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SHESTAKOV-UMIRBAEV REDUCTIONS AND NAGATA'S CONJECTURE ON A POLYNOMIAL AUTOMORPHISM

Dedicated to the memory of Professor Masayoshi Nagata

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Abstract. In 2003, Shestakov-Umirbaev solved Nagata's conjecture on an automorphism of a polynomial ring. In the present paper, we reconstruct their theory by using the "generalized Shestakov-Umirbaev inequality", which was recently given by the author. As a consequence, we obtain a more precise tameness criterion for polynomial automorphisms. In particular, we deduce that no tame automorphism of a polynomial ring admits a reduction of type IV.

1. Introduction. Let k be a field, n a natural number, and $k[\mathbf{x}] = k[x_1, \dots, x_n]$ the polynomial ring in n variables over k. In the present paper, we discuss the structure of the automorphism group $\mathrm{Aut}_k\, k[\mathbf{x}]$ of $k[\mathbf{x}]$ over k. Let $F : k[\mathbf{x}] \to k[\mathbf{x}]$ be an endomorphism of $k[\mathbf{x}]$ over k. We identify F with the n-tuple $(f_1, \dots, f_n)$ of elements of $k[\mathbf{x}]$, where $f_i = F(x_i)$ for each i. Then, F is an automorphism of $k[\mathbf{x}]$ if and only if the k-algebra $k[\mathbf{x}]$ is generated by $f_1, \dots, f_n$. Note that the sum $\deg F := \sum_{i=1}^{n} \deg f_i$ of the total degrees of $f_1, \dots, f_n$ is at least n whenever F is an automorphism. An automorphism F is said to be *affine* if $\deg F = n$, in which case there exist $(a_{i,j})_{i,j} \in GL_n(k)$ and $(b_i)_i \in k^n$ such that $f_i = \sum_{j=1}^{n} a_{i,j} x_j + b_i$ for each i. We say that F is *elementary* if there exist $l \in \{1, \dots, n\}$ and $\phi \in k[x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_n]$ such that $f_l = x_l + \phi$ and $f_i = x_i$ for each $i \neq l$. The subgroup $\mathrm{T}_k\, k[\mathbf{x}]$ of $\mathrm{Aut}_k\, k[\mathbf{x}]$ generated by affine automorphisms and elementary automorphisms is called the *tame subgroup*, and elements of $\mathrm{T}_k\, k[\mathbf{x}]$ are called *tame automorphisms*.

It is a fundamental question in polynomial ring theory whether $\mathrm{T}_k\, k[\mathbf{x}] = \mathrm{Aut}_k\, k[\mathbf{x}]$ holds for each n. The equality is obvious if $n = 1$. It also holds true if $n = 2$, which was shown by Jung [4] in 1942 when k is a field of characteristic zero, and by van der Kulk [5] in 1953 when k is an arbitrary field. This is a consequence of the result that every automorphism but an affine automorphism of $k[\mathbf{x}]$ admits an elementary reduction if $n = 2$. Here, we say that F *admits an elementary reduction* if $\deg F \circ E < \deg F$ for some elementary automorphism E, that is, there exist $l \in \{1, \dots, n\}$ and $\phi \in k[f_1, \dots, f_{l-1}, f_{l+1}, \dots, f_n]$ such that $\deg(f_l + \phi) < \deg f_l$. In the case of $n = 2$, it follows from the result that, for each $F \in \mathrm{Aut}_k\, k[\mathbf{x}]$ with

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$\deg F > 2$, there exist elementary automorphisms $E_1, \ldots, E_r$ for some $r \in \boldsymbol{N}$ such that

$$\deg F > \deg F \circ E_1 > \cdots > \deg F \circ E_1 \circ \cdots \circ E_r = 2\,.$$

This implies that F is tame.

When $n = 3$, the structure of $\mathrm{Aut}_k\, k[\mathbf{x}]$ becomes far more difficult. In 1972, Nagata [8] conjectured that the automorphism

$$F = (x_1 - 2(x_1x_3 + x_2^2)x_2 - (x_1x_3 + x_2^2)^2x_3,\ \ x_2 + (x_1x_3 + x_2^2)x_3,\ \ x_3) \tag{1.1}$$

is not tame. This famous conjecture was finally solved in the affirmative by Shestakov-Umirbaev [10] in 2003 for a field k of characteristic zero. Therefore, $\mathrm{T}_k\, k[\mathbf{x}]$ is not equal to $\mathrm{Aut}_k\, k[\mathbf{x}]$ if $n = 3$. We note that the question remains open for $n \geq 4$.

Shestakov-Umirbaev [10] showed that, if $\deg F > 3$ for $F \in \mathrm{T}_k\, k[\mathbf{x}]$, then F admits an elementary reduction, or there exists a sequence of elementary automorphisms $E_1, \ldots, E_r$ such that $\deg F \circ E_1 \circ \cdots \circ E_r < \deg F$, where $r \in \{2, 3, 4\}$. In the latter case, F satisfies some special conditions, and is said to *admit a reduction of type I, II, III or IV* according to the conditions. Nagata's automorphism is not affine, and does not admit neither an elementary reduction nor any one of the four types of reductions. Therefore, Shestakov-Umirbaev concluded that Nagata's automorphism is not tame. We note that there exist tame automorphisms which admit reductions of type I (see [1], [7] and [10]). However, it is not known whether there exist automorphisms admitting reductions of the other types.

To prove the criterion above, Shestakov-Umirbaev [9, Theorem 3] showed an inequality concerning the total degrees of polynomials, which was used as a crucial tool. This inequality was recently generalized by the author [6]. The purpose of this paper is to reconstruct the Shestakov-Umirbaev theory using the generalized inequality. As a consequence, we obtain a more precise tameness criterion for polynomial automorphisms. In particular, we deduce that no tame automorphism of $k[\mathbf{x}]$ admits a reduction of type IV (Theorem 7.1).

The main theorem (Theorem 2.1) is formulated in Section 2 using the notion of the weighted degree of a differential form. In Section 3, we derive some consequences of the generalized Shestakov-Umirbaev inequality. In Section 4, we investigate properties of "Shestakov-Umirbaev reductions", which is roughly speaking a generalization and refinement of the notions of reductions of types I, II and III. In Section 5, we prove some technical propositions which form the core of the proof of the main theorem. The main theorem is proved in Section 6 with the aid of the results in Sections 3, 4 and 5. In Section 7, we discuss relations with the original theory of Shestakov-Umirbaev. We conclude this paper with some remarks and an appendix.

2. Main result. In what follows, we assume that the field k is of characteristic zero. Let Γ be a finitely generated totally ordered $\mathbf{Z}$-module, and $\mathbf{w} = (w_1, \ldots, w_n)$ an n-tuple of elements of Γ with $w_i > 0$ for $i = 1, \ldots, n$. Since a finitely generated totally ordered $\mathbf{Z}$-module is necessarily free, we sometimes regard Γ as a $\mathbf{Z}$-submodule of $\boldsymbol{Q} \otimes_{\mathbf{Z}} \Gamma$. We define the $\mathbf{w}$-*weighted grading* $k[\mathbf{x}] = \bigoplus_{\gamma \in \Gamma} k[\mathbf{x}]_\gamma$ by setting $k[\mathbf{x}]_\gamma$ to be the k-vector subspace of $k[\mathbf{x}]$ generated by monomials $x_1^{a_1} \cdots x_n^{a_n}$ with $\sum_{i=1}^n a_i w_i = \gamma$ for each $\gamma \in \Gamma$. For

$f \in k[\mathbf{x}] \setminus \{0\}$, we define the **w**-*degree* $\deg_{\mathbf{w}} f$ of f to be the maximum among $\gamma \in \Gamma$ with $f_\gamma \neq 0$, where $f_\gamma \in k[\mathbf{x}]_\gamma$ for each γ such that $f = \sum_{\gamma \in \Gamma} f_\gamma$. We define $f^{\mathbf{w}} = f_\delta$, where $\delta = \deg_{\mathbf{w}} f$. In case $f = 0$, we set $\deg_{\mathbf{w}} f = -\infty$, i.e., a symbol which is less than any element of Γ. For example, if $\Gamma = \boldsymbol{Z}$ and $w_i = 1$ for $i = 1, \ldots, n$, then the **w**-degree is the same as the total degree. For each k-vector subspace V of $k[\mathbf{x}]$, we define $V^{\mathbf{w}}$ to be the k-vector subspace of $k[\mathbf{x}]$ generated by $\{f^{\mathbf{w}}; f \in V \setminus \{0\}\}$. For each l-tuple $F = (f_1, \ldots, f_l)$ of elements of $k[\mathbf{x}]$ for $l \in \boldsymbol{N}$, we define $\deg_{\mathbf{w}} F = \sum_{i=1}^{l} \deg_{\mathbf{w}} f_i$. For each $\sigma \in \mathfrak{S}_l$, we define $F_\sigma = (f_{\sigma(1)}, \ldots, f_{\sigma(l)})$, where $\mathfrak{S}_l$ is the symmetric group of $\{1, \ldots, l\}$. The identity permutation is denoted by id. For distinct $i_1, \ldots, i_r \in \{1, \ldots, l\}$, the cyclic permutation with $i_1 \mapsto i_2, i_2 \mapsto i_3, \ldots, i_r \mapsto i_1$ is denoted by $(i_1, \ldots, i_r)$.

