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CANONICAL FIXED PARTS OF FIBRED ALGEBRAIC SURFACES

Dedicated to Professor Sampei Usui on his sixtieth birthday

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Abstract. It is shown that the fixed part of the canonical linear system of a fibre in a relatively minimal fibred surface supports at most exceptional sets of weakly elliptic singularities.

Introduction. Let S be a non-singular projective surface and $f : S \to C$ a surjective morphism of S onto a non-singular projective curve C with connected fibres. We call f a relatively minimal fibration of genus g if a general fibre is a non-singular projective curve of genus g and there are no (-1)-curves contained in fibres. We assume that $g \geq 2$ throughout the paper. Let F be a fibre of f. Then the intersection form is negative semi-definite on $\mathrm{Supp}(F)$ by Zariski's lemma. Furthermore, there exist a positive integer m and a 1-connected curve D such that $F = mD$. When m is strictly greater than one, F is called a multiple fibre of multiplicity m and $\mathcal{O}_D(D)$ is a torsion of order m.

In [8], we considered the canonical linear system on the minimal resolution of a normal surface singularity and showed that the fixed part supports at most exceptional sets of rational singular points (cf. [1] and [2]). The present article is an extension of it to the semi-global case and we study the fixed part of the canonical linear system $|K_F|$ which we call the canonical fixed part in this paper. Recall that the canonical fixed part is closely related to the Horikawa index (see [3, p. 12]), an analytic invariant of a singular fibre germ. In fact, according to [6, Lemma 10 and Theorem 3], if $g = 2$, the canonical fixed part is a chain of (-2)-curves (of type A) and the Horikawa index is almost equivalent to the number of its irreducible components.

Let Z be a (non-zero) subcurve of D such that the restriction map $H^0(F, K_F) \to H^0(Z, K_F)$ is the zero map. Then we have $p_a(Z) \leq 1$ by a result in [9]. Since the intersection form is negative semi-definite on fibres, we can expect a stronger assertion. We shall show in Theorem 3.1 that $\mathrm{Supp}(Z)$ contracts to rational singular points when F is a non-multiple fibre, and to rational or weakly elliptic singular points [17] when F is multiple. The most delicate part in the proof is to see that the support of the canonical fixed part is strictly smaller than that of the whole fibre. Though this may sound strange, one should realize that it do happens when $g = 1$ as a simple example shows: if $F = mD$ is a multiple fibre in an elliptic fibration with D being a smooth elliptic curve, then K_F is a torsion of order $m \geq 2$

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on D, and we have $H^0(D, K_F) = 0$ implying that $D \subseteq \mathrm{Bs}|K_F|$. If such a phenomenon were happen, then the intersection form would not be negative definite on the fixed part and we would fail to contract it to normal surface singularities. Another point to be noticed is that we do not know a priori whether the fundamental cycles on the connected components of Z are subcurves of Z or not. The proof of Theorem 3.1 goes similarly as in [8] in spirit, looking a fixed component through a particular curve called a loupe if available. For multiple fibres, we compare $|K_F|$ and $|K_D|$ to see how the torsion sheaf $\mathcal{O}_D(D)$ affects the base locus.

The geometric genus of a weakly elliptic singularity can be arbitrarily big. So, it is another problem to have a bound on $h^1(Z, \mathcal{O}_Z)$. In Theorem 4.6, we shall show that $h^1(Z, \mathcal{O}_Z) \leq m - 1$ holds, where m denotes the multiplicity of F. Furthermore, it is shown that, if $H^1(Z, \mathcal{O}_Z) \neq 0$, Z contains the unique fundamental cycle of a minimally elliptic singularity [12], though Z may have several connected components.

In the global situation, our results can be applied to the fixed part of $|K_S + f^*\mathfrak{d}|$, where $\mathfrak{d}$ is a divisor on C which is ample enough for the restriction map $H^0(S, K_S + f^*\mathfrak{d}) \to H^0(F, K_F)$ to be surjective. It is a very interesting question to ask whether a similar assertion holds for the fixed part of the canonical linear system on a projective algebraic surface of general type with $p_g \gg 0$, especially when the canonical map is not composed of a pencil.

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1. Preliminaries. By a *curve*, we mean a non-zero effective divisor on a smooth surface. If a curve D decomposes as the sum of two curves D_1, D_2, then $p_a(D) = p_a(D_1) + p_a(D_2) - 1 + D_1 D_2$. A curve D is called (numerically) *k-connected*, if $D_1 D_2 \geq k$ holds for any decomposition $D = D_1 + D_2$ with D_1, D_2 being curves. The following can be found in [5, (A.4)Lemma] (see also [13]).

LEMMA 1.1. *Let D be a k-connected curve and $D = D_1 + D_2$ an effective decomposition such that $D_1 D_2 = k$. Then D_1 and D_2 are $[(k+1)/2]$-connected, where $[x]$ denotes the integer part of x.*

A line bundle L on D is called *nef* if L is of non-negative degree on any component of D. We will tacitly use the fact that $H^0(D, -L) \neq 0$ implies $L = \mathcal{O}_D$ when D is *chain-connected* and L is nef. Here, a curve D is called chain-connected [16] if either D is irreducible or $\mathcal{O}_{D-\Gamma}(-\Gamma)$ is not nef for any proper subcurve $\Gamma \prec D$. It is clear that every 1-connected curve is chain-connected.

As to the base points of linear systems, we have the following fundamental result due to Catanese and Franciosi [4, Proposition 2.4].

THEOREM 1.2. *Let L be a line bundle on a 1-connected curve D with $p_a(D) > 0$ such that $L - K_D$ is nef. If $p \in \mathrm{Bs}|L|$, then there exists a subcurve Δ of D satisfying one of the following.*

(1) *p is a non-singular point of Δ and $\mathcal{O}_\Delta(L) \simeq \mathcal{O}_\Delta(K_\Delta + p)$.*

(2) *$\Delta = D$, $L \neq K_D$ and p is a non-singular point of D. Furthermore, there exists another non-singular point $q \in D$ such that $\mathcal{O}_D(L) \simeq \mathcal{O}_D(K_D + p - q)$.*

(3) *$\Delta = D$, $L \neq K_D$, p is a double point of D and $\mathcal{O}_{\hat{D}}(\nu^*(L - K_D)) \simeq \mathcal{O}_{\hat{D}}$, where $\nu : \hat{D} \to D$ denotes the blowing-up of the maximal ideal $\mathfrak{m}_p$.*

The following can be found in [9, Theorem 5.4] (see also [14, Theorem 4.1], [8, Theorem 1.1] and [11, Theorem A]).

THEOREM 1.3. *Let L be a line bundle on a 1-connected curve D which is numerically equivalent to K_D, and let Z be a proper subcurve of D such that the restriction map $H^0(D, L) \to H^0(Z, L)$ is the zero map. Then*

$$p_a(Z) \leq \begin{cases} 0 & \text{if } L = K_D\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

If the equality holds here, then Z is 1-connected and D decomposes as

$$D = Z + \Gamma_1 + \cdots + \Gamma_n\,,$$

where $n = Z(D - Z) = h^0(D - Z, \mathcal{O}_{D-Z})$, $\mathcal{O}_{D-Z}(L) \simeq \mathcal{O}_{D-Z}(K_D)$, each Γ_i is a 1-connected curve with $(D - \Gamma_i)\Gamma_i = Z\Gamma_i = 1$, $\mathcal{O}_{\Gamma_j + \cdots + \Gamma_n}(-\Gamma_{j-1})$ is trivial for $2 \leq j \leq n$ and either $\Gamma_j \preceq \Gamma_i$ or $\mathrm{Supp}(\Gamma_i) \cap \mathrm{Supp}(\Gamma_j) = \emptyset$ for $i < j$.

Let $\mathcal{A} = \bigcup_i A_i$ be a connected bunch of irreducible curves A_i. The intersection form is negative semi-definite on $\mathcal{A}$ if and only if there exists a curve Z such that $\mathrm{Supp}(Z) = \mathcal{A}$ and $-Z$ is nef on $\mathcal{A}$. The smallest curve with such a property exists and we call it the *numerical cycle* on $\mathcal{A}$ according to [15, Chapter 4]. When the intersection form is negative definite, it is usually called the *fundamental cycle* (cf. [1], [2]). A numerical cycle is not necessarily 1-connected, but it is chain-connected. If a chain-connected curve D is such that $\mathcal{O}_D(-D)$ is nef, then it is the numerical cycle on its support. This is a consequence of the following fact which can be found in [9, Proposition 1.5].

LEMMA 1.4. *Let D_1, D_2 be curves such that $\mathcal{O}_{D_1}(-D_2)$ is nef. If D_1 is chain-connected, then either $\mathrm{Supp}(D_1) \cap \mathrm{Supp}(D_2) = \emptyset$ or $D_1 \preceq D_2$.*

As for multiple fibres in a fibred surface, we have the following which is an analogue of Theorem 1.3 (see [9, Corollary 6.2]).

THEOREM 1.5. *Let F be a multiple fibre and Z a subcurve of F such that the restriction map $H^0(F, K_F) \to H^0(Z, K_F)$ is the zero map. Then $p_a(Z) \leq 1$. If $p_a(Z) = 1$, then Z is 0-connected and F decomposes as*

$$F = Z + \Gamma_0 + \Gamma_1 + \cdots + \Gamma_n\,,$$

where $n = -Z^2 = h^0(F - Z, \mathcal{O}_{F-Z}) - 1$,

(1) *for $1 \leq i \leq n$, Γ_i is a 1-connected curve with $\Gamma_i^2 = -1$, $Z\Gamma_i = 1$, and $\mathcal{O}_{\Gamma_i}(-(\Gamma_0 + \cdots + \Gamma_{i-1})) \simeq \mathcal{O}_{\Gamma_i}$, $\mathcal{O}_{\Gamma_j}(-\Gamma_i)$ is numerically trivial when $i < j$, and*

(2) *Γ_0 is a positive multiple of the numerical cycle D on $\mathrm{Supp}(F)$.*

2. Loupes in fibres. Hereafter, F denotes a fibre in a relatively minimal fibred surface of genus $g \geq 2$. We are interested in the fixed part of $|K_F|$, that is, the biggest subcurve Z_{can} of F such that the restriction map $H^0(F, K_F) \to H^0(Z_{\mathrm{can}}, K_F)$ is the zero map. We call Z_{can} the canonical fixed part of F.

Let D be the numerical cycle on $\mathrm{Supp}(F)$. Then there exists a positive integer m such that $F = mD$. When $m \geq 2$, F is called a multiple fibre and $\mathcal{O}_D(D)$ is a torsion of order m. We have $g - 1 = m(p_a(D) - 1)$ and $h^0(D, K_F) = p_a(D) - 1$. Recall that D is 1-connected and that the restriction map $H^0(F, K_F) \to H^0(D, K_F)$ is surjective (see, e.g. [7, Lemma 4.2.1]).

