<j\le 5}((\mathrm{Tr}(PP_{i,j}))^2+(\mathrm{Tr}(PQ_{i,j}))^2)=-\frac{1}{8}\mathrm{Tr}(P^2)\,.$$

This follows from

$$\sum_{1\le i<j\le 5}((\mathrm{Tr}(PP_{i,j}))^2+(\mathrm{Tr}(PQ_{i,j}))^2)$$

$$(6.28)\qquad \begin{aligned}&=\sum\left(\left(-\frac{1}{8}\langle P,P_{i,j}\rangle\right)^2+\left(-\frac{1}{8}\langle P,Q_{i,j}\rangle\right)^2\right)\\&=\frac{1}{64}\sum(\langle P,P_{i,j}\rangle^2+\langle P,Q_{i,j}\rangle^2)\\&=\frac{1}{64}\|P\|^2\\&=-\frac{1}{8}\mathrm{Tr}(P^2).\end{aligned}$$

By substituting (6.21), (6.24) and (6.27) for (6.20), we finish the proof. □

We find a pair (a,b) of real numbers so that $f_{a,b}$ is the Cartan-Münzner polynomial.

THEOREM 6.5. *Assume that* $G/K=\mathrm{SO}(10)/\mathrm{U}(5)$. *If* $(a,b)=(112,-128)$, *then the squared-norm* $f_{a,b}$ *of the moment map* μ *is a Cartan-Münzner polynomial.*

PROOF. If $(a, b) = (112, -128)$, then Propositions 6.3 and 6.4 show that

$$\begin{cases} \|\mathrm{grad}\, f_{a,b}\|^2 = -8192\left(\mathrm{Tr}(P^2)\right)^3 = 16\|P\|^6\,, \\ \Delta f_{a,b} = -64\,\mathrm{Tr}\left(P^2\right) = 4\|P\|^2\,. \end{cases} \tag{6.29}$$

Therefore, from (1.2), $f_{a,b}(P)$ is a Cartan-Münzner polynomial for $g = 4$ and $(m_1, m_2) = (4, 5)$. □

Thus, we obtain Theorem 3.1.

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