The **w**-degree of a differential form was defined by the author [6]. Let $\Omega_{k[\mathbf{x}]/k}$ be the module of differentials of $k[\mathbf{x}]$ over k, and $\bigwedge^l \Omega_{k[\mathbf{x}]/k}$ the l-th exterior power of the $k[\mathbf{x}]$-module $\Omega_{k[\mathbf{x}]/k}$ for $l \in \boldsymbol{N}$. Then, we may uniquely express each $\omega \in \bigwedge^l \Omega_{k[\mathbf{x}]/k}$ as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_l \leq n} f_{i_1, \ldots, i_l} dx_{i_1} \wedge \cdots \wedge dx_{i_l} ,$$

where $f_{i_1, \ldots, i_l} \in k[\mathbf{x}]$ for each $i_1, \ldots, i_l$. Here, df denotes the differential of f for each $f \in k[\mathbf{x}]$. We define the **w**-degree of ω by

$$\deg_{\mathbf{w}} \omega = \max\{\deg_{\mathbf{w}} f_{i_1, \ldots, i_l} x_{i_1} \cdots x_{i_l}; 1 \leq i_1 < \cdots < i_l \leq n\} .$$

By the assumption that $\omega_i > 0$ for $i = 1, \ldots, n$, it follows that

$$\deg_{\mathbf{w}} \omega \geq \min\{w_{i_1} + \cdots + w_{i_l}; 1 \leq i_1 < \cdots < i_l \leq n\} > 0 \tag{2.1}$$

if $\omega \neq 0$. For each $f \in k[\mathbf{x}] \setminus k$, we have

$$\deg_{\mathbf{w}} df = \max\{\deg_{\mathbf{w}} f_{x_i} x_i; i = 1, \ldots, n\} = \deg_{\mathbf{w}} f , \tag{2.2}$$

since $df = \sum_{i=1}^{n} f_{x_i} dx_i$. Here, f_{x_i} denotes the partial derivative of f in x_i for each $f \in k[\mathbf{x}]$ and $i \in \{1, \ldots, n\}$. We remark that $df_1 \wedge \cdots \wedge df_l \neq 0$ if and only if $f_1, \ldots, f_l$ are algebraically independent over k for $f_1, \ldots, f_l \in k[\mathbf{x}]$ (cf. [3, Proposition 1.2.9]). By definition, it follows that

$$\deg_{\mathbf{w}} df_1 \wedge \cdots \wedge df_l \leq \sum_{i=1}^{l} \deg_{\mathbf{w}} df_i = \sum_{i=1}^{l} \deg_{\mathbf{w}} f_i . \tag{2.3}$$

In (2.3), the equality holds if and only if $f_1^{\mathbf{w}}, \ldots, f_l^{\mathbf{w}}$ are algebraically independent over k. Actually, we can write $df_1 \wedge \cdots \wedge df_l = df_1^{\mathbf{w}} \wedge \cdots \wedge df_l^{\mathbf{w}} + \eta$, where $\eta \in \bigwedge^l \Omega_{k[\mathbf{x}]/k}$ with $\deg_{\mathbf{w}} \eta < \sum_{i=1}^{l} \deg_{\mathbf{w}} df_i$, and $\deg_{\mathbf{w}} df_1^{\mathbf{w}} \wedge \cdots \wedge df_l^{\mathbf{w}} = \sum_{i=1}^{l} \deg_{\mathbf{w}} df_i$ if $df_1^{\mathbf{w}} \wedge \cdots \wedge df_l^{\mathbf{w}} \neq 0$. Therefore, if $f_1, \ldots, f_n \in k[\mathbf{x}]$ are algebraically independent over k, then

$$\sum_{i=1}^{n} \deg_{\mathbf{w}} f_i = \sum_{i=1}^{n} \deg_{\mathbf{w}} df_i \geq \deg_{\mathbf{w}} df_1 \wedge \cdots \wedge df_n \geq \sum_{i=1}^{n} w_i =: |\mathbf{w}| \tag{2.4}$$

by (2.1), (2.2) and (2.3). As will be shown in Lemma 6.1(i), F is tame if $\deg_{\mathbf{w}} F = |\mathbf{w}|$ for $F \in \mathrm{Aut}_k\, k[\mathbf{x}]$.

Now, assume that $n \geq 3$, and let $\mathcal{T}$ be the set of triples $F = (f_1, f_2, f_3)$ of elements of $k[\mathbf{x}]$ such that f_1, f_2 and f_3 are algebraically independent over k. We identify each $F \in \mathcal{T}$ with the injective homomorphism $F : k[\mathbf{y}] \to k[\mathbf{x}]$ of k-algebras defined by $F(y_i) = f_i$ for $i = 1, 2, 3$, where $k[\mathbf{y}] = k[y_1, y_2, y_3]$ is the polynomial ring in three variables over k. In the case where $n = 3$, we identify $k[\mathbf{y}]$ with $k[\mathbf{x}]$ by the identification $y_i = x_i$ for each i. Let $\mathcal{E}_i$ denote the set of elementary automorphisms E of $k[\mathbf{y}]$ such that $E(y_j) = y_j$ for each $j \neq i$ for $i \in \{1, 2, 3\}$, and set $\mathcal{E} = \bigcup_{i=1}^{3} \mathcal{E}_i$. We say that $F \in \mathcal{T}$ *admits an elementary reduction* for the weight $\mathbf{w}$ if $\deg_{\mathbf{w}} F \circ E < \deg_{\mathbf{w}} F$ for some $E \in \mathcal{E}$, and call $F \circ E$ an *elementary reduction* of F for the weight $\mathbf{w}$.

Let $F = (f_1, f_2, f_3)$ and $G = (g_1, g_2, g_3)$ be elements of $\mathcal{T}$. We say that the pair (F, G) satisfies the *Shestakov-Umirbaev condition* for the weight $\mathbf{w}$ if the following conditions hold:
(SU1) $g_1 = f_1 + af_3^2 + cf_3$ and $g_2 = f_2 + bf_3$ for some $a, b, c \in k$, and $g_3 - f_3$ belongs to $k[g_1, g_2]$,
(SU2) $\deg_{\mathbf{w}} f_1 \leq \deg_{\mathbf{w}} g_1$ and $\deg_{\mathbf{w}} f_2 = \deg_{\mathbf{w}} g_2$,
(SU3) $(g_1^{\mathbf{w}})^2 \approx (g_2^{\mathbf{w}})^s$ for some odd number $s \geq 3$,
(SU4) $\deg_{\mathbf{w}} f_3 \leq \deg_{\mathbf{w}} g_1$, and $f_3^{\mathbf{w}}$ does not belong to $k[g_1^{\mathbf{w}}, g_2^{\mathbf{w}}]$,
(SU5) $\deg_{\mathbf{w}} g_3 < \deg_{\mathbf{w}} f_3$,
(SU6) $\deg_{\mathbf{w}} g_3 < \deg_{\mathbf{w}} g_1 - \deg_{\mathbf{w}} g_2 + \deg_{\mathbf{w}} dg_1 \wedge dg_2$.

Here, $h_1 \approx h_2$ (resp. $h_1 \not\approx h_2$) denotes that h_1 and h_2 are linearly dependent (resp. linearly independent) over k for $h_1, h_2 \in k[\mathbf{x}] \setminus \{0\}$. We say that $F \in \mathcal{T}$ *admits a Shestakov-Umirbaev reduction* for the weight $\mathbf{w}$ if there exist $G \in \mathcal{T}$ and $\sigma \in \mathfrak{S}_3$ such that (F_σ, G_σ) satisfies the Shestakov-Umirbaev condition, and call this G a *Shestakov-Umirbaev reduction* of F for the weight $\mathbf{w}$. As will be discussed in Section 4, F and G have various properties when (F, G) satisfies the Shestakov-Umirbaev condition. For example, it follows from (SU1) through (SU6) that $\deg_{\mathbf{w}} G < \deg_{\mathbf{w}} F$ (Property (P6)).

Here is the main theorem.

THEOREM 2.1. *Assume that $n = 3$, and $\mathbf{w} = (w_1, w_2, w_3)$ is a triple of elements of Γ with $w_i > 0$ for $i = 1, 2, 3$. If $\deg_{\mathbf{w}} F > |\mathbf{w}|$ for a tame automorphism F of $k[\mathbf{x}]$, then F admits an elementary reduction for the weight $\mathbf{w}$ or a Shestakov-Umirbaev reduction for the weight $\mathbf{w}$.*

In the case where $n = 3$ and $\Gamma = \mathbf{Z}$, the condition that F admits a Shestakov-Umirbaev reduction for the weight $\mathbf{w} = (1, 1, 1)$ implies that F admits an elementary reduction or a reduction of one of the types I, II and III (Proposition 7.2). In view of this, the reader who is familiar with the theory of Shestakov-Umirbaev may notice that no tame automorphism of $k[\mathbf{x}]$ admits a reduction of type IV (Theorem 7.1). In fact, if F admits a reduction of type IV, then there exists an elementary automorphism E such that $F \circ E$ admits a reduction of type IV, but does not admit an elementary reduction nor any of the reductions of types I, II and III (cf. Appendix). In Section 7, however, we prove this result more directly.

We remark that F admits an elementary reduction for the weight $\mathbf{w}$ if and only if $f_i^{\mathbf{w}}$ belongs to $k[f_j, f_l]^{\mathbf{w}}$ for some $i \in \{1, 2, 3\}$, where $j, l \in \{1, 2, 3\} \setminus \{i\}$ with $j < l$. In the case where $\deg_{\mathbf{w}} f_1$, $\deg_{\mathbf{w}} f_2$ and $\deg_{\mathbf{w}} f_3$ are pairwise linearly independent over $\boldsymbol{Z}$, this condition implies that $\deg_{\mathbf{w}} f_i$ belongs to the subsemigroup of Γ generated by $\deg_{\mathbf{w}} f_j$ and $\deg_{\mathbf{w}} f_l$. Indeed, each $\phi \in k[f_j, f_l] \setminus \{0\}$ is a linear combination of $f_j^p f_l^q$ for $(p, q) \in (\boldsymbol{Z}_{\geq 0})^2$ over k, in which $\deg_{\mathbf{w}} f_j^p f_l^q \neq \deg_{\mathbf{w}} f_j^{p'} f_l^{q'}$ if and only if $(p, q) \neq (p', q')$. Here, $\boldsymbol{Z}_{\geq 0}$ denotes the set of nonnegative integers. Hence, $\deg_{\mathbf{w}} \phi = \deg_{\mathbf{w}} f_j^p f_l^q = p \deg_{\mathbf{w}} f_i + q \deg_{\mathbf{w}} f_l$ for some $p, q \in \boldsymbol{Z}_{\geq 0}$.

Note that $\delta := (1/2) \deg_{\mathbf{w}} f_2 = (1/2) \deg_{\mathbf{w}} g_2$ belongs to Γ by (SU2) and (SU3). As will be shown in Section 4, (SU1) through (SU6) imply that $\delta < \deg_{\mathbf{w}} f_i \leq s\delta$ for each $i \in \{1, 2, 3\}$ (Property (P7)). Since $\delta > 0$, it follows that $\deg_{\mathbf{w}} f_i < s \deg_{\mathbf{w}} f_j$ for each $i, j \in \{1, 2, 3\}$. Therefore, if F admits a Shestakov-Umirbaev reduction for the weight $\mathbf{w}$, then F satisfies the following conditions:

(i) One of $(1/2) \deg_{\mathbf{w}} f_1$, $(1/2) \deg_{\mathbf{w}} f_2$ and $(1/2) \deg_{\mathbf{w}} f_3$ belongs to Γ.

(ii) For each $i, j \in \{1, 2, 3\}$, there exists $l \in \boldsymbol{N}$ such that $\deg_{\mathbf{w}} f_i < l \deg_{\mathbf{w}} f_j$.