The following easy lemma is useful in the sequel.

LEMMA 2.1. *Let D be the numerical cycle of a fibre. Let Δ be a subcurve of D with $\Delta^2 = -1$. Then it is 1-connected and the restriction map $H^0(D, K_D) \to H^0(\Delta, K_D)$ is surjective. If Δ_1 is another subcurve with $\Delta_1^2 = -1$, then either Δ and Δ_1 are disjoint or one is a subcurve of the other, except in the following cases*:

(1) $D = \Delta + \Delta_1$.

(2) $D = \Delta + \Delta_1 - \gcd(\Delta, \Delta_1)$ *and* $\gcd(\Delta, \Delta_1) \neq 0$.

In particular, if $\mathrm{Supp}(\Delta + \Delta_1)$ *is strictly smaller than* $\mathrm{Supp}(D)$, *then either* $\mathrm{Supp}(\Delta) \cap \mathrm{Supp}(\Delta_1) = \emptyset$ *or* $\Delta \preceq \Delta_1$ *or* $\Delta_1 \preceq \Delta$.

PROOF. If $\Delta^2 = -1$, then $\Delta(D - \Delta) = 1$. Hence Δ and $D - \Delta$ are both 1-connected by Lemma 1.1, since so is D. The second assertion follows from the cohomology long exact sequence for

$$0 \to \mathcal{O}_{D-\Delta}(K_{D-\Delta}) \to \mathcal{O}_D(K_D) \to \mathcal{O}_\Delta(K_D) \to 0\,,$$

since $H^1(D - \Delta, K_{D-\Delta}) \to H^1(D, K_D)$ is an isomorphism as the dual of $H^0(D, \mathcal{O}_D) \to H^0(D - \Delta, \mathcal{O}_{D-\Delta})$.

Let Δ_1 be a subcurve of D with $\Delta_1^2 = -1$. If $\Delta + \Delta_1 \neq D$, then we have $0 > (\Delta + \Delta_1)^2 = -2 + 2\Delta\Delta_1$. It follows that $\Delta\Delta_1 \leq 0$, which implies that either Δ and Δ_1 are disjoint, or they have a common component. Assume the latter and put $G = \gcd(\Delta, \Delta_1)$, $B = \Delta - G$ and $B_1 = \Delta_1 - G$. Then

$$(2.1) \qquad (G + B + B_1)^2 = \Delta^2 + \Delta_1^2 + G^2 + 2\Delta\Delta_1 - 2G(\Delta + \Delta_1) = -2 - G^2 + 2BB_1$$

and

$$(2.2) \qquad \Delta\Delta_1 = G^2 + (B + B_1)G + BB_1\,.$$

Since B and B_1 do not have a common component, we have $BB_1 \geq 0$. If D is not equal to $G + B + B_1 = \Delta + \Delta_1 - \gcd(\Delta, \Delta_1)$, then $(G + B + B_1)^2 < 0$. It follows $G^2 = -1$ and $BB_1 = 0$, because we have $G^2 < 0$ and $BB_1 \geq 0$. By $\Delta\Delta_1 \leq 0$, this implies that $(B + B_1)G \leq 1$. Since Δ and Δ_1 are 1-connected, we conclude that either B or B_1 must be zero. □

We apply Theorem 1.2 for $L = K_F$:

LEMMA 2.2. *Let $E \preceq Z_{\mathrm{can}}$ be an irreducible component. Then one of the following holds*:

(1) *There exists a strict subcurve Δ of D with $E \prec \Delta$ such that $\mathcal{O}_\Delta(-\Delta) \simeq \mathcal{O}_\Delta(p)$ holds for any point $p \in E$ which is a non-singular point of Δ. Furthermore*, $E \subseteq \mathrm{Bs}|K_\Delta|$ *and* $E \simeq \boldsymbol{P}^1$.

(2) *F is a multiple fibre*, *D is of multiplicity one along E and*, *for a general $p \in E$ which is a non-singular point of D*, *there exists a non-singular point $q \in D$ such that $\mathcal{O}_D(D) \simeq \mathcal{O}_D(q-p)$.*

PROOF. We take a moving point p on E in such a way that it is a non-singular point of E as well as that of D_{red}. Then we can immediately drop the case (3) of Theorem 1.2 and see that one of (1) and (2) of Theorem 1.2 holds for such general p's. If (2) of Theorem 1.2 is the case, then we are in (2). So we assume that (1) of Theorem 1.2 is the case. Then Δ is 1-connected, because $\Delta^2 = -1$. Furthermore, we may assume that $\mathcal{O}_\Delta(-\Delta) \simeq \mathcal{O}_\Delta(p)$ holds for infinitely many such p's, since we have only a finite number of choices of Δ's. It follows $h^0(\Delta, \mathcal{O}_\Delta(p)) \geq 2$ and we have $h^0(\Delta, K_\Delta - p) \geq p_a(\Delta)$ by the Riemann-Roch theorem. This implies that $p \in \mathrm{Bs}|K_\Delta|$ and, therefore, $E \subseteq \mathrm{Bs}|K_\Delta|$. Then $E \simeq \boldsymbol{P}^1$ by Theorem 1.3. Furthermore, we have $h^0(\Delta, \mathcal{O}_\Delta(p)) = 2$. From the exact sequence of sheaves

$$0 \to \mathcal{O}_{\Delta-E}(-E+p) \to \mathcal{O}_\Delta(p) \to \mathcal{O}_E(p) \to 0\,,$$

we know that $H^0(\Delta, \mathcal{O}_\Delta(p)) \simeq H^0(E, \mathcal{O}_E(p))$. Then, for a given point $q \in E$ which is a non-singular point of Δ, we have $\mathcal{O}_\Delta(-\Delta) \simeq \mathcal{O}_\Delta(q)$, since p and q are linearly equivalent on Δ. □

An irreducible component $E \preceq Z_{\mathrm{can}}$ will be referred to as a *canonical fixed component*. It is said to be of type (I) if there exists a curve Δ for E as in (1) of Lemma 2.2. In this case, we call Δ a *loupe* for E. It should be noticed, however, that a loupe is not necessarily unique if exists. If $E \preceq Z_{\mathrm{can}}$ is not of type (I), we call it of type (II). We have $p_a(E) \leq 1$ for any component of type (II) by Theorem 1.3.

Let Δ be a loupe for a type (I) component $E \preceq Z_{\mathrm{can}}$. Then it is 1-connected by Lemma 2.1. As one sees from $\mathcal{O}_\Delta(-\Delta) \simeq \mathcal{O}_\Delta(p)$, Δ is the numerical cycle on its support. Since $\Delta^2 = -1$ while $D^2 = 0$, the support of Δ is strictly smaller than that of D. Hence the intersection form is negative definite on $\mathrm{Supp}(\Delta)$. In other words, any subcurve of Δ contracts to normal surface singularities. By virtue of Theorem 1.3, the fact that $E \subseteq \mathrm{Bs}|K_\Delta|$ gives us a particular decomposition of Δ:

$$\Delta = E + C_1 + \cdots + C_{n-1} \tag{2.3}$$

where $n = -E^2$, $h^0(\Delta - E, \mathcal{O}_{\Delta-E}) = n-1$, each C_i is 1-connected and $EC_i = -C_i^2 = 1$, $\mathcal{O}_{C_j+\cdots+C_{n-1}}(-C_{j-1})$ is trivial for $j \geq 2$ and, when $i < j$, either C_i and C_j are disjoint or $C_j \prec C_i$. In particular, any two maximal curves in $\{C_i\}_{i=1}^n$ are disjoint and the support of a maximal curve is a connected component of $\mathrm{Supp}(\Delta - E)$. Furthermore, it is shown in [8, Lemma 1.4] that each C_i is the fundamental cycle on its support, using the fact that $E \not\preceq C_i$.

We remark also that we have $n \geq 2$, because our fibration is relatively minimal. We know that D is 1-connected and $(D - \Delta)\Delta = 1$. Then $D - \Delta$ is also 1-connected.

LEMMA 2.3. *Let Δ be a loupe for a component $E \preceq Z_{\mathrm{can}}$ of type* (I). *If $E \npreceq D - \Delta$, then $D - \Delta$ is the fundamental cycle on its support.*

PROOF. Let A be a component of $D - \Delta$. If $A \prec \Delta$, then $A\Delta = 0$ by $\mathcal{O}_{\Delta - E}(\Delta) \simeq \mathcal{O}_{\Delta - E}$ and $A \neq E$, and it follows $A(D - \Delta) = 0$. If $A \not\prec \Delta$, then $A\Delta \geq 0$ and $A(D - \Delta) \leq 0$. Therefore, $-(D - \Delta)$ is nef on $D - \Delta$. Since $D - \Delta$ is 1-connected, we see from Lemma 1.4 that $D - \Delta$ is the fundamental cycle on its support. □

LEMMA 2.4. *Assume that F is a multiple fibre and let Δ be a loupe for a component $E \preceq Z_{\mathrm{can}}$ of type* (I). *Then the following hold.*

(1) *For a point $p \in E$ which is a non-singular point of Δ, $p \in \mathrm{Bs}|K_D|$ holds if and only if $\mathcal{O}_\Delta(D) \simeq \mathcal{O}_\Delta$.*

(2) *The restriction map $H^0(D, K_F) \to H^0(\Delta, K_F)$ is not surjective if and only if $\mathcal{O}_{D-\Delta}(D) \simeq \mathcal{O}_{D-\Delta}$.*

PROOF. (1) Since $\Delta^2 = -1$, the restriction map $H^0(D, K_D) \to H^0(\Delta, K_D)$ is surjective by Lemma 2.1. Let $p \in E$ be a point which is a non-singular point of Δ. Consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_\Delta(K_D - p) \to \mathcal{O}_\Delta(K_D) \to \mathcal{O}_p \to 0\,.$$

We have $H^1(\Delta, K_D) = 0$. Hence $p \in \mathrm{Bs}|K_D|$ if and only if $H^1(\Delta, K_D - p)^\vee \simeq H^0(\Delta, -D + \Delta + p) \neq 0$. Since $\mathcal{O}_\Delta(-\Delta) \simeq \mathcal{O}_\Delta(p)$, the last condition becomes $H^0(\Delta, -D) \neq 0$. Since Δ is 1-connected and $\mathcal{O}_\Delta(D)$ is numerically trivial, this happens if and only if $\mathcal{O}_\Delta(D) \simeq \mathcal{O}_\Delta$. Note that we have shown that the three conditions $p \in \mathrm{Bs}|K_D|$, $E \subseteq \mathrm{Bs}|K_D|$ and $\mathcal{O}_\Delta(D) \simeq \mathcal{O}_\Delta$ are equivalent.