Now, we deduce that Nagata's automorphism is not tame by means of Theorem 2.1. Let $\Gamma = \boldsymbol{Z}^3$ equipped with the lexicographic order, i.e., the ordering defined by $a \leq b$ for $a, b \in \boldsymbol{Z}^3$ if the first nonzero component of $b - a$ is positive, and let $\mathbf{w} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, where $\mathbf{e}_i$ is the i-th standard unit vector of $\boldsymbol{R}^3$ for each i. Then, we have

$$\deg_{\mathbf{w}} f_1 = (2, 0, 3), \quad \deg_{\mathbf{w}} f_2 = (1, 0, 2), \quad \deg_{\mathbf{w}} f_3 = (0, 0, 1).$$

Hence, $\deg_{\mathbf{w}} F = (3, 0, 6) > (1, 1, 1) = |\mathbf{w}|$. The three vectors above are pairwise linearly independent over $\boldsymbol{Z}$, while any one of them is not contained in the subsemigroup of $\boldsymbol{Z}^3$ generated by the other two vectors. Hence, F does not admit an elementary reduction for the weight $\mathbf{w}$. Since $(1/2) \deg_{\mathbf{w}} f_i$ does not belong to $\boldsymbol{Z}^3$ for each $i \in \{1, 2, 3\}$, we know that F does not admit a Shestakov-Umirbaev reduction for the weight $\mathbf{w}$. By the definition of the lexicographic order, $l \deg_{\mathbf{w}} f_3 = (0, 0, l)$ is less than $\deg_{\mathbf{w}} f_i$ for $i = 1, 2$ for any $l \in \boldsymbol{N}$, which also implies that F does not admit a Shestakov-Umirbaev reduction for the weight $\mathbf{w}$. Therefore, we have the following corollary to Theorem 2.1.

COROLLARY 2.2. *Nagata's automorphism defined in* (1.1) *is not tame.*

We define the *rank* of $\mathbf{w}$ as the rank of the $\boldsymbol{Z}$-submodule of Γ generated by $w_1, \dots, w_n$. If $\operatorname{rank} \mathbf{w} = n$, then the dimension of the k-vector space $k[\mathbf{x}]_\gamma$ is at most one for each γ. Consequently, $\deg_{\mathbf{w}} f = \deg_{\mathbf{w}} g$ if and only if $f^{\mathbf{w}} \approx g^{\mathbf{w}}$ for each $f, g \in k[\mathbf{x}] \setminus \{0\}$. In such a case, the assertion of Theorem 2.1 can be proved more easily than the general case. In fact, a few steps can be skipped in the proof. We note that $\mathbf{w} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ has the maximal rank three, and therefore it suffices to prove the assertion of Theorem 2.1 in this special case to conclude that Nagata's automorphism is not tame.

3. Inequalities. In this section, we derive some consequences from the generalized Shestakov-Umirbaev inequality [6, Theorem 2.1]. In what follows, we denote "$\deg_{\mathbf{w}}$" by "deg" for the sake of simplicity. Let g be a nonzero element of $k[\mathbf{x}]$, and $\Phi = \sum_i \phi_i y^i$ a nonzero polynomial in a variable y over $k[\mathbf{x}]$, where $\phi_i \in k[\mathbf{x}]$ for each $i \in \boldsymbol{Z}_{\geq 0}$. We define

$\deg_{\mathbf{w}}^{g} \Phi$ to be the maximum among $\deg \phi_i g^i$ for $i \in \mathbf{Z}_{\geq 0}$. Then, $\deg_{\mathbf{w}}^{g} \Phi$ is not less than the $\mathbf{w}$-degree of $\Phi(g) := \sum_i \phi_i g^i$ in general. On the other hand, $\deg_{\mathbf{w}}^{g} \Phi^{(i)} = \deg \Phi^{(i)}(g)$ holds for sufficiently large i, where $\Phi^{(i)}$ denotes the i-th order derivative of Φ in y. We define $m_{\mathbf{w}}^{g}(\Phi)$ to be the minimal $i \in \mathbf{Z}_{\geq 0}$ such that $\deg_{\mathbf{w}}^{g} \Phi^{(i)} = \deg \Phi^{(i)}(g)$.

In the notation above, the generalized Shestakov-Umirbaev inequality is stated as follows. This inequality plays an important role in our theory, yet the proof is quite simple and short.

THEOREM 3.1 ([6, Theorem 2.1]). *Assume that* $f_1, \dots, f_r \in k[\mathbf{x}]$ *are algebraically independent over* k, *where* $1 \leq r \leq n$. *Then,*

$$\deg \Phi(g) \geq \deg_{\mathbf{w}}^{g} \Phi + m_{\mathbf{w}}^{g}(\Phi)(\deg \omega \wedge dg - \deg \omega - \deg g)$$

holds for each $\Phi \in k[f_1, \dots, f_r][y] \setminus \{0\}$ *and* $g \in k[\mathbf{x}] \setminus \{0\}$, *where* $\omega = df_1 \wedge \cdots \wedge df_r$.

Here is a remark (see [6, Section 3] for detail). Define $\Phi^{\mathbf{w},g} = \sum_{i \in I} \phi_i^{\mathbf{w}} y^i$ for each $\Phi \in k[\mathbf{x}][y]$, where I is the set of $i \in \mathbf{Z}_{\geq 0}$ such that $\deg \phi_i g^i = \deg_{\mathbf{w}}^{g} \Phi$. Then, $(\Phi^{(i)})^{\mathbf{w},g} = (\Phi^{\mathbf{w},g})^{(i)}$ holds for each i. Moreover, $\deg_{\mathbf{w}}^{g} \Phi = \deg \Phi(g)$ if and only if $\Phi^{\mathbf{w},g}(g^{\mathbf{w}}) \neq 0$. Hence, $m_{\mathbf{w}}^{g}(\Phi)$ is equal to the minimal $i \in \mathbf{Z}_{\geq 0}$ such that $(\Phi^{\mathbf{w},g})^{(i)}(g^{\mathbf{w}}) \neq 0$. Since k is of characteristic zero, this implies that $g^{\mathbf{w}}$ is a multiple roof of $\Phi^{\mathbf{w},g}$ of order $m_{\mathbf{w}}^{g}(\Phi)$.

Now, let $S = \{f, g\}$ be a subset of $k[\mathbf{x}]$ such that f and g are algebraically independent over k, and ϕ a nonzero element of $k[S]$. We can uniquely express $\phi = \sum_{i,j} c_{i,j} f^i g^j$, where $c_{i,j} \in k$ for each $i, j \in \mathbf{Z}_{\geq 0}$. We define $\deg^S \phi$ to be the maximum among $\deg f^i g^j$ for $i, j \in \mathbf{Z}_{\geq 0}$ with $c_{i,j} \neq 0$. We will frequently use the fact that, if $\phi^{\mathbf{w}}$ does not belong to $k[f^{\mathbf{w}}, g^{\mathbf{w}}]$, or if $\deg \phi < \deg f$ and ϕ does not belong to $k[g]$, then $\deg \phi < \deg^S \phi$.

The following lemma is a consequence of Theorem 3.1. The statement (i) is an analogue of Shestakov-Umirbaev [10, Corollary 1], while the statement (ii) is essentially new.

LEMMA 3.2. *Let* $S = \{f, g\}$ *be as above, and* ϕ *a nonzero element of* $k[S]$ *such that* $\deg \phi < \deg^S \phi$. *Then, there exist* $p, q \in \mathbf{N}$ *with* $\gcd(p, q) = 1$ *such that* $(g^{\mathbf{w}})^p \approx (f^{\mathbf{w}})^q$. *Furthermore, the following assertions hold:*

(i) $\deg \phi \geq q \deg f + \deg df \wedge dg - \deg f - \deg g$.

(ii) *Let* h *be an element of* $k[\mathbf{x}]$ *such that* f, g *and* h *are algebraically independent over* k. *If* $\deg(h + \phi) < \deg h$, *then*

$$\deg(h + \phi) \geq q \deg f + \deg df \wedge dg \wedge dh - \deg df \wedge dh - \deg g\,.$$

PROOF. Let $\Phi = \sum_{i,j} c_{i,j} f^i y^j$ be an element of $k[f][y]$ such that $\Phi(g) = \phi$, where $c_{i,j} \in k$ for each $i, j \in \mathbf{Z}_{\geq 0}$, and let J be the set of $(i, j) \in (\mathbf{Z}_{\geq 0})^2$ such that $c_{i,j} \neq 0$ and $\deg f^i g^j = \deg^S \phi$. Then, we have $\deg_{\mathbf{w}}^{g} \Phi = \deg^S \phi$ and

$$\Phi^{\mathbf{w},g} = \sum_{(i,j) \in J} c_{i,j} (f^{\mathbf{w}})^i y^j\,.$$

Since $\deg \phi < \deg^S \phi$ by assumption, we get $\deg \Phi(g) < \deg_{\mathbf{w}}^{g} \Phi$. Hence, $m_{\mathbf{w}}^{g}(\Phi) \geq 1$ and $\Phi^{\mathbf{w},g}(g^{\mathbf{w}}) = 0$ as mentioned. In particular, J contains at least two elements, say (i, j) and

(i', j'), since $\Phi^{\mathbf{w},g} \neq 0$, $g^{\mathbf{w}} \neq 0$ and $\Phi^{\mathbf{w},g}(g^{\mathbf{w}}) = 0$. Then, $(i - i') \deg f = (j' - j) \deg g$. Since $\deg f > 0$ and $\deg g > 0$, this implies that $q \deg f = p \deg g$ for some $p, q \in \boldsymbol{N}$ with $\gcd(p, q) = 1$. For each $(i_1, j_1), (i_2, j_2) \in J$, there exists $l \in \boldsymbol{Z}$ such that $i_2 - i_1 = -ql$ and $j_2 - j_1 = pl$. Hence, we can find $(i_0, j_0) \in J$ and $m \in \boldsymbol{N}$ such that J is contained in $\{(i_0 - ql, j_0 + pl); l = 0, \ldots, m\}$, and $(i_0 - qm, j_0 + pm)$ belongs to J. Note that $qm \leq i_0$. Putting $c'_l = c_{i_0 - ql, j_0 + pl}$ for each l, we get

$$\Phi^{\mathbf{w},g} = (f^{\mathbf{w}})^{i_0} y^{j_0} \sum_{l=0}^{m} c'_l (f^{\mathbf{w}})^{-ql} y^{pl} = c'_m (f^{\mathbf{w}})^{i_0} y^{j_0} \prod_{l=1}^{m} ((f^{\mathbf{w}})^{-q} y^p - \alpha_l)\,,$$

where $\alpha_1, \ldots, \alpha_m$ are the roots of the equation $\sum_{l=0}^{m} c'_l y^l = 0$ in an algebraic closure of k. Since $\Phi^{\mathbf{w},g}(g^{\mathbf{w}}) = 0$, we get $(f^{\mathbf{w}})^{-q}(g^{\mathbf{w}})^p = \alpha_l$ for some l. Then, α_l belongs to $k \setminus \{0\}$, because $f^{\mathbf{w}}$ and $g^{\mathbf{w}}$ are in $k[\mathbf{x}] \setminus \{0\}$. Therefore, $(g^{\mathbf{w}})^p \approx (f^{\mathbf{w}})^q$. This proves the first assertion. By the expression above, we know that $\Phi^{\mathbf{w},g}$ cannot have a multiple root of order greater than m. Hence, $m_{\mathbf{w}}^g(\Phi) \leq m$. Thus, it follows that

$$\deg_{\mathbf{w}}^g \Phi = \deg^S \phi = \deg f^{i_0} g^{j_0} \geq i_0 \deg f \geq qm \deg f \geq qm_{\mathbf{w}}^g(\Phi) \deg f. \tag{3.1}$$

By Theorem 3.1, together with (2.2) and (3.1), we get

$$\begin{aligned} \deg \phi = \deg \Phi(g) &\geq \deg_{\mathbf{w}}^g \Phi + m_{\mathbf{w}}^g(\Phi)(\deg df \wedge dg - \deg f - \deg g) \\ &\geq qm_{\mathbf{w}}^g(\Phi) \deg f + m_{\mathbf{w}}^g(\Phi)(\deg df \wedge dg - \deg f - \deg g) \geq m_{\mathbf{w}}^g(\Phi) M\,, \end{aligned}$$

where $M = q \deg f + \deg df \wedge dg - \deg f - \deg g$. Since $m_{\mathbf{w}}^g(\Phi) \geq 1$, the assertion (i) follows from the inequality above if $M > 0$. If $M \leq 0$, then (i) is clear, since $\deg \phi \geq 0$.

To show (ii), consider the polynomial $\Psi := h + \Phi$ in y over $k[f, h]$. Recall that $\deg \phi < \deg^S \phi = \deg_{\mathbf{w}}^g \Phi$. By the assumption that $\deg(h + \phi) < \deg h$, we get $\deg \phi = \deg h$. Hence, $\deg h < \deg_{\mathbf{w}}^g \Phi$. Thus, we have $\deg_{\mathbf{w}}^g \Psi = \deg_{\mathbf{w}}^g \Phi$ and $\Psi^{\mathbf{w},g} = \Phi^{\mathbf{w},g}$, and so $m_{\mathbf{w}}^g(\Psi) = m_{\mathbf{w}}^g(\Phi)$. Therefore, $\deg_{\mathbf{w}}^g \Psi \geq qm_{\mathbf{w}}^g(\Psi) \deg f$ by (3.1). By Theorem 3.1, we obtain

$$\begin{aligned} &\deg(h + \phi) = \deg \Psi(g) \\ &\quad \geq \deg_{\mathbf{w}}^g \Psi + m_{\mathbf{w}}^g(\Psi) M' \geq qm_{\mathbf{w}}^g(\Psi) \deg f + m_{\mathbf{w}}^g(\Psi) M' \geq m_{\mathbf{w}}^g(\Psi)(q \deg f + M')\,, \end{aligned}$$

where $M' = \deg df \wedge dh \wedge dg - \deg df \wedge dh - \deg g$. Since $m_{\mathbf{w}}^g(\Psi) = m_{\mathbf{w}}^g(\Phi) \geq 1$, the inequality above implies the inequality in (ii). □

Let p and q be natural numbers such that $\gcd(p, q) = 1$ and $2 \leq p < q$. The following assertions are checked easily.

(1) $pq - p - q > 0$.
(2) If $pq - p - q \leq q$, then $p = 2$ and $q \geq 3$ is an odd number.
(3) If $pq - p - q \leq p$, then $p = 2$ and $q = 3$.

LEMMA 3.3. *Let f, g, ϕ and p, q be as in Lemma 3.2.*
(i) *Assume that $f^{\mathbf{w}}$ does not belong to $k[g^{\mathbf{w}}]$, and $g^{\mathbf{w}}$ does not belong to $k[f^{\mathbf{w}}]$. Then,* $\deg \phi > \deg df \wedge dg$.

(ii) *Assume that* $\deg f < \deg g$, $\deg\phi \leq \deg g$, *and* $g^{\mathbf{w}}$ *does not belong to* $k[f^{\mathbf{w}}]$. *Then,* $p = 2$, $q \geq 3$ *is an odd number,* $\delta := (1/2)\deg f$ *belongs to* Γ, *and*

$$\deg\phi \geq (q-2)\delta + \deg df \wedge dg\,, \quad \deg d\phi \wedge df \geq q\delta + \deg df \wedge dg\,. \tag{3.2}$$

If $\deg\phi \leq \deg f$, *then* $q = 3$.

PROOF. Since $p\deg g = q\deg f$ and $\gcd(p, q) = 1$, it follows that $\delta := (1/p)\deg f$ belongs to Γ. By Lemma 3.2(i), we have

$$\deg\phi \geq p\deg g + \deg df \wedge dg - \deg f - \deg g = (pq - p - q)\delta + \deg df \wedge dg\,. \tag{3.3}$$

Since $(g^{\mathbf{w}})^p \approx (f^{\mathbf{w}})^q$ and $\gcd(p, q) = 1$, the assumptions of (i) imply $2 \leq p < q$ or $2 \leq q < p$. Hence, $pq - p - q > 0$ as claimed above. Therefore, $\deg\phi > \deg df \wedge dg$ by (3.3), proving (i).

If the assumptions of (ii) are satisfied, we have $2 \leq p < q$, since $g^{\mathbf{w}}$ does not belong to $k[f^{\mathbf{w}}]$. Since $\deg\phi \leq \deg g = q\delta$ by assumption, (3.3) yields $pq - p - q < q$. Thus, $p = 2$, and $q \geq 3$ is an odd number by the claim. By substituting $p = 2$, we obtain from (3.3) the first inequality of (3.2). To show the second inequality of (3.2), consider $\Phi \in k[f][y]$ defined in the proof of Lemma 3.2. Recall that $m_{\mathbf{w}}^g(\Phi) \geq 1$, and $pm_{\mathbf{w}}^g(\Phi)\deg g = qm_{\mathbf{w}}^g(\Phi)\deg f \leq \deg_{\mathbf{w}}^g \Phi$ by (3.1). By definition, $\deg_{\mathbf{w}}^g \Phi^{(1)} = \deg_{\mathbf{w}}^g \Phi - \deg g$ and $m_{\mathbf{w}}^g(\Phi^{(1)}) = m_{\mathbf{w}}^g(\Phi) - 1$. Since $p = 2$ and $\deg f < \deg g$, it follows from Theorem 2.1 that

$$\begin{aligned}
\deg\Phi^{(1)}(g) &\geq \deg_{\mathbf{w}}^g \Phi^{(1)} + m_{\mathbf{w}}^g(\Phi^{(1)})M'' \\
&= \deg_{\mathbf{w}}^g \Phi - \deg g + (m_{\mathbf{w}}^g(\Phi) - 1)M'' \\
&\geq 2m_{\mathbf{w}}^g(\Phi)\deg g - \deg g + (m_{\mathbf{w}}^g(\Phi) - 1)M'' \\
&= (m_{\mathbf{w}}^g(\Phi) - 1)(\deg df \wedge dg - \deg f + \deg g) + \deg g \\
&\geq \deg g = q\delta\,,
\end{aligned}$$

where $M'' = \deg df \wedge dg - \deg f - \deg g$. Since $d\phi = (\sum_{i,j} c_{i,j} i f^{i-1} g^j)df + \Phi^{(1)}(g)dg$, we have $d\phi \wedge df = \Phi^{(1)}(g)dg \wedge df$. Therefore,

$$\deg d\phi \wedge df = \deg\Phi^{(1)}(g) + \deg df \wedge dg \geq q\delta + \deg df \wedge dg\,.$$

This proves the second inequality of (3.2). If $\deg\phi \leq \deg f$, then $pq - p - q < p$ by (3.3), since $\deg f = p\delta$. Hence, $q = 3$ as claimed above. □

The following remark is useful. Assume that $f, g, h \in k[\mathbf{x}]$ and $\phi \in k[S]$ satisfy the following (i) through (iv), where $S = \{f, g\}$:

(i) f and g are algebraically independent over k,
(ii) $\deg f < \deg g$ and $\deg h < \deg g$,
(iii) $g^{\mathbf{w}}$ and $h^{\mathbf{w}}$ do not belong to $k[f^{\mathbf{w}}]$,
(iv) $\deg(h + \phi) < \deg h$.

Then, we claim that $\deg\phi < \deg^S \phi$, and that f, g and ϕ satisfy the assumptions of Lemma 3.3(ii). In fact, $\phi^{\mathbf{w}} \approx h^{\mathbf{w}}$ by (iv), and $h^{\mathbf{w}}$ does not belong to $k[f^{\mathbf{w}}, g^{\mathbf{w}}]$, since $h^{\mathbf{w}}$ does not belong to $k[f^{\mathbf{w}}]$ by (iii), and $\deg h < \deg g$ by (ii). Hence, $\phi^{\mathbf{w}}$ does not belong to $k[f^{\mathbf{w}}, g^{\mathbf{w}}]$.

Because ϕ is an element of $k[f, g]$, we get $\deg \phi < \deg^S \phi$. By (ii) and (iii), it follows that $\deg f < \deg g$, $\deg \phi = \deg h < \deg g$, and $g^{\mathbf{w}}$ does not belong to $k[f^{\mathbf{w}}]$. Thus, f, g and ϕ satisfy the required conditions. Therefore, the conclusion of Lemma 3.3(ii) holds in this situation.

The following theorem is a generalization of Shestakov-Umirbaev [9, Lemma 5].

THEOREM 3.4 ([6, Theorem 5.2]). *For any $\eta_1, \ldots, \eta_l \in \Omega_{k[\mathbf{x}]/k}$ with $l \geq 2$, there exist $1 \leq i_1 < i_2 \leq l$ such that*

$$\deg \eta_{i_1} + \deg \tilde{\eta}_{i_1} = \deg \eta_{i_2} + \deg \tilde{\eta}_{i_2} \geq \deg \eta_i + \deg \tilde{\eta}_i$$

for $i = 1, \ldots, l$, where $\tilde{\eta}_i = \eta_1 \wedge \cdots \wedge \eta_{i-1} \wedge \eta_{i+1} \wedge \cdots \wedge \eta_l$ for each i.

Using Theorem 3.4, we prove a lemma needed later. Assume that $k_1, k_2, k_3 \in k[\mathbf{x}]$ are algebraically independent over k, and $k_1' := k_1 + ak_3^2 + ck_3 + \psi$ and $k_2' := k_2 + \phi$ satisfy the following conditions for some $a, c \in k$, $\psi \in k[k_2]$ and $\phi \in k[k_3]$:

(a) $\deg k_2' < \deg k_1'$,
(b) $\deg k_1' - \deg k_2' < \deg k_3$,
(c) $\deg \psi < \deg k_1' - \deg k_2' + \deg k_2$,
(d) $\deg k_3 + \deg dk_1' \wedge dk_2' < \deg k_1' + \deg dk_2' \wedge dk_3$.