(2) Consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_{D-\Delta}(K_F - \Delta) \to \mathcal{O}_D(K_F) \to \mathcal{O}_\Delta(K_F) \to 0\,.$$

We have $H^1(D, K_F) = 0$, since F is a multiple fibre and $\mathcal{O}_D(D)$ is a non-trivial torsion on a 1-connected curve D. Therefore, $H^0(D, K_F) \to H^0(\Delta, K_F)$ is not surjective if and only if $H^1(D - \Delta, K_F - \Delta) \simeq H^0(D - \Delta, D)^\vee \neq 0$. Since $D - \Delta$ is 1-connected, we have $H^0(D - \Delta, D) \neq 0$ if and only if $\mathcal{O}_{D-\Delta}(D) \simeq \mathcal{O}_{D-\Delta}$. □

LEMMA 2.5. *Let Δ be a loupe for a component $E \preceq Z_{\mathrm{can}}$ of type* (I). *Put $G = \gcd(\Delta, D - \Delta)$, $B = \Delta - G$ and $B_1 = D - \Delta - G$. If G and B are both non-zero, then the following hold.*

(1) *$G^2 = -1$, $GB = GB_1 = 1$ and $\mathrm{Supp}(B) \cap \mathrm{Supp}(B_1) = \emptyset$.*

(2) *D is of multiplicity one along E, and $D - \Delta$ is the fundamental cycle on its support.*

(3) *If $\Delta = E + C_1 + \cdots + C_{n-1}$ is the decomposition as in* (2.3), *then G is one of the maximal curves in $\{C_i\}_{i=1}^{n-1}$. Furthermore, G is the fundamental cycle on its support.*

(4) *B is the fundamental cycle on its support and $\mathcal{O}_{B-E}(D) \simeq \mathcal{O}_{B-E}$.*

(5) *The restriction map $H^0(D, K_D) \to H^0(E, K_D)$ is of rank at most one.*

PROOF. We have $(D-G)^2 = G^2$. On the other hand, we have $(D-G)^2 = (G+B+B_1)^2 = -2 - G^2 + 2BB_1$ by (2.1). It follows $-G^2 + BB_1 = 1$. Then we get $G^2 = -1$ and $BB_1 = 0$, since $BB_1 \geq 0$ and $G^2 < 0$. We have $G\Delta = G^2 + GB = -1 + GB \geq 0$ by the 1-connectedness of Δ. Since $G\Delta \leq 0$, we have $GB = 1$ and $G\Delta = 0$. From the last, we see that $E \preceq B$ and $E \not\preceq G$. We know from $BB_1 = 0$ that B and B_1 are disjoint. Therefore, D is of multiplicity one along E. Furthermore, since $E \not\prec D - \Delta$, it follows from Lemma 2.3 that $D - \Delta$ is the fundamental cycle on its support. We have $G(B + B_1) = 2$ by (2.2) and $\Delta(D - \Delta) = 1$. So $GB_1 = 1$. Since Δ and $D - \Delta$ are both 1-connected, the condition $GB = GB_1 = 1$ shows that G, B and B_1 are all 1-connected.

It follows from $\mathcal{O}_\Delta(D-\Delta) \simeq \mathcal{O}_\Delta(D+p)$ that $\mathcal{O}_B(G) \simeq \mathcal{O}_B(D+p)$ and $\mathcal{O}_{B-E}(G) \simeq \mathcal{O}_{B-E}(D)$, since $\mathcal{O}_B(B_1) \simeq \mathcal{O}_B$ and $p \notin B - E$. On the other hand, it follows from $\mathcal{O}_\Delta(-\Delta) \simeq \mathcal{O}_\Delta(p)$ that $\mathcal{O}_B(-B) \simeq \mathcal{O}_B(G+p)$. We get $\mathcal{O}_B(-B) \simeq \mathcal{O}_B(D+2p)$. This shows that $\mathcal{O}_B(-B)$ is nef. Hence, B is the fundamental cycle on its support. Note that we have $\mathcal{O}_{B-E}(-B) \simeq \mathcal{O}_{B-E}(G) \simeq \mathcal{O}_{B-E}(D)$.

We claim that G is the fundamental cycle on its support. To see this, let Γ be a component of G. Then $\Gamma \neq E$ and we have $0 = \Delta\Gamma = G\Gamma + B\Gamma$. If Γ is a component of B, then $B\Gamma = 0$ which shows $G\Gamma = 0$. If Γ is not a component of B, then $B\Gamma \geq 0$ which implies $G\Gamma \leq 0$. In sum, we see that $\mathcal{O}_G(-G)$ is nef. Since G is 1-connected, it is the fundamental cycle on its support. Then, since $G^2 = -1$, we can find an irreducible component G_0 of G such that $G_0G = -1$ and $G\Gamma = 0$ for any other components $\Gamma \prec G$. Since $G_0B = 1$ and $\mathcal{O}_B(-B)$ is nef, G_0 is not a component of B. It follows that G_0 is a non-multiple component of Δ.

Let $\Delta = E + C_1 + \cdots + C_{n-1}$ be the decomposition as in (2.3), where $n = -E^2$. Recall that we have $EC_i = -C_i^2 = 1$ for $1 \leq i \leq n-1$ and $\mathcal{O}_{C_j+\cdots+C_{n-1}}(C_{j-1}) \simeq \mathcal{O}_{C_j+\cdots+C_{n-1}}$ for $2 \leq j \leq n-1$. There exists a C_i which contains G_0. Since Δ is of multiplicity one along G_0, such a C_i is unique and, hence, C_i is a maximal curve in $\{C_j\}_{j=1}^{n-1}$.

We claim that $G = C_i$. This can be seen as follows. We have $G^2 = C_i^2 = -1$ and $G + C_i \prec D$. Hence, either $G \preceq C_i$ or $C_i \preceq G$ by Lemma 2.1. Suppose that C_i is a strict subcurve of G. We have $GC_i = -1$, because $G_0 \preceq C_i$ and $\mathcal{O}_{G-G_0}(-G)$ is numerically trivial. Then we get $(G - C_i)C_i = 0$ by $C_i^2 = GC_i = -1$. This is absurd, because G is 1-connected. Therefore, we get $G \preceq C_i$. We have $C_i^2 = (D-\Delta)^2 = -1$ and $C_i + (D-\Delta) \prec D$. Since $B_1 \neq 0$, we have $C_i \preceq D - \Delta$ by Lemma 2.1. Then, since $G = \gcd(\Delta, D - \Delta)$ and $G \preceq C_i$, we conclude that $G = C_i$.

We have shown that $B = E + \sum_{j \neq i} C_j$. It follows that $\mathcal{O}_{B-E}(G) \simeq \mathcal{O}_{B-E}$, since $G = C_i$ and it is maximal in $\{C_j\}_{j=1}^{n-1}$. Then $\mathcal{O}_{B-E}(D) \simeq \mathcal{O}_{B-E}$, since we already know that $\mathcal{O}_{B-E}(G) \simeq \mathcal{O}_{B-E}(D)$.

Consider the restriction map $H^0(D, K_D) \to H^0(E, K_D)$. It is easy to see that its cokernel is of dimension $h^0(D - E, \mathcal{O}_{D-E}) - 1$. Recall that $E \subseteq \mathrm{Bs}|K_\Delta|$. Then $h^0(\Delta - E, \mathcal{O}_{\Delta-E}) = n - 1$ by Theorem 1.3. We have $h^0(B - E, \mathcal{O}_{B-E}) = n - 2$ from

$$0 \to \mathcal{O}_{B-E}(-G) \to \mathcal{O}_{\Delta-E} \to \mathcal{O}_G \to 0$$

and $\mathcal{O}_{B-E}(-G) \simeq \mathcal{O}_{B-E}$. We have the exact sequence

$$0 \to H^0(B-E, -D+\Delta-G) \to H^0(D-E, \mathcal{O}_{D-E}) \to H^0(D-\Delta+G, \mathcal{O})$$

in which the last map is non-trivial. Hence

$$h^0(D-E, \mathcal{O}_{D-E}) - 1 \geq h^0(B-E, -D+\Delta-G) = h^0(B-E, \mathcal{O}_{B-E}) = n-2\,.$$

Since $h^0(E, K_D) = \deg K_D|_E + 1 = -E^2 - 1 = n-1$, the restriction map in question is of rank at most one. □

LEMMA 2.6. *Let Δ and Δ_1 be loupes for distinct canonical fixed components E and E_1, respectively. Assume that $D = \Delta + \Delta_1 - \gcd(\Delta, \Delta_1)$ and $\gcd(\Delta, \Delta_1) \neq 0$. Then the following hold.*

(1) *D has multiplicity one along E and E_1, and $E, E_1 \prec \gcd(\Delta, \Delta_1)$,*

(2) *$\mathcal{O}_{D-\Delta}(D) \simeq \mathcal{O}_{D-\Delta}$ and $\mathcal{O}_{D-\Delta_1}(D) \simeq \mathcal{O}_{D-\Delta_1}$.*

PROOF. Put $G = \gcd(\Delta, \Delta_1)$, $B = \Delta - G$ and $B_1 = \Delta_1 - G$. Then $D = \Delta + B_1 = \Delta_1 + B$. Since $B^2 = B_1^2 = -1$ and they have no common components, it follows from Lemma 2.1 that B and B_1 are disjoint 1-connected curves. We have $0 \geq B\Delta = B^2 + BG = -1 + BG$. Since Δ is 1-connected, we get $BG = 1$ and $B\Delta = 0$. Then $E \preceq G$, $G^2 = -2$ and G is 1-connected. We have $\mathcal{O}_B(D) \simeq \mathcal{O}_B(\Delta + B_1) \simeq \mathcal{O}_B$. Quite similarly, $B_1G = 1$, $B_1\Delta_1 = 0$, $E_1 \preceq G$ and $\mathcal{O}_{B_1}(D) \simeq \mathcal{O}_{B_1}$.