LEMMA 3.5. *Under the assumption above, we have*

$$\deg dk_1 \wedge dk_3 = \deg k_1' - \deg k_2' + \deg dk_2 \wedge dk_3 \,. \tag{3.4}$$

If furthermore $\phi = bk_3 + d$ for some $b, d \in k$, then the following assertions hold:

(i) *If $a \neq 0$ and $\deg dk_1' \wedge dk_2' < \deg k_3$, then*

$$\deg dk_1 \wedge dk_2 = \deg k_3 + \deg dk_2 \wedge dk_3 \,.$$

(ii) *Assume that $\deg dk_1' \wedge dk_2' < \deg dk_2 \wedge dk_3$. Then,*

$$\deg dk_1 \wedge dk_2 = \begin{cases} \deg k_3 + \deg dk_2 \wedge dk_3 & \text{if } a \neq 0 \\ \deg dk_1 \wedge dk_3 & \text{if } a = 0 \text{ and } b \neq 0 \\ \deg dk_2 \wedge dk_3 & \text{if } a = b = 0 \text{ and } c \neq 0 \\ \deg dk_1' \wedge dk_2' & \text{if } a = b = c = 0 \,. \end{cases}$$

(iii) *Assume that ψ belongs to k. Set $k_1'' = k_1 + a'k_3^2 + c'k_3 + \psi'$ and $k_2'' = k_2 + b'k_3 + d'$ for $a', b', c', d', \psi' \in k$. If $\deg dk_1' \wedge dk_2'$ and $\deg dk_1'' \wedge dk_2''$ are less than $\deg dk_2 \wedge dk_3$, then $(a', b', c') = (a, b, c)$.*

PROOF. Put $\eta_1 = dk_1'$, $\eta_2 = dk_2'$ and $\eta_3 = dk_3$. Then, $\deg \eta_3 + \deg \tilde{\eta}_3 < \deg \eta_1 + \deg \tilde{\eta}_1$ by (d), since $\deg dk_i' = \deg k_i'$ for $i = 1, 2$ and $\deg dk_3 = \deg k_3$ by (2.2). Hence, we have $\deg \eta_1 + \deg \tilde{\eta}_1 = \deg \eta_2 + \deg \tilde{\eta}_2$ by Theorem 3.4. Since ϕ is an element of $k[k_3]$, we get $d\phi \wedge dk_3 = 0$. Hence, $dk_2' \wedge dk_3 = d(k_2 + \phi) \wedge dk_3 = dk_2 \wedge dk_3$. Thus, we obtain

$$\begin{aligned} \deg dk_1' \wedge dk_3 = \deg \tilde{\eta}_2 &= \deg \eta_1 - \deg \eta_2 + \deg \tilde{\eta}_1 \\ &= \deg k_1' - \deg k_2' + \deg dk_2' \wedge dk_3 \\ &= \deg k_1' - \deg k_2' + \deg dk_2 \wedge dk_3 \,. \end{aligned} \tag{3.5}$$

We show that $\deg dk_1' \wedge dk_3 = \deg dk_1 \wedge dk_3$, which implies (3.4). Set $\psi_1 = \Psi^{(1)}(k_2)$, where Ψ is an element of $k[y]$ satisfying $\Psi(k_2) = \psi$. Then, $\deg \psi_1 \leq \deg \psi - \deg k_2$, and so $\deg \psi_1 < \deg k_1' - \deg k_2'$ by (c). Hence,

$$\begin{aligned} \deg \psi_1 dk_2 \wedge dk_3 &= \deg \psi_1 + \deg dk_2 \wedge dk_3 \\ &< \deg k_1' - \deg k_2' + \deg dk_2 \wedge dk_3 = \deg dk_1' \wedge dk_3 \end{aligned} \tag{3.6}$$

by (3.5). Since $d\psi = \psi_1 dk_2$, it follows that

$$\begin{aligned} dk_1' \wedge dk_3 &= dk_1 \wedge dk_3 + 2ak_3 dk_3 \wedge dk_3 + c dk_3 \wedge dk_3 + d\psi \wedge dk_3 \\ &= dk_1 \wedge dk_3 + \psi_1 dk_2 \wedge dk_3 . \end{aligned}$$

This equality and (3.6) imply $\deg dk_1 \wedge dk_3 = \deg dk_1' \wedge dk_3$. This proves (3.4).

Next, assume that $\phi = bk_3 + d$ for some $b, d \in k$. Then, we have

$$\begin{aligned} dk_1 \wedge dk_2 = dk_1' \wedge dk_2' &+ 2ak_3 dk_2 \wedge dk_3 \\ &- b(dk_1 \wedge dk_3 + \psi_1 dk_2 \wedge dk_3) + c dk_2 \wedge dk_3 . \end{aligned} \tag{3.7}$$

By (b), (a) and (3.6), it follows that

$$\begin{aligned} \deg k_3 dk_2 \wedge dk_3 = \deg k_3 + \deg dk_2 \wedge dk_3 &> \deg k_1' - \deg k_2' + \deg dk_2 \wedge dk_3 \\ &> \max\{\deg dk_2 \wedge dk_3, \deg \psi_1 dk_2 \wedge dk_3\} . \end{aligned}$$

Since the right-hand side of the first inequality is equal to $\deg dk_1 \wedge dk_3$ by (3.4), we get

$$\deg k_3 dk_2 \wedge dk_3 > \deg dk_1 \wedge dk_3 > \max\{\deg dk_2 \wedge dk_3, \deg \psi_1 dk_2 \wedge dk_3\} . \tag{3.8}$$

In view of (3.8), the assertions (i) and (ii) easily follow from (3.7).

Finally, we verify (iii). A direct computation shows that

$$\begin{aligned} dk_1'' \wedge dk_2'' - dk_1' \wedge dk_2' = 2(a - a')k_3 dk_2 \wedge dk_3 &- (b - b')dk_1 \wedge dk_3 \\ &+ (c - c')dk_2 \wedge dk_3 . \end{aligned}$$

By assumption, the **w**-degree of the left-hand side of this equality is less than that of $dk_2 \wedge dk_3$, while those of $k_3 dk_2 \wedge dk_3$ and $dk_1 \wedge dk_3$ are greater than that of $dk_2 \wedge dk_3$ by (3.8). Therefore, it follows that $a = a'$, $b = b'$ and $c = c'$. □

4. Shestakov-Umirbaev reductions. In this section, we study the properties of Shestakov-Umirbaev reductions. In what follows, unless otherwise stated, $F = (f_1, f_2, f_3)$ and $G = (g_1, g_2, g_3)$ denote elements of $\mathcal{T}$, and $S_i := \{f_1, f_2, f_3\} \setminus \{f_i\}$ for each i. We say that the pair (F, G) satisfies the *weak Shestakov-Umirbaev condition* for the weight **w** if (SU4), (SU5), (SU6) and the following three conditions hold:

(SU1′) $g_1 - f_1$, $g_2 - f_2$ and $g_3 - f_3$ belong to $k[f_2, f_3]$, $k[f_3]$ and $k[g_1, g_2]$, respectively,

(SU2′) $\deg f_i \leq \deg g_i$ for $i = 1, 2$,

(SU3′) $\deg g_2 < \deg g_1$, and $g_1^{\mathbf{w}}$ does not belong to $k[g_2^{\mathbf{w}}]$.

It is easy to see that (SU1), (SU2) and (SU3) imply (SU1′), (SU2′) and (SU3′), respectively. Hence, if (F, G) satisfies the Shestakov-Umirbaev condition for the weight **w**, then (F, G) satisfies the weak Shestakov-Umirbaev condition for the weight **w**. We say that

$F \in \mathcal{T}$ *admits a weak Shestakov-Umirbaev reduction* for the weight $\mathbf{w}$ if (F_σ, G_σ) satisfies the weak Shestakov-Umirbaev condition for the weight $\mathbf{w}$ for some $\sigma \in \mathfrak{S}_3$ and $G \in \mathcal{T}$, and call this G a *weak Shestakov-Umirbaev reduction* of F for the weight $\mathbf{w}$. The weight $\mathbf{w}$ is fixed throughout, and so is not explicitly mentioned in what follows.

We show that F and G have the following properties (P1) through (P12) if (F, G) satisfies the weak Shestakov-Umirbaev condition:

(P1) $(g_1^{\mathbf{w}})^2 \approx (g_2^{\mathbf{w}})^s$ for some odd number $s \geq 3$, and so $\delta := (1/2)\deg g_2$ belongs to Γ.

(P2) $\deg f_3 \geq (s-2)\delta + \deg dg_1 \wedge dg_2$.

(P3) $\deg f_2 = \deg g_2$.

(P4) If $\deg \phi \leq \deg g_1$ for $\phi \in k[S_1]$, then there exist $a', c' \in k$ and $\psi' \in k[f_2]$ with $\deg \psi' \leq (s-1)\delta$ such that $\phi = a' f_3^2 + c' f_3 + \psi'$.

(P5) If $\deg f_1 < \deg g_1$, then $s = 3$, $g_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$, $\deg f_3 = (3/2)\delta$ and

$$\deg f_1 \geq \frac{5}{2}\delta + \deg dg_1 \wedge dg_2 .$$

(P6) $\deg G < \deg F$.

(P7) $\deg f_2 < \deg f_1$, $\deg f_3 \leq \deg f_1$, and $\delta < \deg f_i \leq s\delta$ for $i = 1, 2, 3$.

(P8) $f_i^{\mathbf{w}}$ does not belong to $k[f_j^{\mathbf{w}}]$ if $i \neq j$ and $(i, j) \neq (1, 3)$. If $f_1^{\mathbf{w}}$ belongs to $k[f_3^{\mathbf{w}}]$, then $s = 3$, $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$ and $\deg f_3 = (3/2)\delta$.

(P9) If $\deg \phi \leq \deg f_2$ for $\phi \in k[S_2]$, then there exist $b', d' \in k$ such that $\phi = b' f_3 + d'$.

(P10) Assume that $k[g_1, g_2] \neq k[S_3]$. If $\deg \phi \leq \deg f_1$ for $\phi \in k[S_3]$, then there exist $c'' \in k$ and $\psi'' \in k[f_2]$ with $\deg \psi'' \leq \min\{(s-1)\delta, \deg \phi\}$ such that $\phi = c'' f_1 + \psi''$. If $\deg \phi < \deg f_1$, then $c'' = 0$.