Since $\mathcal{O}_\Delta(-\Delta) \simeq \mathcal{O}_\Delta(p)$ for a general point $p \in E$, restricting it to G, we have $\mathcal{O}_G(-G) \simeq \mathcal{O}_G(B+p)$. We have $\mathcal{O}_G(B_1) = \mathcal{O}_G(D-\Delta) \simeq \mathcal{O}_G(D+p)$. Restricting it to $E \prec G$, we get $EB_1 = 1$. Similarly, $\mathcal{O}_G(-G) \simeq \mathcal{O}_G(B_1 + p_1)$, $\mathcal{O}_G(B) \simeq \mathcal{O}_G(D+p_1)$ and $E_1B = 1$ for a general point $p_1 \in E_1$. Hence $\mathcal{O}_G(-G) \simeq \mathcal{O}_G(D+p+p_1)$, which shows that G is the fundamental cycle on its support. Furthermore, $\mathcal{O}_{G-E}(B_1)$ and $\mathcal{O}_{G-E_1}(B)$ are numerically trivial. We claim that $E_1 \not\preceq B$ and $E \not\preceq B_1$. This can be seen as follows. If Δ and B_1 have no common components, then we clearly have $E \not\preceq B_1$. If Δ and B_1 have a common component, then it follows from Lemma 2.5 that $E \preceq \Delta - \gcd(\Delta, B_1)$, that is, $E \not\preceq B_1$. Similarly, we get $E_1 \not\preceq B$. In particular, we see that D is of multiplicity one along both E and E_1. □

3. Rationality and ellipticity. Let (V, o) be (a germ of) a normal surface singularity and $\pi : X \to V$ its resolution. The arithmetic genus of (V, o) is defined as $p_a(V, o) := \sup\{p_a(\Gamma)\ ;\ 0 \prec \Gamma,\ \mathrm{Supp}(\Gamma) \subseteq \pi^{-1}(o)\}$. We call (V, o) a rational (resp. weakly elliptic) singular point when $p_a(V, o) = 0$ (resp. 1). Let Z be the fundamental cycle on $\pi^{-1}(o)$. It is known that $p_a(Z) = 0$ (resp. 1) implies that (V, o) is rational (resp. weakly elliptic). See [2, Theorem 3], [17, p. 443] and [12, Corollary 4.2] for the detail.

In this section, we shall show our first main result:

THEOREM 3.1. *Let F be a fibre in a relatively minimal fibred surface of genus $g \geq 2$. Then the following hold.*

(1) *When F is a non-multiple fibre, $\mathrm{Bs}|K_F|$ supports at most exceptional sets of rational singular points.*

(2) *When F is a multiple fibre*, $\mathrm{Bs}|K_F|$ *supports at most exceptional sets of rational or weakly elliptic singular points.*

This theorem has an obvious corollary.

COROLLARY 3.2. *Let F be a fibre in a relatively minimal fibred surface of genus $g \geq 2$, and let Z be an arbitrary curve with* $\mathrm{Supp}(Z) \subseteq \mathrm{Bs}|K_F|$. *Then* $\chi(Z, \mathcal{O}_Z) \geq 0$. *If F is a non-multiple fibre, then* $\chi(Z, \mathcal{O}_Z) \geq 1$.

In order to show Theorem 3.1, we need the following lemma which is a version of the main result in [8] applied to the present situation:

LEMMA 3.3. *Let D be the numerical cycle of the fibre F in a relatively minimal fibred algebraic surface of genus $g \geq 2$. Let L be a line bundle on D such that $L - K_F$ is nef and Γ a strict subcurve of D. If Γ is the fundamental cycle on its support and if the restriction map $H^0(D, L) \to H^0(\Gamma, L)$ is surjective, then $H^1(Z, \mathcal{O}_Z) = 0$ holds for any curve Z with support in* $\mathrm{Bs}|L| \cap \mathrm{Supp}(\Gamma)$.

Let $\mathcal{E} = \bigcup_{i=1}^{\nu} E_i$ be a connected bunch of canonical fixed components $E_i \preceq Z_{\mathrm{can}}$. We denote by $Z_{\mathcal{E}}$ the numerical cycle on $\mathcal{E}$. By Lemma 1.4, we have $Z_{\mathcal{E}} \preceq D$.

We shall show that $h^1(Z_{\mathcal{E}}, \mathcal{O}_{Z_{\mathcal{E}}}) \leq 1$ with several lemmas.

LEMMA 3.4. *Let $\mathcal{E} = \bigcup_i E_i$ and $Z_{\mathcal{E}}$ be as above. If each E_i is a multiple component of D, then $h^1(Z_{\mathcal{E}}, \mathcal{O}_{Z_{\mathcal{E}}}) = 0$.*

PROOF. Since D is of multiplicity one along a canonical fixed component of type (II), all the E_i's are of type (I) by the assumption. We take a loupe Δ_i for each E_i. Then, for any two loupes Δ_i and Δ_j, we have either $\mathrm{Supp}(\Delta_i) \cap \mathrm{Supp}(\Delta_j) = \emptyset$ or $\Delta_i \preceq \Delta_j$ or $\Delta_j \preceq \Delta_i$. This can be seen as follows. By Lemma 2.1, we have only to exclude the possibilities that $D = \Delta_i + \Delta_j$ and $D = \Delta_i + \Delta_j - \gcd(\Delta_i, \Delta_j)$. In both cases, however, we already know from Lemmas 2.5 and 2.6 that D should be of multiplicity one along E_i and E_j, which is forbidden. Then, since $\mathcal{E}$ is connected, we can find the biggest loupe, say Δ, in $\{\Delta_i\}$. Let E be the canonical fixed component whose loupe is Δ.

If the restriction map $H^0(D, K_F) \to H^0(\Delta, K_F)$ is surjective, then we get $h^1(Z_{\mathcal{E}}, \mathcal{O}_{Z_{\mathcal{E}}}) = 0$ by Lemma 3.3 applied to $(\Gamma, L) = (\Delta, K_F)$. This allows us to assume that F is a multiple fibre by Lemma 2.1, and that $H^0(D, K_F) \to H^0(\Delta, K_F)$ is not surjective. Then $\mathcal{O}_{D-\Delta}(D) \simeq \mathcal{O}_{D-\Delta}$ by Lemma 2.4.

Assume first that $E \subseteq \mathrm{Bs}|K_D|$. Then $\mathcal{O}_{\Delta}(D) \simeq \mathcal{O}_{\Delta}$ by Lemma 2.4. Since Δ is the biggest, we have $\Delta_i \preceq \Delta$ and it follows $\mathcal{O}_{\Delta_i}(D) \simeq \mathcal{O}_{\Delta_i}$ for any i. Again by Lemma 2.4, this implies that $E_i \subseteq \mathrm{Bs}|K_D|$. Therefore, $\mathcal{E} \subseteq \mathrm{Bs}|K_D|$. Since $H^0(D, K_D) \to H^0(\Delta, K_D)$ is surjective by Lemma 2.1, we can apply Lemma 3.3 to $(\Gamma, L) = (\Delta, K_D)$ and obtain $h^1(Z_{\mathcal{E}}, \mathcal{O}_{Z_{\mathcal{E}}}) = 0$.

Assume next that $E \not\subseteq \mathrm{Bs}|K_D|$. Then $\mathcal{O}_{\Delta}(D) \not\simeq \mathcal{O}_{\Delta}$ by Lemma 2.4. Since $\mathcal{O}_{D-\Delta}(D) \simeq \mathcal{O}_{D-\Delta}$, we cannot have $\Delta \preceq D - \Delta$. Since E is a multiple component of D by the assumption, we must have $E \preceq \gcd(\Delta, D - \Delta)$. However, Lemma 2.5 forbids it. □

LEMMA 3.5. *Let $\mathcal{E} = \bigcup_i E_i$ and $Z_{\mathcal{E}}$ be as above. Assume that one of the following conditions holds when F is a multiple fibre*:

(1) *There exists a component E_i along which D is of multiplicity one and $\mathcal{O}_{D-E_i}(D) \simeq \mathcal{O}_{D-E_i}$.*

(2) *Every E_i along which D is of multiplicity one is a fixed component of $|K_D|$.*

Then $h^1(Z_{\mathcal{E}}, \mathcal{O}_{Z_{\mathcal{E}}}) \leq 1$ with the equality holding only when F is a multiple fibre.

PROOF. Recall that $Z_{\mathcal{E}} \preceq D$. We may assume that D is of multiplicity one along some E_i's by Lemma 3.4. We denote by A the sum of all such components in $\mathcal{E}$.

Suppose first that $\mathrm{Supp}(A) = \mathcal{E}$. Then we have $Z_{\mathcal{E}} = A$ by $Z_{\mathcal{E}} \preceq D$. We know that $H^0(F, K_F) \to H^0(A, K_F)$ is the zero map. Since $H^0(F, K_F) \to H^0(D, K_F)$ is surjective and $h^0(D, K_F) \neq 0$, we see that $A \neq D$. It follows from Theorem 1.3 that we have $p_a(Z_{\mathcal{E}}) = 0$ when F is a non-multiple fibre, and $p_a(Z_{\mathcal{E}}) \leq 1$ when F is a multiple fibre. Since $h^0(Z_{\mathcal{E}}, \mathcal{O}_{Z_{\mathcal{E}}}) = 1$, we get $h^1(Z_{\mathcal{E}}, \mathcal{O}_{Z_{\mathcal{E}}}) = p_a(Z_{\mathcal{E}}) \leq 1$.

Next, suppose that $\mathrm{Supp}(A)$ is strictly smaller than $\mathcal{E}$. Then $Z_{\mathcal{E}} - A$ consists of canonical fixed components of type (I) along which D is of multiplicity at least two. Let $Z_{\mathcal{E}} - A = \Gamma_1 + \cdots + \Gamma_k$ be the decomposition into connected components. We claim that $h^1(\Gamma_i, \mathcal{O}_{\Gamma_i}) = 0$ for $i = 1, \ldots, k$. This can be seen as follows. Let Z_i be the numerical cycle on $\mathrm{Supp}(\Gamma_i)$. We have $h^1(Z_i, \mathcal{O}_{Z_i}) = 0$ by Lemma 3.4. Then we also have $h^1(\Gamma_i, \mathcal{O}_{\Gamma_i}) = 0$ by a result of Artin, since we now know that Z_i is the fundamental cycle of a rational singular point (see, [1] and [2]).

We have seen that $H^1(Z_{\mathcal{E}} - A, \mathcal{O}_{Z_{\mathcal{E}}-A}) = 0$. Let η be a non-zero section of $\mathcal{O}([D - Z_{\mathcal{E}}])$ defining $D - Z_{\mathcal{E}}$. Since $\gcd(A, D - Z_{\mathcal{E}}) = 0$, we see that $\eta|_A$ is non-zero and defines an injection $\mathcal{O}_A(-(D - Z_{\mathcal{E}})) \hookrightarrow \mathcal{O}_A$. We consider the commutative diagram

$$\begin{array}{ccc} H^0(Z_{\mathcal{E}}, K_{Z_{\mathcal{E}}}) & \longrightarrow & H^0(A, K_{Z_{\mathcal{E}}}) \\ {\scriptstyle \cdot\eta}\big\downarrow & & \big\downarrow{\scriptstyle \cdot\eta|_A} \\ H^0(D, K_D) & \longrightarrow & H^0(A, K_D)\,, \end{array} \tag{3.1}$$

where the vertical maps are injections induced by η and the horizontal maps are the restriction maps.

We first examine the case that F is a non-multiple fibre. Then $D = F$ and the map at the bottom row is the zero map by $A \preceq Z_{\mathrm{can}}$. It follows that the map at the top row is also zero. Since the kernel of $H^0(Z_{\mathcal{E}}, K_{Z_{\mathcal{E}}}) \to H^0(A, K_{Z_{\mathcal{E}}})$ is isomorphic to $H^0(Z_{\mathcal{E}} - A, K_{Z_{\mathcal{E}}-A}) \simeq H^1(Z_{\mathcal{E}} - A, \mathcal{O}_{Z_{\mathcal{E}}-A})^{\vee}$, which is zero as we saw above, we get $H^0(Z_{\mathcal{E}}, K_{Z_{\mathcal{E}}}) = 0$. By duality, $H^1(Z_{\mathcal{E}}, \mathcal{O}_{Z_{\mathcal{E}}}) = 0$.

We next consider the case that F is a multiple fibre. We shall show that the rank of the restriction map $H^0(D, K_D) \to H^0(A, K_D)$ is at most one. The assertion is obvious when (2) holds. So, we assume (1) holds. Let E be a component of A satisfying $\mathcal{O}_{D-E}(D) \simeq \mathcal{O}_{D-E}$. Then $\mathcal{O}_{D-A}(D) \simeq \mathcal{O}_{D-A}$. Since we know that $H^0(D, K_F) \to H^0(A, K_F)$ is the zero map, we have $h^0(D - A, D) = p_a(A) + A(D - A) - 1$, that is, $h^0(D - A, \mathcal{O}_{D-A}) = p_a(A) - A^2 - 1$. Since A is a strict subcurve of D, we have $h^1(A, K_D) = 0$. Then the dimension of the cokernel of $H^0(D, K_D) \to H^0(A, K_D)$ is $h^1(D - A, K_{D-A}) - h^1(D, K_D) = h^0(D - A, \mathcal{O}_{D-A}) - 1 = p_a(A) - A^2 - 2$. By the Riemann-Roch theorem, we have $h^0(A, K_D) = \deg K_A + A(D-A) + 1 - p_a(A) = p_a(A) - A^2 - 1$. This shows that the rank of $H^0(D, K_D) \to H^0(A, K_D)$ is one. Then the rank of $H^0(Z_{\mathcal{E}}, K_{Z_{\mathcal{E}}}) \to H^0(A, K_{Z_{\mathcal{E}}})$ is at most one by (3.1) and we get $h^1(Z_{\mathcal{E}}, \mathcal{O}_{Z_{\mathcal{E}}}) \le 1$ as in the previous case. □

Recall that any canonical fixed component of type (II) is as in (1) of Lemma 3.5.

LEMMA 3.6. *Let F be a multiple fibre. Let $\mathcal{E} = \bigcup E_i$ be a connected bunch of canonical fixed components of type* (I). *Assume that there exists a component E of $\mathcal{E}$ along which D is of multiplicity one, $E \not\subset \mathrm{Bs}|K_D|$ and $\mathcal{O}_{D-E}(D) \not\simeq \mathcal{O}_{D-E}$. Then $h^1(Z_{\mathcal{E}}, \mathcal{O}_{Z_{\mathcal{E}}}) \le 1$ holds for the numerical cycle $Z_{\mathcal{E}}$ on $\mathcal{E}$.*

PROOF. Let Δ be a loupe for E. Then $\mathcal{O}_\Delta(D) \not\simeq \mathcal{O}_\Delta$ by Lemma 2.4, since $E \not\subseteq \mathrm{Bs}|K_D|$. We have $\Delta \not\preceq D - \Delta$, since D is of multiplicity one along E. Assume that Δ has no common components with $D - \Delta$. Since $\mathcal{O}_\Delta(D - \Delta) \simeq \mathcal{O}_\Delta(D + p)$ for a general point $p \in E$, we see that $D - \Delta$ meets Δ at a point $q \in E$ which is a non-singular point of Δ. Then, since $\mathcal{O}_\Delta(D - \Delta) \simeq \mathcal{O}_\Delta(q)$, we get $\mathcal{O}_\Delta(D) \simeq \mathcal{O}_\Delta(q - p)$. However, we already know that p and q are linearly equivalent on Δ. So we get $\mathcal{O}_\Delta(D) \simeq \mathcal{O}_\Delta$, which is inadequate. Hence Δ has a common component with $D - \Delta$.

Let $\Delta = E + C_1 + \cdots + C_{n-1}$ be the decomposition as in (2.3). We put $C_0 = D - \Delta$. Then, for $0 \le i < j < n$, $\mathcal{O}_{C_j}(-C_i)$ is numerically trivial and it follows from Lemma 1.4 that either $\mathrm{Supp}(C_i) \cap \mathrm{Supp}(C_j) = \emptyset$ or $C_j \prec C_i$. Then C_0 is a maximal element in $\{C_i\}_{i=0}^{n-1}$ and $\mathrm{Supp}(C_0)$ is a connected component of $\mathrm{Supp}(D - E)$. This gives us a decomposition $D - E = D_1 + D_2$ with $\mathrm{Supp}(D_1) \cap \mathrm{Supp}(D_2) = \emptyset$, if D_1 denotes the sum of all the C_i's such that $C_i \preceq C_0$. By Lemma 2.5, we have $\mathcal{O}_{D_2}(D) \simeq \mathcal{O}_{D_2}$. Since D_1 is disjoint from D_2, we have $\mathcal{O}_{D_1}(D) \not\simeq \mathcal{O}_{D_1}$ by $\mathcal{O}_{D-E}(D) \not\simeq \mathcal{O}_{D-E}$.

We denote by $\{C_{i_\alpha}\}_{\alpha=1}^{l}$ the set of all maximal curves in $\{C_i\}_{i=0}^{n-1}$. Then the C_{i_α}'s are mutually disjoint and $\bigcup_{\alpha=1}^{l} \mathrm{Supp}(C_{i_\alpha})$ is nothing but the decomposition of $\mathrm{Supp}(D - E)$ into its connected components. Furthermore, we have $C_{i_\alpha} \preceq D_2$ unless $C_{i_\alpha} = C_0$. We claim that $\mathcal{O}_{D-C_{i_\alpha}}(D) \not\simeq \mathcal{O}_{D-C_{i_\alpha}}$. This can be seen as follows. If $C_{i_\alpha} = C_0 = D - \Delta$, then the assertion is nothing but the assumption $\mathcal{O}_\Delta(D) \not\simeq \mathcal{O}_\Delta$. For the other C_{i_α}, we have $D_1 \preceq D - C_{i_\alpha}$ and, hence, $\mathcal{O}_{D-C_{i_\alpha}}(D) \simeq \mathcal{O}_{D-C_{i_\alpha}}$ immediately contradicts $\mathcal{O}_{D_1}(D) \not\simeq \mathcal{O}_{D_1}$. Therefore, $\mathcal{O}_{D-C_{i_\alpha}}(D) \not\simeq \mathcal{O}_{D-C_{i_\alpha}}$ for $\alpha = 1, 2, \ldots, l$.

Now, we have $\mathcal{E} - E \subseteq \bigcup_{\alpha=1}^{l} \mathrm{Supp}(C_{i_\alpha})$, since D is of multiplicity one along E. Put $\mathcal{E}_\alpha = (\mathcal{E} - E) \bigcap \mathrm{Supp}(C_{i_\alpha})$. Note that, when $\mathcal{E}_\alpha \ne \emptyset$, $\mathcal{E}_\alpha$ is connected, since so is $\mathcal{E}$ and $EC_{i_\alpha} = 1$. Let Z_α be the numerical cycle on $\mathcal{E}_\alpha$. Since C_{i_α} is the fundamental cycle on its support, we have $Z_\alpha \preceq C_{i_\alpha}$ by Lemma 1.4. As we saw above, we have $\mathcal{O}_{D-C_{i_\alpha}}(D) \not\simeq \mathcal{O}_{D-C_{i_\alpha}}$. Since $C_{i_\alpha}^2 = -1$, we see that $D - C_{i_\alpha}$ is 1-connected. So, we can show that the restriction map $H^0(D, K_F) \to H^0(C_{i_\alpha}, K_F)$ is surjective as in Lemma 2.4. Then, by Lemma 3.3 applied to $(\Gamma, L) = (C_{i_\alpha}, K_F)$, we get $H^1(Z_\alpha, \mathcal{O}_{Z_\alpha}) = 0$ and see that Z_α is the fundamental cycle of a rational singular point. This is sufficient to imply that $H^1(Z_{\mathcal{E}} - E, \mathcal{O}_{Z_{\mathcal{E}}-E}) = 0$. Then, because the restriction map $H^0(D, K_D) \to H^0(E, K_D)$ is of rank at most one by Lemma 2.5, we can show that $h^1(Z_{\mathcal{E}}, \mathcal{O}_{Z_{\mathcal{E}}}) \le 1$ by using (3.1) as in Lemma 3.5. $\square$

PROOF OF THEOREM 3.1. We have shown that $p_a(Z_{\mathcal{E}}) = h^1(Z_{\mathcal{E}}, \mathcal{O}_{Z_{\mathcal{E}}}) \le 1$ for the numerical cycle $Z_{\mathcal{E}}$ on a given connected bunch $\mathcal{E}$ of canonical fixed components. Since $p_a(D) \ge 2$, we see that $\mathcal{E}$ is strictly smaller than $\mathrm{Supp}(D)$. Therefore, the intersection form is negative definite on $\mathcal{E}$, and we obtain a normal surface singularity by contracting $\mathcal{E}$. As we already remarked at the begining of the section, by well-known results due to Artin [2] and Wagreich [17], see also [12], rational and (weakly) elliptic singularities are characterized by their fundamental genera, that is, the arithmetic genus of the fundamental cycle. Therefore, by contracting $\mathcal{E}$, we obtain a rational singularity when $p_a(Z_{\mathcal{E}}) = 0$, and an elliptic singularity when $p_a(Z_{\mathcal{E}}) = 1$. Note that we have $p_a(Z_{\mathcal{E}}) = 1$ only when F is a multiple fibre. Hence $\mathcal{E}$ contracts to a rational singular point if F is non-multiple. $\square$

4. Further remarks on the fixed part.

We give a few comments on Z_{can} detected from the considerations in the previous sections.

LEMMA 4.1. *Let $\mathcal{E}$ be a connected bunch of canonical fixed components. If it supports an exceptional set of a rational double point, then the dual graph of $\mathcal{E}$ is of Dynkin type* A *or* D.