(P11) There exist $a, b, c, d \in k$ and $\psi \in k[f_2]$ with $\deg \psi \leq (s-1)\delta$ such that $g_1 = f_1 + a f_3^2 + c f_3 + \psi$ and $g_2 = f_2 + b f_3 + d$. If $a \neq 0$ or $b \neq 0$, then $\deg f_3 \leq \deg f_2$. If $\deg f_3 \leq \deg f_2$, then $s = 3$. Furthermore, if ψ belongs to k, then a, b and c are uniquely determined by F in the following sense: If (F, G') satisfies the weak Shestakov-Umirbaev condition for $G' = (g_1', g_2', g_3') \in \mathcal{T}$, where $g_1' = f_1 + a' f_3^2 + c' f_3 + \psi'$ and $g_2' = f_2 + b' f_3 + d'$ with $a', b', c', d', \psi' \in k$, then $a' = a$, $b' = b$ and $c' = c$.

(P12) The following equalities and inequality hold:

$$\deg df_1 \wedge df_2 = \begin{cases} \deg f_3 + \deg df_2 \wedge df_3 & \text{if } a \neq 0 \\ \deg df_1 \wedge df_3 & \text{if } a = 0 \text{ and } b \neq 0 \\ \deg df_2 \wedge df_3 & \text{if } a = b = 0 \text{ and } c \neq 0 \\ \deg dg_1 \wedge dg_2 & \text{if } a = b = c = 0 \end{cases}$$

$$\deg df_1 \wedge df_3 = (s-2)\delta + \deg df_2 \wedge df_3$$

$$\deg df_2 \wedge df_3 \geq s\delta + \deg dg_1 \wedge dg_2 .$$

To show these properties, we set $\phi_i = g_i - f_i$ for $i = 1, 2, 3$. Since $\deg g_3 < \deg f_3$ by (SU5), we have $\phi_3^{\mathbf{w}} = -f_3^{\mathbf{w}}$ and $\deg \phi_3 = \deg f_3$. Hence, $\deg \phi_3 \leq \deg g_1$ and $\phi_3^{\mathbf{w}}$ does not belong to $k[g_1^{\mathbf{w}}, g_2^{\mathbf{w}}]$ by (SU4). Set $U = \{g_1, g_2\}$. Since ϕ_3 is contained in $k[U]$ by (SU1$'$), it follows that $\deg \phi_3 < \deg^U \phi_3$. In view of (SU3$'$), we know that the assumptions of

Lemma 3.3(ii) hold for $f = g_2$, $g = g_1$ and $\phi = \phi_3$. Therefore, there exists an odd number $s \geq 3$ such that $(g_1^{\mathbf{w}})^2 \approx (g_2^{\mathbf{w}})^s$ and

$$\deg f_3 = \deg \phi_3 \geq (s-2)\delta + \deg dg_1 \wedge dg_2 \,, \tag{4.1}$$

$$\deg dg_2 \wedge d\phi_3 \geq s\delta + \deg dg_1 \wedge dg_2 \,, \tag{4.2}$$

where $\delta = (1/2)\deg g_2$. This proves (P1) and (P2).

We show that g_2 is expressed as in (P11). By (SU1$'$), $\phi_2 = g_2 - f_2$ belongs to $k[f_3]$. Hence, $\phi_2 = \sum_{i=0}^{p} b_i f_3^i$ for some $b_0, \dots, b_p \in k$ with $b_p \neq 0$, where $p \in \mathbf{Z}_{\geq 0}$. By (SU2$'$), $\deg \phi_2 \leq \max\{\deg g_2, \deg f_2\} = \deg g_2 = 2\delta$. By (4.1), we get $\deg f_3 > \delta$, since $s \geq 3$. Thus, we must have $p \leq 1$ and $\phi_2 = b_1 f_3 + b_0$, for otherwise $\deg \phi_2 = p \deg f_3 > p\delta \geq 2\delta$, a contradiction. Therefore, g_2 is expressed as stated.

We show (P3) and the first assertion of (P8) for $(i, j) = (2, 3), (3, 2)$ by contradiction. Supposing that $\deg f_2 \neq \deg g_2$, we have $\deg f_2 < \deg g_2$ by (SU2$'$). Since $g_2 = f_2 + bf_3 + d$ as shown above, it follows that $g_2^{\mathbf{w}} = bf_3^{\mathbf{w}}$ and $b \neq 0$. Hence, $f_3^{\mathbf{w}}$ belongs to $k[g_1^{\mathbf{w}}, g_2^{\mathbf{w}}]$, a contradiction to (SU4). Therefore, $\deg f_2 = \deg g_2$, proving (P3). Next, we show that $f_2^{\mathbf{w}} \not\approx f_3^{\mathbf{w}}$. Supposing that $f_2^{\mathbf{w}} \approx f_3^{\mathbf{w}}$, we have $\deg f_2 = \deg f_3$. Hence, $\deg g_2 = \deg f_3$ by (P3). Thus, $g_2^{\mathbf{w}} = f_2^{\mathbf{w}} + bf_3^{\mathbf{w}}$. Since $f_2^{\mathbf{w}} \approx f_3^{\mathbf{w}}$, we get $g_2^{\mathbf{w}} \approx f_3^{\mathbf{w}}$. This contradicts (SU4). Therefore, $f_2^{\mathbf{w}} \not\approx f_3^{\mathbf{w}}$. Now, suppose that $f_3^{\mathbf{w}}$ belongs to $k[f_2^{\mathbf{w}}]$. Then, $f_3^{\mathbf{w}} \approx (f_2^{\mathbf{w}})^l$ for some $l \geq 2$, since $f_2^{\mathbf{w}} \not\approx f_3^{\mathbf{w}}$. Hence, $\deg f_2 < \deg f_3$. From $\deg f_2 = \deg g_2 = \deg(f_2 + bf_3 + d)$, we get $b = 0$, and $f_2^{\mathbf{w}} = g_2^{\mathbf{w}}$. Since $f_3^{\mathbf{w}} \approx (f_2^{\mathbf{w}})^l$, it follows that $f_3^{\mathbf{w}} \approx (g_2^{\mathbf{w}})^l$, a contradiction to (SU4). Therefore, $f_3^{\mathbf{w}}$ does not belong to $k[f_2^{\mathbf{w}}]$. Suppose that $f_2^{\mathbf{w}}$ belongs to $k[f_3^{\mathbf{w}}]$. Then, $f_2^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^l$ for some $l \in \boldsymbol{N}$, where $l \geq 2$ as above. This is impossible, because $\deg f_2 = 2\delta$ by (P3) and $\deg f_3 > \delta$ by (4.1). Therefore, $f_2^{\mathbf{w}}$ does not belong to $k[f_3^{\mathbf{w}}]$.

Since $g_2 - f_2$ is contained in $k[f_3]$ by (SU1$'$), it follows that $df_2 \wedge df_3 - dg_2 \wedge df_3 = d(f_2 - g_2) \wedge df_3 = 0$. Moreover, $df_3 = dg_3 - d\phi_3$. Hence,

$$df_2 \wedge df_3 = dg_2 \wedge df_3 = dg_2 \wedge dg_3 - dg_2 \wedge d\phi_3 \,. \tag{4.3}$$

By (2.3), (SU6), (P1) and (4.2), we get

$$\begin{aligned} \deg dg_2 \wedge dg_3 \leq \deg g_2 + \deg g_3 &< \deg g_1 + \deg dg_1 \wedge dg_2 \\ &= s\delta + \deg dg_1 \wedge dg_2 \leq \deg dg_2 \wedge d\phi_3 \,. \end{aligned}$$

Then, it follows from (4.3) that $\deg df_2 \wedge df_3 = \deg dg_2 \wedge d\phi_3$. Therefore, we obtain

$$\deg df_2 \wedge df_3 \geq s\delta + \deg dg_1 \wedge dg_2 \tag{4.4}$$

by (4.2). This proves the last inequality of (P12).

The following lemma is useful in proving (P4), (P9), (P10) and (P11).

LEMMA 4.1. *Assume that* $\deg f_2 = 2\delta$ *and* $(s-2)\delta < \deg f_3 \leq s\delta$ *for some* $\delta \in \Gamma$ *and an odd number* $s \geq 3$. *Then, the following assertions hold*:

(i) *If* $\deg^{S_1} \phi \leq s\delta$ *for* $\phi \in k[S_1]$, *then there exist* $a, c \in k$ *and* $\psi \in k[f_2]$ *with* $\deg \psi \leq (s-1)\delta$ *such that* $\phi = af_3^2 + cf_3 + \psi$. *If* $a \neq 0$, *then* $\deg f_3 < \deg f_2$.

(ii) *Assume that* $\deg f_1 > \deg f_2$. *If* $\deg^{S_2} \phi \leq \deg f_2$ *for* $\phi \in k[S_2]$, *then there exist* $b, d \in k$ *such that* $\phi = bf_3 + d$.
(iii) *Assume that* $\deg f_1 \leq s\delta$. *If* $\deg^{S_3} \phi \leq \deg f_1$ *for* $\phi \in k[S_3]$, *then there exist* $c' \in k$ *and* $\psi' \in k[f_2]$ *with* $\deg \psi' \leq \min\{(s-1)\delta, \deg^{S_3} \phi\}$ *such that* $\phi = c' f_1 + \psi'$. *If* $\deg^{S_3} \phi < \deg f_1$, *then* $c' = 0$.
(iv) *If* $\deg f_3 \leq \deg f_2$, *then* $s = 3$.

PROOF. To show (i), write $\phi = \sum_{i,j} c_{i,j} f_2^i f_3^j$, where $c_{i,j} \in k$ for each $i, j \in \mathbf{Z}_{\geq 0}$. Since $\deg^{S_1} \phi \leq s\delta$ by assumption, $\deg f_2^i f_3^j \leq s\delta$ if $c_{i,j} \neq 0$ for $i, j \in \mathbf{Z}_{\geq 0}$. We verify that, if $\deg f_2^i f_3^j \leq s\delta$, then $i \leq (s-1)/2$ and $j = 0$, or $i = 0$ and $j = 1, 2$. This shows that ϕ can be expressed as in (i). It follows that $\deg f_2^i f_3 > 2i\delta + (s-2)\delta \geq s\delta$ if $i \geq 1$. If $i > (s-1)/2$, then $2i > s$, since s is an odd number. Hence, $\deg f_2^i = 2i\delta > s\delta$. If $j \geq 3$, then $\deg f_3^j > j(s-2)\delta \geq s\delta$, since $s \geq 3$. Thus, if $\deg f_2^i f_3^j \leq s\delta$, then (i, j) must be as stated above. Therefore, ϕ can be expressed as in (i). Assume that $a \neq 0$. Then, $\deg f_3^2 \leq \deg^{S_1} \phi \leq s\delta$. Since $(s-2)\delta < \deg f_3$, we get $2(s-2) < s$. Thus, $s < 4$, and hence $s = 3$. Therefore, $\deg f_3 \leq (s/2)\delta = (3/2)\delta < 2\delta = \deg f_2$. This proves (i).