PROOF. We may assume that it is not of type A. Let $\mathcal{E}'$ be the subset of $\mathcal{E}$ consisting of all the multiple components of $Z_{\mathcal{E}}$. Then $\mathcal{E}'$ is connected and any component of $\mathcal{E}'$ is also a multiple component of D, since $Z_{\mathcal{E}} \prec D$. We can find a loupe Δ for some $E \subset \mathcal{E}'$ such that $\mathcal{E}' \subset \mathrm{Supp}(\Delta)$ as in the proof of Lemma 3.4. Let Z denote the fundamental cycle on $\mathcal{E}'$. We know that $Z \prec \Delta$ and Δ is of multiplicity one along E. Furthermore, we have $-1 = E\Delta = EZ + E(\Delta - Z) \ge EZ$. Then Z has to be of type A, because otherwise any non-multiple component E' of Z satisfies $ZE' = 0$ in view of the A-D-E classification. This happens only when the dual graph of $\mathcal{E}$ is of type D. $\square$

PROPOSITION 4.2. *Suppose that $|K_F|$ has a fixed component. Then the following hold.*

(1) *The numerical cycle D is not* 3*-connected. If it is* 2*-connected, then F is a multiple fibre and the canonical fixed part consists of* (-2)*-curves of type* (II) *each of whose connected component forms a Dynkin diagram of type* A.

(2) *A canonical fixed component E with $p_a(E) = 1$ is unique if exists. It is either a non-singular elliptic curve or a rational curve with a node. Furthermore, the other canonical fixed components, if exist, are of type* (I).

(3) *If a canonical fixed component of type* (II) *exists, then D is of hyperelliptic type.*

PROOF. (1) If there is a canonical fixed component of type (I), then its loupe Δ satisfies $\Delta^2 = -1$ and D is not 2-connected. So we may assume that any canonical fixed components are of type (II). Let Z be the fixed part of $|K_F|$. Then it is a reduced curve, since D is of multiplicity one along any component of type (II) and $Z \prec D$. We may assume that the support of Z is connected. Then, as we showed in the proof of Lemma 3.5, the restriction map $H^0(D, K_D) \to H^0(Z, K_D)$ is of rank one. Therefore, if there exists a component $E \preceq Z$ with $\deg K_D|_E > 0$, then $|K_D|$ should have a base point, which implies that D is not 2-connected by Theorem 1.2. This allows us to assume that $\deg K_D|_Z = 0$. Then Z consists of (-2)-curves, which in particular shows that D is not 3-connected. Since $p_a(D) \geq 2$, we have $Z^2 < 0$. Now, since the fundamental cycle on the support of Z, which is Z itself in the present case, is reduced, it must be contracted to a rational double point of type A_n for some n.

(2) Let $E \preceq Z_{\mathrm{can}}$ be a component such that $p_a(E) = 1$. Then it is of type (II) and is a non-multiple component of D. Note that the image of $H^0(D, K_F) \to H^0(E, K_F)$ contains that of $H^0(E, K_F - (D - E)) \to H^0(E, K_F)$. Hence we have $H^0(E, K_F - (D - E)) = 0$. Since $\mathcal{O}_E(K_F - (D - E)) \simeq \mathcal{O}_E(-D)$, we see that $\mathcal{O}_E(D) \not\simeq \mathcal{O}_E$. If E_1 is another canonical fixed component of type (II), then we have $\mathcal{O}_D(D) \simeq \mathcal{O}_D(q_1 - p_1)$ with two points $p_1, q_1 \in E_1$ and we cannot have $\mathcal{O}_E(D) \not\simeq \mathcal{O}_E$. Therefore, there are no other type (II) components. Furthermore, by $\mathcal{O}_E(D) \not\simeq \mathcal{O}_E$, we see that E is not simply connected. Hence E has a node if it is singular.

(3) Let E be a component of type (II). Then we can find distinct pairs of points (p, p') and (q, q') on E such that $\mathcal{O}_D(D) \simeq \mathcal{O}_D(p' - p) \simeq \mathcal{O}_D(q' - q)$. We have $\mathcal{O}_D(p + q') \simeq \mathcal{O}_D(q + p')$, which gives us a base-point-free g^1_2 on D. Hence D is of hyperelliptic type. □

We study the decomposition of Z_{can} especially when F is a multiple fibre. Recall that, by Theorem 1.5, it has the property that $p_a(Z') \leq 1$ for any subcurve $Z' \preceq Z_{\mathrm{can}}$. The following lemma can be found in [9, Lemma 5.6].

LEMMA 4.3. *Let C be a curve such that $p_a(C') \leq 1$ holds for any $0 \prec C' \preceq C$. Assume that $p_a(C) = 1$. Then C is 0-connected and decomposes as $C = \Gamma_1 + \cdots + \Gamma_n$, where each Γ_i is a chain-connected curve with $p_a(\Gamma_i) = 1$ and $\mathcal{O}_{\Gamma_j}(-\Gamma_i)$ is numerically trivial for $i < j$. In particular, $\Gamma_i\Gamma_j = 0$ and, either $\Gamma_j \preceq \Gamma_i$ or $\mathrm{Supp}(\Gamma_i) \cap \mathrm{Supp}(\Gamma_j) = \emptyset$ for $i < j$. Furthermore, $h^0(C, \mathcal{O}_C) \leq n$ with equality holding only when $\mathcal{O}_{\Gamma_i + \cdots + \Gamma_n}(-\Gamma_{i-1})$ is trivial for $2 \leq i \leq n$.*

The following can be found in [11, Lemma 1.6].

LEMMA 4.4. *Let L be a line bundle on a curve C such that $\deg L|_{C'} \geq 2p_a(C') - 2$ holds for any subcurve $C' \preceq C$. If $H^1(C, L) \neq 0$, then there exists a subcurve $\Gamma \preceq C$ such that $\mathcal{O}_\Gamma(L) \simeq \mathcal{O}_\Gamma(K_\Gamma)$ and $h^0(\Gamma, \mathcal{O}_\Gamma) = 1$.*

PROPOSITION 4.5. *Let C be a curve such that $p_a(C') \leq 1$ for any $0 \prec C' \preceq C$. If $h^1(C, \mathcal{O}_C) \neq 0$, then C decomposes as $C = C_1 + \cdots + C_k + \Gamma$, where*

(1) *C_i is a 0-connected curve with $p_a(C_i) = 1$ for $1 \leq i \leq k$,*

(2) *$\mathcal{O}_{C_j}(-C_i)$ is nef of positive degree and $C_j \preceq C_i$ for $i < j$,*

(3) *$h^1(C_1, \mathcal{O}_{C_1}) = h^1(C, \mathcal{O}_C)$, $h^1(C_i, \mathcal{O}_{C_i}) = h^1(C - \sum_{j=1}^{i-1} C_j, \mathcal{O})$ for $2 \leq i \leq k$, and*

(4) *either* $\Gamma = 0$ *or* Γ *is a curve with* $h^1(\Gamma, \mathcal{O}_\Gamma) = 0$ *and* $\mathcal{O}_\Gamma(-C_i)$ *is nef for* $1 \leq i \leq k$.

PROOF. By the assumption, we can apply Lemma 4.4 to any nef line bundle on C. By Lemma 4.4 applied to $L = \mathcal{O}_C$, there exists a subcurve of arithmetic genus one. Let C_1 be a maximal subcurve of C with $p_a(C_1) = 1$. It is 0-connected by Lemma 4.3. If $C_1 = C$, then we stop with $k = 1$ and $\Gamma = 0$. So, we assume that $C_1 \neq C$.

Take any curve $A \preceq C - C_1$. By the maximality of C_1, we have $0 \geq p_a(A + C_1) = p_a(A) + p_a(C_1) - 1 + AC_1 = p_a(A) + AC_1$. If $h^0(A, \mathcal{O}_A) = 1$, then $p_a(A) \geq 0$ and we get $AC_1 \leq 0$. In particular, we have $AC_1 \leq 0$ for any irreducible component $A \preceq C - C_1$. Therefore, $-C_1$ is nef on $C - C_1$. We also remark that $p_a(A) \leq 0$ holds when $AC_1 = 0$.

We claim that $h^1(C, \mathcal{O}_C) = h^1(C_1, \mathcal{O}_{C_1})$. To see this, consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_{C-C_1}(-C_1) \to \mathcal{O}_C \to \mathcal{O}_{C_1} \to 0 .$$

Assume that $H^1(C-C_1, -C_1) \neq 0$. It follows from Lemma 4.4 applied to $L = \mathcal{O}_{C-C_1}(-C_1)$ that there exists a curve $B \preceq C - C_1$ with $h^0(B, \mathcal{O}_B) = 1$ and $\mathcal{O}_B(-C_1) \simeq \mathcal{O}_B(K_B)$. Note that $\mathcal{O}_B(-C_1)$ is numerically trivial, because it is nef while $\deg K_B \leq 0$. It follows that $BC_1 = 0$ and $p_a(B) = 1$, which is impossible as remarked above. Therefore, $h^1(C - C_1, -C_1) = 0$ and we get $h^1(C_1, \mathcal{O}_{C_1}) = h^1(C, \mathcal{O}_C)$.

If $h^1(C-C_1, \mathcal{O}_{C-C_1}) = 0$, we stop by putting $\Gamma = C - C_1$. If $h^1(C-C_1, \mathcal{O}_{C-C_1}) \neq 0$, then we repeat the above argument with $C - C_1$ instead of C. If we let C_2 be a maximal subcurve of $C - C_1$ with $p_a(C_2) = 1$, then $-C_2$ is nef on $C - C_1 - C_2$ and $h^1(C - C_1, \mathcal{O}_{C-C_1}) = h^1(C_2, \mathcal{O}_{C_2})$. Now, by an obvious inductive argument, we can find curves C_i with $p_a(C_i) = 1$ and $h^1(C_i, \mathcal{O}_{C_i}) = h^1(C - \sum_{j=1}^{i-1} C_j, \mathcal{O})$ for $i = 1, \ldots, k$, until we get $H^1(\Gamma, \mathcal{O}_\Gamma) = 0$ for $\Gamma = C - \sum_{i=1}^{k} C_i$.