We can prove (ii) and (iii), similarly. Actually, if $\deg f_1 > \deg f_2$ and if $\deg f_1^i f_3^j \leq \deg f_2$ for $i, j \in \mathbf{Z}_{\geq 0}$, then $i = 0$. Moreover, we have $j \leq 1$, since $\deg f_3^2 > 2(s-2)\delta \geq 2\delta = \deg f_2$. Therefore, $\phi = bf_3 + d$ for some $b, d \in k$ in the case (ii). To show (iii), assume that $\deg^{S_3} \phi \leq \deg f_1$ for $\phi \in k[S_3]$. Clearly, $i = 0$ or $(i, j) = (1, 0)$ if $\deg f_1^i f_2^j \leq \deg f_1$, while $i = 0$ if $\deg f_1^i f_2^j < \deg f_1$. Hence, $\phi = c' f_1 + \psi'$ for some $c' \in k$ and $\psi' \in k[f_2]$ where $c' = 0$ if $\deg^{S_3} \phi < \deg f_1$. We note that $\deg \psi' \leq \deg^{S_3} \phi$. Since $\deg^{S_3} \phi \leq \deg f_1 \leq s\delta$ by assumption, it follows that $\deg \psi' \leq s\delta$. This implies that $\deg \psi' \leq (s-1)\delta$, because s is an odd number, and $\deg \psi' = \deg f_2^l = 2l\delta$ with $l \in \mathbf{Z}_{\geq 0}$ if $\psi' \neq 0$. Therefore, we obtain $\deg \psi' \leq \min\{(s-1)\delta, \deg^{S_3} \phi\}$.

The assertion (iv) follows from $(s-2)\delta < \deg f_3 \leq \deg f_2 = 2\delta$. □

We show (P4) using Lemma 4.1(i). Since $\deg f_2 = \deg g_2 = 2\delta$ by (P3), and since $(s-2)\delta < \deg f_3 \leq s\delta$ by (4.1) and (SU4), it suffices to check that $\deg^{S_1} \phi \leq s\delta$. Supposing the contrary, we have $\deg \phi < \deg^{S_1} \phi$, since $\deg \phi \leq \deg g_1 = s\delta$ by the assumption of (P3). As shown above, $f_i^{\mathbf{w}}$ does not belong to $k[f_j^{\mathbf{w}}]$ for $(i, j) = (2, 3), (3, 2)$. Hence, $\deg \phi > \deg df_2 \wedge df_3$ by Lemma 3.3(i). Since $\deg df_2 \wedge df_3 > s\delta$ by (4.4), we get $\deg \phi > s\delta$, a contradiction. Thus, $\deg^{S_1} \phi \leq s\delta$, and thereby proving (P4).

We complete the proof of the former part of (P11). Since $\phi_1 = g_1 - f_1$ belongs to $k[S_1]$ by (SU1′), and since $\deg \phi_1 \leq \max\{\deg g_1, \deg f_1\} = \deg g_1 = s\delta$ by (SU2′), we know by (P4) that $g_1 = f_1 + \phi_1$ is expressed as in (P11). If $a \neq 0$, then $\deg f_3 < \deg f_2$ by the last assertion of Lemma 4.1(i). Since $\deg f_2 = \deg g_2$ and $g_2 = f_2 + bf_3 + d$, we get $\deg f_3 \leq \deg f_2$ if $b \neq 0$. By Lemma 4.1(iv), $\deg f_3 \leq \deg f_2$ implies $s = 3$. We have thus proved the former part of (P11).

We show that the conditions listed before Lemma 3.5 and the inequality $\deg dk_1' \wedge dk_2' < \deg dk_2 \wedge dk_3$ hold for $k_i = f_i$ for $i = 1, 2, 3$ and $k_i' = g_i$ for $i = 1, 2$. By the former part of

(P11), k_1' and k_2' are expressed in terms of k_1, k_2 and k_3 as required. Since $\deg g_2 < \deg g_1$ by (SU3$'$), we get (a). Since $\deg g_1 - \deg g_2 = (s-2)\delta$, (b) follows from (4.1). By (P3), (c) is equivalent to $\deg \psi < \deg k_1'$, which follows from $\deg \psi \leq (s-1)\delta < \deg g_1$. The rest of the conditions are due to (4.4), since $df_2 \wedge df_3 = dg_2 \wedge df_3$ as mentioned. Therefore, we obtain the estimation of $\deg df_1 \wedge df_2$ described in (P12) from Lemma 3.5(ii). Owing to (3.4), we have

$$\deg df_1 \wedge df_3 = (s-2)\delta + \deg df_2 \wedge df_3 \,, \tag{4.5}$$

the second equality of (P12). The uniqueness of a, b and c claimed in (P11) follows from Lemma 3.5(iii). This completes the proofs of (P11) and (P12).

Here, we remark that

$$\deg df_1 \wedge df_3 \geq 2(s-1)\delta + \deg dg_1 \wedge dg_2 \tag{4.6}$$

follows from (4.4) and (4.5). Since $\deg f_1 + \deg f_3 \geq \deg df_1 \wedge df_3$, we obtain that

$$\deg f_1 \geq 2(s-1)\delta + \deg dg_1 \wedge dg_2 - \deg f_3 \,. \tag{4.7}$$

Now, we show (P5). By the assumption of (P5), we have $\deg f_1 < \deg g_1$. Hence, $g_1^{\mathbf{w}} = (f_1 + \phi_1)^{\mathbf{w}} = \phi_1^{\mathbf{w}}$, and so $\deg \phi_1 = s\delta$. Since $g_1^{\mathbf{w}} \not\approx f_3^{\mathbf{w}}$ by (SU4), we get $\phi_1^{\mathbf{w}} \not\approx f_3^{\mathbf{w}}$. By (P11), we have $\phi_1 = af_3^2 + cf_3 + \psi$, in which $\deg \psi \leq (s-1)\delta$. From this, it follows that $a \neq 0$, for otherwise $\phi_1^{\mathbf{w}} = cf_3^{\mathbf{w}}$, a contradiction. Hence, $s = 3$ by (P11). Moreover, $\phi_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$, and thus $g_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$. Therefore, $\deg f_3 = (1/2)\deg g_1 = (3/2)\delta$. The last inequality of (P5) follows from (4.7).

We show (P6) and (P7) with the aid of (P5). If $\deg g_1 = \deg f_1$, then (P6) is clear, since $\deg g_2 = \deg f_2$ by (P3), and $\deg g_3 < \deg f_3$ by (SU5). Assume that $\deg f_1 < \deg g_1$. Then,

$$\deg f_1 + \deg f_3 > \frac{5}{2}\delta + \deg dg_1 \wedge dg_2 + \frac{3}{2}\delta = 4\delta + \deg dg_1 \wedge dg_2$$

by (P5). On the other hand, since $\deg g_2 = 2\delta$, and $\deg g_1 = s\delta = 3\delta$ by (P5), it follows from (SU6) that

$$\deg g_1 + \deg g_3 < \deg g_1 + \deg g_1 - \deg g_2 + \deg dg_1 \wedge dg_2 = 4\delta + \deg dg_1 \wedge dg_2 \,.$$

Therefore, $\deg G < \deg F$ by (P3). This proves (P6). If $\deg f_1 = \deg g_1$, then $\deg f_2 < \deg f_1$ by (SU3$'$), and $\deg f_3 \leq \deg f_1$ by (SU4). If $\deg f_1 < \deg g_1$, then $\deg f_1 > (5/2)\delta$ and $\deg f_3 = (3/2)\delta$ by (P5). Hence, $\deg f_i < \deg f_1$ for $i = 2, 3$. This proves the first two statements of (P7). The last statement of (P7) follows from the conditions that $(5/2)\delta < \deg f_1 \leq \deg g_1 = s\delta$, $\deg f_2 = 2\delta$ and $(s-2)\delta < \deg f_3 \leq \deg g_1$.

Let us complete the proof of (P8). First, we show that $\deg f_i \neq l \deg f_j$ holds for any $l \in \boldsymbol{N}$ for $(i, j) = (1, 2), (2, 1)$, which proves that $f_i^{\mathbf{w}}$ does not belong to $k[f_j^{\mathbf{w}}]$. In the case $\deg f_1 = \deg g_1$, we have $2 \deg f_1 = s \deg f_2$ by (P1) and (P3). Since $s \geq 3$ is an odd number, the assertion is true. In the case $\deg f_1 < \deg g_1$, we have $(5/2)\delta < \deg f_1 < 3\delta$ by (P5). Since $\deg f_2 = 2\delta$, the assertion is readily verified. Thus, $f_i^{\mathbf{w}}$ does not belong to $k[f_j^{\mathbf{w}}]$ for $(i, j) = (1, 2), (2, 1)$. Next, suppose to the contrary that $f_3^{\mathbf{w}}$ belongs to $k[f_1^{\mathbf{w}}]$. Since $\deg f_3 \leq \deg f_1$ by (P7), it follows that $f_3^{\mathbf{w}} \approx f_1^{\mathbf{w}}$. In view of (P5), we get $\deg f_1 = \deg g_1$.

Hence, $g_1^{\mathbf{w}} = f_1^{\mathbf{w}} + cf_3^{\mathbf{w}}$. Consequently, we obtain $f_3^{\mathbf{w}} \approx g_1^{\mathbf{w}}$, a contradiction to (SU4). Therefore, $f_3^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}]$. Since the cases $(i, j) = (2, 3), (3, 2)$ are done, this completes the proof of the former part of (P8). For the latter part, assume that $f_1^{\mathbf{w}}$ belongs to $k[f_3^{\mathbf{w}}]$. Then, $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^l$ for some $l \in \boldsymbol{N}$. Since $f_3^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}]$, it follows that $l \geq 2$. Then, we must have $s = 3$ and $l = 2$. In fact, if $s \geq 5$ or $l \geq 3$, then $s \leq l(s-2)$, and so

$$\deg f_1 \leq \deg g_1 = s\delta \leq l(s-2)\delta < l \deg f_3 ,$$

which contradicts $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^l$. Thus, $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$. If $\deg f_3 \neq (3/2)\delta$, then $\deg f_1 = \deg g_1$ by (P5), and hence

$$\deg f_3 = \frac{1}{2} \deg f_1 = \frac{1}{2} \deg g_1 = \frac{1}{2} s\delta = \frac{3}{2}\delta ,$$

a contradiction. Therefore, $\deg f_3 = (3/2)\delta$. This completes the proof of (P8).