We claim that $C_j \preceq C_i$ when $i < j$. Recall that $-C_i$ is nef on C_j. But we cannot have $C_iC_j = 0$, since $p_a(C_j) = 1$. Hence $C_iC_j < 0$ and we see that C_i and C_j have a common component. Put $G = \gcd(C_i, C_j)$ and $B_i = C_i - G$, $B_j = C_j - G$. Assume that $B_j \neq 0$. We have $B_jG \geq 0$, since C_j is 0-connected. On the other hand, since $-C_i$ is nef on C_j, we get $0 \geq C_iB_j = B_iB_j + GB_j \geq GB_j$. Hence $GB_j = B_iB_j = 0$, implying that $C_iB_j = 0$. Then one should have $p_a(B_j) \leq 0$. This leads us to a contradiction, because we would have $p_a(G) \geq 2$ from $1 = p_a(C_j) = p_a(G) + p_a(B_j) - 1 + GB_j$. Therefore, $B_j = 0$ and $C_j \preceq C_i$. □

Let Z be a subcurve of Z_{can} such that $\mathcal{E} = \text{Supp}(Z)$ is connected. We denote by $Z_\mathcal{E}$ the fundamental cycle on $\mathcal{E}$. Then $Z_\mathcal{E} \prec D$. Though we do not know whether $Z_\mathcal{E}$ is a subcurve of Z, we shall show that the "essential part" of $Z_\mathcal{E}$ is in fact a subcurve of Z when $H^1(Z, \mathcal{O}_Z) \neq 0$. This can be seen as follows. Assume that $h^1(Z, \mathcal{O}_Z) \neq 0$. Let $Z = C_1 + \cdots + C_k + \Gamma$ be the decomposition as in Proposition 4.5. Then we have $p_a(C_i) = 1$ for each i. If $C_i = \Gamma_{i,1} + \cdots + \Gamma_{i,n_i}$ denotes the decomposition as in Lemma 4.3, then $\mathcal{O}_{\Gamma_{i,j}}(-Z_\mathcal{E})$ is nef. Since $\Gamma_{i,j}$ is chain-connected, we have $\Gamma_{i,j} \preceq Z_\mathcal{E}$ by Lemma 1.4. Since $1 = p_a(\Gamma_{i,j}) \leq p_a(Z_\mathcal{E}) \leq 1$, we get $p_a(\Gamma_{i,j}) = p_a(Z_\mathcal{E}) = 1$. Hence every $\Gamma_{i,j}$ contains the minimal model Z_0 of $Z_{\mathcal{E}}$ (see [9, §3]), which is the fundamental cycle of a minimally elliptic singularity (cf. [12]) in the present case. Then $Z_0 \preceq \Gamma_{i,n_i} \prec \Gamma_{i,n_i-1} \prec \cdots \prec \Gamma_{i,1} \preceq Z_{\mathcal{E}}$ for each i, $1 \le i \le k$. Recall that there exists a reduced subcurve A of $Z_{\mathcal{E}}$ along which D is of multiplicity one, $H^0(D, K_D) \to H^0(A, K_D)$ is of rank one and $H^1(Z_{\mathcal{E}} - A, \mathcal{O}) = 0$. The last condition implies that there exists an irreducible component E of A satisfying $E \preceq Z_0$. Hence, Z is of multiplicity at least $\sum_{i=1}^k n_i$ along E. Since $(\sum_{i=1}^k n_i)Z_0 \preceq Z \prec F = mD$, we have $\sum_{i=1}^k n_i \le m$ by comparing the respective multiplicities along E. Since $p_a(C_1) = 1$ and $H^0(F, K_F) \to H^0(C_1, K_F)$ is zero, it follows from Theorem 1.5 that $F - C_1$ contains a positive multiple of D. Then $n_1 Z_0 \preceq C_1 \preceq (m-1)D$, which gives us $n_1 \le m-1$. By Proposition 4.5 and Lemma 4.3, we get $h^1(Z, \mathcal{O}_Z) = h^1(C_1, \mathcal{O}_{C_1}) = h^0(C_1, \mathcal{O}_{C_1}) \le n_1 \le m-1$.

THEOREM 4.6. *Let $F = mD$ be a fibre in a relatively minimal fibred surface and Z_{can} the fixed part of $|K_F|$. Then $h^1(Z_{\mathrm{can}}, \mathcal{O}_{Z_{\mathrm{can}}}) \le m-1$. If $h^1(Z_{\mathrm{can}}, \mathcal{O}_{Z_{\mathrm{can}}}) \ne 0$, then Z_{can} contains the unique fundamental cycle Z_0 of a minimally elliptic singular point.*

PROOF. The assertion follows from what we have seen above, when $\mathrm{Supp}(Z_{\mathrm{can}})$ is connected. Suppose that it has several connected components. We let Z and Z' be connected subcurves of Z_{can} with $h^1(Z, \mathcal{O}_Z) > 0$, $h^1(Z', \mathcal{O}_{Z'}) > 0$ and $\mathrm{Supp}(Z) \cap \mathrm{Supp}(Z') = \emptyset$. Let Z_0 and Z_0' be the fundamental cycles of minimally elliptic singularities such that $Z_0 \preceq Z$ and $Z_0' \preceq Z'$. Since $Z_0 \prec D$, $Z_0' \prec D$ and $\mathrm{Supp}(Z_0) \cap \mathrm{Supp}(Z_0') = \emptyset$, we see that $Z_0 + Z_0' \prec D$. We have $p_a(Z_0 + Z_0') = p_a(Z_0) + p_a(Z_0') - 1 + Z_0 Z_0' = 1$ and $H^0(D, K_F) \to H^0(Z_0 + Z_0', K_F)$ is the zero map. By Theorem 1.3, $Z_0 + Z_0'$ must be 1-connected. This is impossible, since Z_0 and Z_0' are disjoint. Therefore, Z_{can} has at most one connected component Z with $h^1(Z, \mathcal{O}_Z) > 0$. □

Let Z_0 be as above. Recall that $-Z_0^2$ is closely related to the embedded dimension of the singularity. It is shown in [9, Proposition 6.3] that $-Z_0^2 \le (g-1)/m = p_a(D) - 1$.

5. Examples. Here, we give examples of hyperelliptic fibrations $f : S \to C$ of odd genus $g > 1$ with a double fibre F such that $\mathrm{Bs}|K_F|$ contains a particular curve, in order to see actual pictures predicted by results in the previous sections.

We shall use the following notation. Let Σ_d be the Hirzebruch surface of degree $d \ge 0$. We respectively denote by Δ_0 and Γ a minimal section and a fibre of $\Sigma_d \to \boldsymbol{P}^1$. Take a sufficiently large integer m and consider the linear system $|(2g+2)\Delta_0 + 2(m+1)\Gamma|$, where g is a positive odd integer. Fix a fibre Γ_0 and take a point $p_1 \in \Gamma_0$. We can take a reduced member $B_0 \in |(2g+2)\Delta_0 + (2m+1)\Gamma|$ such that $\Gamma_0 \cap B_0 = \{p_1\}$. Put $B = \Gamma_0 + B_0$. We assume that B is smooth except at p_1 which is a $(g+2)$-ple point. We consider the minimal resolution of the surface obtained as the double covering of Σ_d with branch locus B.

EXAMPLE 5.1. (A (-1)-elliptic curve in the fixed part.) Assume that the local analytic equation of B around p_1 is of the form

$$b(x, y) = x(x + y^2)\{(x - y^2)^g - x^g y^{2g}\},$$

where (x, y) is a system of local coordinates around p_1 such that y induces a inhomogeneous fibre coordinate on $\Gamma_0 = \{x = 0\}$.

We take an even resolution of B. Let $\sigma_1 : W_1 \to \Sigma_d$ be the blowing-up at p_1. Then we still have a $(g+2)$-ple point on the proper transform of B by σ_1. In fact, putting $x = uv$, $y = u$, we have

$$x(x+y^2)\{(x-y^2)^g - x^g y^{2g}\} = u^{g+2} \cdot v(v+u)\{(v-u)^g - v^g u^{2g}\}.$$

Since g is odd, the even transform B_1 of B is defined locally by $uv(v+u)\{(v-u)^g - v^g u^{2g}\} = 0$. Hence the point p_2 corresponding to $(u, v) = (0, 0)$ is a $(g+3)$-ple point of B_1. Let $\sigma_2 : W_2 \to W_1$ be the blowing-up at p_2. If we put $v = st$, $u = s$, then

$$uv(v+u)\{(v-u)^g - v^g u^{2g}\} = s^{g+3} \cdot t(t+1)\{(t-1)^g - t^g s^{2g}\}.$$

Hence the even transform B_2 of B_1 is given locally by $t(t+1)\{(t-1)^g - t^g s^{2g}\} = 0$, which has an infinitely near g-ple point p_3 at $(s, t) = (0, 1)$ whose local analytic equation is like $x^g = y^{2g}$. Let $\sigma_3 : W_3 \to W_2$ be the blowing-up at p_3. Since g is odd, the even transform B_3 of B_2 is given by $w(z^g - (wz+1)^g w^g) = 0$ locally over p_3, where $t - 1 = wz$, $s = w$. Hence B_3 has an ordinary $(g+1)$-ple point p_4 at $(w, z) = (0, 0)$. Let $\sigma_4 : W_4 \to W_3$ be the blowing-up at p_4. Then the even transform B_4 of B_3 becomes non-singular, which completes the even resolution of B.

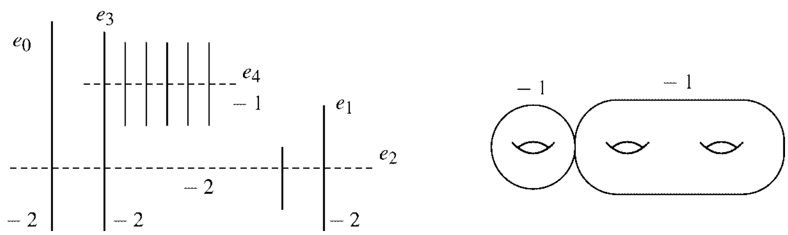

FIGURE 1. (-1)-elliptic.

Now, the double covering $\tilde{S}$ of W_4 with branch locus B_4 has a fibration $\tilde{f} : \tilde{S} \to \boldsymbol{P}^1$ of genus g induced by the ruling of Σ_d. The fibre $\tilde{F}$ of $\tilde{f}$ derived from Γ_0 is a double fibre consisting of 5 irreducible components three of which are (-1)-curves. By contracting them all, we get a relatively minimal fibration $f : S \to \boldsymbol{P}^1$ with a double fibre F consisting of two irreducible components meeting transversally at one point. One of them is a (-1)-elliptic curve and the other is a curve of genus $(g-1)/2$ with self-intersection number -1. To be more precise, we let $\hat{e}_i$ be the inverse image of p_i on W_4. Since the multiplicity sequence during the even resolution is $\{g+2, g+3, g, g+1\}$, we have

$$\begin{aligned} &K_{W_4} + \frac{1}{2}B_4 \\ &\quad \sim \sigma^*\left(K_{\Sigma_d} + \frac{1}{2}B\right) - \left(\frac{g-1}{2}\hat{e}_1 + \frac{g+1}{2}\hat{e}_2 + \frac{g-3}{2}\hat{e}_3 + \frac{g-1}{2}\hat{e}_4\right) \end{aligned}$$

$$= \sigma^* ((g-1)\Delta_0 + (m-d-g)\Gamma) + (g-1)e_0 + \frac{g-1}{2}(\hat{e}_1 - \hat{e}_4) + \frac{g-3}{2}(\hat{e}_2 - \hat{e}_3)$$
$$\sim \sigma^* ((g-1)\Delta_0 + (m-d-g)\Gamma) + (g-1)e_0 + \frac{g-1}{2}(e_1 + e_3) + (g-2)e_2$$

where $\sigma = \sigma_1 \circ \sigma_2 \circ \sigma_3 \circ \sigma_4$ and e_0 is the proper transform of Γ_0 while e_i for $i > 0$ stands for the proper transform of the exceptional (-1)-curve appeared in σ_i. We remark that e_i $(0 \le i \le 3)$ is a (-2)-curve, $e_0 + e_1 + e_3 \prec B_4$ and $e_4 = \hat{e}_4$ is a (-1)-curve. Since

$$(K_{W_4} + (1/2)B_4)e_0 = (K_{W_4} + (1/2)B_4 - e_0)e_1 = (K_{W_4} + (1/2)B_4 - e_0 - e_1)e_3$$
$$= (K_{W_4} + (1/2)B_4 - e_0 - e_1 - e_3)e_2 = -1\,,$$

we see that $e_0 + e_1 + e_2 + e_3 \subseteq \mathrm{Bs}|K_{W_4} + (1/2)B_4|$. Hence, if we put $\Delta = \sigma^*(K_{W_4} + (1/2)B_4) - e_0 - e_1 - e_2 - e_3$, then

$$|K_{\tilde{S}}| = \pi^*|K_{W_4} + (1/2)B_4| = \pi^*|\Delta| + \pi^*(e_0 + e_1 + e_2 + e_3)\,,$$

where $\pi : \tilde{S} \to W_4$ denotes the covering map.

There are irreducible curves $\tilde{E}_i$, $0 \le i \le 4$, such that $\pi^* e_i = 2\tilde{E}_i$ when $i = 0, 1, 3$, and $\pi^* e_i = \tilde{E}_i$ when $i = 2, 4$. Among them, $\tilde{E}_0$, $\tilde{E}_1$ and $\tilde{E}_3$ are (-1)-curves which should be contracted to obtain S. On the other hand, $\tilde{E}_2$ is an elliptic curve with $\tilde{E}_2^2 = -4$ and $\tilde{E}_4$ is a curve of genus $(g-1)/2$ with $\tilde{E}_4^2 = -2$. We have $\tilde{F} = 2(\tilde{E}_0 + \tilde{E}_1 + \tilde{E}_2 + 2\tilde{E}_3 + \tilde{E}_4)$. If $\rho : \tilde{S} \to S$ denotes the contraction map and $E_i = \rho_* \tilde{E}_i$ for $i = 2, 4$, then $F = 2(E_2 + E_4)$ with $E_2 E_4 = 1$ and $E_2^2 = E_4^2 = -1$. We have $\rho^* K_S \sim K_{\tilde{S}} - \tilde{E}_0 - \tilde{E}_1 - \tilde{E}_3$. Then, by what we saw above, we get $|\rho^* K_S| = \pi^*|\Delta| + \tilde{E}_0 + \tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3$. Hence the (-1)-elliptic curve E_2 is in the fixed part of $|K_S|$ as well as in that of $|K_S + f^*\mathfrak{d}|$ for any sufficiently ample divisor $\mathfrak{d}$. Since the restriction map $H^0(S, K_S + f^*\mathfrak{d}) \to H^0(F, K_F)$ is surjective, we conclude that $E_2 \subseteq \mathrm{Bs}|K_F|$. Note that $\mathrm{Bs}|K_D|$ is only one point $E_2 \cap E_4$. See, [10] for similar examples.

EXAMPLE 5.2. (A (-2)-elliptic curve in the fixed part.) Let h and h' be odd integers with $h \ge h' > 1$ and put $g = h + h' - 1$. We consider the branch locus B defined locally by

$$b(x, y) = x\{(x - \alpha_1 y^2)^h - \beta_1 x^h y^{2h}\}\{(x - \alpha_2 y^2)^{h'} - \beta_2 x^{h'} y^{2h'}\}\,,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are suitably chosen complex numbers. Then the double fibre F consists of three irreducible components, a (-2)-elliptic curve and two non-singular curves with self-intersection numbers -1 and of respective genus $(h-1)/2$ and $(h'-1)/2$. Furthermore, the elliptic curve is contained in $\mathrm{Bs}|K_F|$.

In fact, after two times of blowing-ups at p_1 with coordinates $(x, y) = (0, 0)$ and at p_2 infinitely near to p_1, we get two singular points p_3 and p_3' on the 2nd exceptional (-1)-curve. Such singular points are locally defined by $x^h = y^{2h}$ and $x^{h'} = y^{2h'}$, respectively. Since h, h' are odd, such singular points can be resolved with two times of blowing-ups, respectively. We let p_4 (resp. p_4') the singular point infinitely near to p_3 (resp. p_3'). Then they are ordinary $(h+1)$-ple and $(h'+1)$-ple points of the even transform, respectively. The multiplicity sequence is thus $\{g+2, g+3, h, h+1, h', h'+1\}$, and the contribution to $K + (1/2)B$ is the

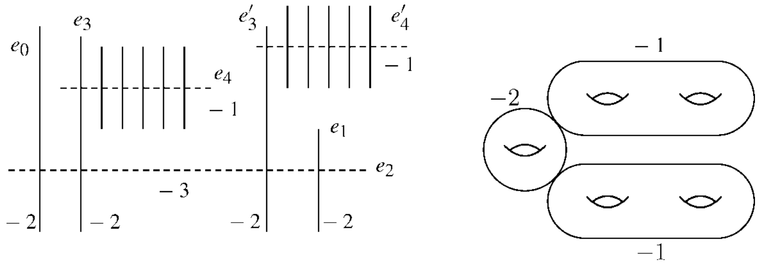

FIGURE 2. (-2)-elliptic.

minus of

$$\frac{g-1}{2}\hat{e}_1+\frac{g+1}{2}\hat{e}_2+\frac{h-3}{2}\hat{e}_3+\frac{h'-3}{2}\hat{e}_3'+\frac{h-1}{2}\hat{e}_4+\frac{h'-1}{2}\hat{e}_4'.$$

Hence, assuming $h\geq h'$, we see that $K_{\tilde{S}}$ is induced by an effective divisor of the form

$$\begin{aligned}\sigma^*((g-1)\varDelta_0+m_0\varGamma)+\frac{g+h-2}{2}e_0+\frac{h-1}{2}e_1\\ +(h-2)e_2+\frac{h-1}{2}e_3+\frac{2h-h'-1}{2}e_3'+(h-h')e_4'\end{aligned}$$

from which we know that $e_0+e_1+e_2+e_3+e_3'$ is in the fixed part. On the canonical resolution $\tilde{S}$, e_2 induces a (-6)-elliptic curve $\tilde{E}_2$ which meets four (-1)-curves $\tilde{E}_0$, $\tilde{E}_1$, $\tilde{E}_3$ and $\tilde{E}_3'$ lying respectively over e_0, e_1, e_3 and e_3'. On the relatively minimal model, we have a double fibre $F=2D$ with $D=E_2+E_4+E_4'$, where E_2 is a (-2)-elliptic curve which meets each of E_4 and E_4' transversally at a point. By what we saw above, $E_2\subseteq \mathrm{Bs}|K_F|$. On the other hand, $\mathrm{Bs}|K_D|$ consists of two points $E_2\cap E_4$, $E_2\cap E_4'$.

EXAMPLE 5.3. (2-connected numerical cycle with A_3-type fixed part.) This serves an example for Proposition 4.2, (1). Here we consider the branch locus locally defined by

$$b(x,y)=x\{(x-y^2)^{g+1}-x^{g+1}y^{2g+2}\}.$$

It has a $(g+2)$-ple point p_1 at $(x,y)=(0,0)$. After blowing-up at p_1 and p_2 infinitely near to p_1, we get a $(g+1)$-ple point p_3 on the second exceptional curve. This singular point is given locally by $x^{g+1}-y^{2g+2}=0$. After blowing-up at p_3, it results in an ordinary $(g+1)$-ple point p_4. The multiplicity sequence is thus $\{g+2, g+3, g+1, g+1\}$ and an effective expression of $K_{W_4}+B_4/2$ is of the form

$$\sigma^*((g-1)\varDelta_0+m_0\varGamma)+ge_0+\frac{g+1}{2}e_1+ge_2+\frac{g+1}{2}e_3+e_4\,,$$

where e_0 and e_i, $1\leq i\leq 3$, are (-2)-curves coming from $\varGamma$ and p_i, respectively, and e_4 is a (-1)-curve over p_4. We see that

$$\begin{aligned}(K_{W_4}+B_4/2)e_0&=(K_{W_4}+B_4/2)e_1=(K_{W_4}+B_4/2-e_0-e_1)e_2\\ &=(K_{W_4}+B_4/2-e_0-e_1-e_2)e_3=-1\,.\end{aligned}$$

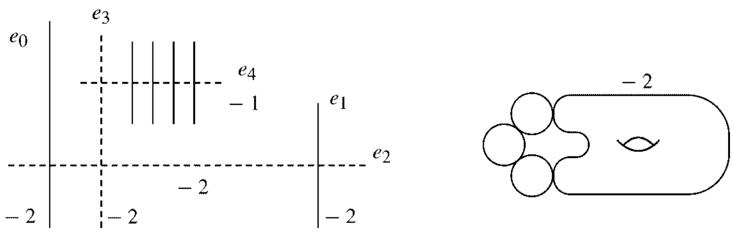

FIGURE 3. A_3 in the fixed part.

Hence $e_0 + e_1 + e_2 + e_3$ is in the fixed part of $|K_{W_4} + B_4/2|$. On the canonical resolution, e_0 and e_1 produce (-1)-curves $\tilde{E}_0$, $\tilde{E}_1$; e_2 gives us a (-4)-curve $\tilde{E}_2$ which meets $\tilde{E}_0$ and $\tilde{E}_1$; e_3 gives us two (-2)-curves $\tilde{E}_3$, $\tilde{E}_3'$ each of which meets $\tilde{E}_2$ and $\tilde{E}_4$ which is a non-singular curve coming from e_4 of genus $(g-1)/2$ with self-intersection -2. On the relatively minimal model, we have a double fibre $F = 2D$ such that $D = E_2 + E_3 + E_3' + E_4$ with $E_2E_3 = E_2E_3' = E_4E_3 = E_4E_3' = 1$ and $E_2E_4 = E_3E_3' = 0$. Then D is numerically 2-connected (and hence $\mathrm{Bs}|K_D| = \emptyset$). Here, $E_3 + E_2 + E_3'$ is a chain of (-2)-curves of type A_3 contained in $\mathrm{Bs}|K_F|$.

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