We show (P9) using Lemma 4.1(ii). Since $\deg f_2 < \deg f_1$ by (P7), we verify that, if $\deg\phi \leq \deg f_2$ for $\phi \in k[S_2]$, then $\deg^{S_2} \phi \leq \deg f_2$. Supposing the contrary, we get $\deg\phi < \deg^{S_2} \phi$. By Lemma 3.2(i), there exist $p, q \in \boldsymbol{N}$ with $\gcd(p, q) = 1$ such that $(f_3^{\mathbf{w}})^p \approx (f_1^{\mathbf{w}})^q$ and

$$\begin{aligned} 2\delta = \deg f_2 \geq \deg\phi &\geq q \deg f_1 + \deg df_1 \wedge df_3 - \deg f_1 - \deg f_3 \\ &\geq (q-1) \deg f_1 - \deg f_3 + 2(s-1)\delta + \deg dg_1 \wedge dg_2 , \end{aligned} \tag{4.8}$$

where the last inequality is due to (4.6). Since $f_3^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}]$ by (P8), we have $p \geq 2$. If $\deg f_1 < \deg g_1$, then $s = 3$, $\deg f_1 > (5/2)\delta$ and $\deg f_3 = (3/2)\delta$ by (P5), and hence the right-hand side of (4.8) is greater than

$$(q-1)\frac{5}{2}\delta - \frac{3}{2}\delta + 4\delta + \deg dg_1 \wedge dg_2 > \frac{5}{2} q\delta > 2\delta ,$$

a contradiction. Thus, $\deg f_1 = \deg g_1 = s\delta$. Then, the right-hand side of (4.8) is at least

$$(q-1)s\delta - \frac{q}{p} s\delta + 2(s-1)\delta + \deg dg_1 \wedge dg_2 > \frac{qs}{p}(p-1)\delta + (s-2)\delta ,$$

which is less than 2δ by (4.8). Hence, $s = 3$ and $(3q/p)(p-1) < 1$. Since $p \geq 2$, it follows that $3 \leq 3q < 1 + 1/(p-1) \leq 2$, a contradiction. Therefore, we conclude that $\deg^{S_2} \phi \leq \deg f_2$, and thereby proving (P9).

To show (P10), assume that $k[S_3] \neq k[g_1, g_2]$, and take $\phi \in k[S_3]$ such that $\deg\phi \leq \deg f_1$. By virtue of Lemma 4.1(iii), it suffices to check that $\deg\phi = \deg^{S_3} \phi$. Supposing the contrary, we get $\deg\phi < \deg^{S_3} \phi$. By (P8), $f_i^{\mathbf{w}}$ does not belong to $k[f_j^{\mathbf{w}}]$ for $(i, j) = (1, 2), (2, 1)$. Hence, $\deg\phi > \deg df_1 \wedge df_2$ by Lemma 3.3(i). Since $k[S_3] \neq k[g_1, g_2]$, we must have $(a, b, c) \neq (0, 0, 0)$. Hence, $\deg df_1 \wedge df_2 \geq \deg df_2 \wedge df_3 > s\delta$ by (P12). Thus, $\deg\phi > s\delta$. This is a contradiction, because $\deg\phi \leq \deg f_1$ and $\deg f_1 \leq \deg g_1 = s\delta$. Therefore, $\deg\phi = \deg^{S_3} \phi$, and thereby (P10) is proved.

We have thus proved the following theorem.

THEOREM 4.2. *If* (F, G) *satisfies the weak Shestakov-Umirbaev condition for* $F, G \in \mathcal{T}$, *then* (P1) *through* (P12) *hold for* F *and* G.

The following proposition is a consequence of Theorem 4.2.

PROPOSITION 4.3. (i) *If* (F, G) *satisfies the weak Shestakov-Umirbaev condition for* $F, G \in \mathcal{T}$, *then there exist* $E_i \in \mathcal{E}_i$ *for* $i = 1, 2$ *with* $\deg G \circ E_1 = \deg G$ *such that* $(F, G \circ E_1 \circ E_2)$ *satisfies the Shestakov-Umirbaev condition.*
(ii) *For* $F \in \mathcal{T}$, *it follows that* F *admits a Shestakov-Umirbaev reduction if and only if* F *admits a weak Shestakov-Umirbaev reduction.*

PROOF. (i) Assume that g_1 and g_2 are expressed as in (P11). Take $\Psi \in k[y]$ such that $\Psi(f_2) = \psi$, and define $E_i \in \mathcal{E}_i$ for $i = 1, 2$ by $E_1(y_1) = y_1 - \Psi(y_2 - d)$ and $E_2(y_2) = y_2 - d$. Then, $(E_1 \circ E_2)(y_i) = E_i(y_i)$ for $i = 1, 2$. Set $G' = G \circ E_1 \circ E_2$ and $g_i' = G'(y_i)$ for each i. We show that (F, G') satisfies (SU1) through (SU6). By definition, $g_2' = g_2 - d = f_2 + bf_3$. If $b = 0$, then $\Psi(g_2 - d) = \Psi(f_2) = \psi$. Hence, $g_1' = g_1 - \Psi(g_2 - d) = f_1 + af_3^2 + cf_3$. Assume that $b \neq 0$. Then, $s = 3$ by (P11). Hence, $\deg \psi \leq (s-1)\delta = 2\delta$. Since ψ belongs to $k[f_2]$ and since $\deg f_2 = 2\delta$ by (P3), we may write $\psi = ef_2 + e'$, where $e, e' \in k$. Then, $\Psi = ey_2 + e'$, and so

$$g_1' = g_1 - (e(g_2 - d) + e') = f_1 + af_3^2 + (c - be)f_3\,. \tag{4.9}$$

Thus, g_1' and g_2' are expressed as in (SU1). From the construction of g_1' and g_2', it follows that $k[g_1', g_2'] = k[g_1, g_2]$. Since (F, G) satisfies (SU1$'$) by assumption, $g_3' - f_3 = g_3 - f_3$ belongs to $k[g_1, g_2]$, and hence belongs to $k[g_1', g_2']$. Therefore, (F, G') satisfies (SU1). We remark that (F, G) satisfies (SU2) and (SU3) on account of (P3), (SU2$'$), and (P1), and satisfies (SU4) through (SU6) by the definition of the weak Shestakov-Umirbaev condition. From this, we can easily conclude that (F, G') satisfies (SU2) through (SU6) on the assumption that $dg_1' \wedge dg_2' = dg_1 \wedge dg_2$ and $(g_i')^{\mathbf{w}} = g_i^{\mathbf{w}}$ for $i = 1, 2$. So, we verify these equalities. Since $g_2' = g_2 - d$, we have $(g_2')^{\mathbf{w}} = g_2^{\mathbf{w}}$ and $dg_2' = dg_2$. Since $dg_1' = dg_1 - \Psi^{(1)}(g_2 - d)dg_2$, we get $dg_1' \wedge dg_2' = dg_1 \wedge dg_2$. If $b = 0$, then $g_1' = g_1 - \psi$. Since $\deg \psi \leq (s-1)\delta < s\delta = \deg g_1$, we have $(g_1')^{\mathbf{w}} = g_1^{\mathbf{w}}$. If $b \neq 0$, then $\deg f_3 \leq \deg f_2$ by (P11), and so $\deg f_3 < \deg g_1$ by (SU3$'$) and (P3). Hence, $(g_1')^{\mathbf{w}} = (g_1 - \psi - bef_3)^{\mathbf{w}} = g_1^{\mathbf{w}}$. Thus, it holds that $dg_1' \wedge dg_2' = dg_1 \wedge dg_2$ and $(g_i')^{\mathbf{w}} = g_i^{\mathbf{w}}$ for $i = 1, 2$. Thereby, (F, G') satisfies (SU2) through (SU6). Therefore, (F, G') satisfies the Shestakov-Umirbaev condition. Since $G \circ E_1 = (g_1', g_2, g_3)$ and $\deg g_1' = \deg g_1$, we have $\deg G \circ E_1 = \deg G$.
(ii) It is clear that F admits a weak Shestakov-Umirbaev reduction if F admits a Shestakov-Umirbaev reduction. The converse follows from (i). □

The following remark is readily verified. If (F, G) satisfies (SU2$'$), (SU3$'$), (SU4), (SU5) and (SU6), then so does (F', G'). Here, $F' = (f_1', f_2', f_3')$ is an element of $\mathcal{T}$ such that $\deg f_i' \leq \deg f_i$ for $i = 1, 2$ and $(f_3')^{\mathbf{w}} \approx f_3^{\mathbf{w}} + h$ for some $h \in k[g_1^{\mathbf{w}}, g_2^{\mathbf{w}}]$, and $G' = (c_1g_1, c_2g_2, c_3g_3)$, where $c_1, c_2, c_3 \in k \setminus \{0\}$. Note that $F' := F \circ E$ satisfies this condition for $E \in \mathcal{E}_i$ such that $\deg F \circ E \leq \deg F$ if $i \in \{1, 2\}$, and $(F \circ E)(y_3)^{\mathbf{w}} \approx f_3^{\mathbf{w}} + h$ for some $h \in k[g_1^{\mathbf{w}}, g_2^{\mathbf{w}}]$ if $i = 3$. Moreover, (F', G') satisfies (SU1$'$) if the following conditions hold:
(i) $c_1g_1 - f_1'$ belongs to $k[f_2, f_3]$ if $i = 1$ and $c_2 = c_3 = 1$,
(ii) $c_1g_1 - f_1$ and $c_2g_2 - f_2'$ respectively belong to $k[f_2', f_3]$ and $k[f_3]$ if $i = 2$ and $c_3 = 1$,

(iii) $c_1g_1 - f_1$, $c_2g_2 - f_2$ and $c_3g_3 - f_3'$ respectively belong to $k[f_2, f_3']$, $k[f_3']$ and $k[g_1, g_2]$ if $i = 3$.

To end this section, we prove a proposition which will be used in the proof of Theorem 2.1. We note that the case (ii) does not arise if rank $\mathbf{w} = n$, since $\deg f_j = \deg f_3$ implies $f_j^{\mathbf{w}} \approx f_3^{\mathbf{w}}$ if rank $\mathbf{w} = n$, while $f_j^{\mathbf{w}} \not\approx f_3^{\mathbf{w}}$ for $j = 1, 2$ by (P8).

PROPOSITION 4.4. *Assume that (F, G) satisfies the weak Shestakov-Umirbaev condition. If $\deg F \circ E \leq \deg F$ for $E \in \mathcal{E}_i$, then the following assertions hold for $F' := F \circ E$, where $i \in \{1, 2, 3\}$.*
(i) *If $i = 1$ or $i = 2$, or if $i = 3$, $k[f_1, f_2] \neq k[g_1, g_2]$ and $\deg f_j \neq \deg f_3$ for $j = 1, 2$, then (F', G) satisfies the weak Shestakov-Umirbaev condition.*
(ii) *If $i = 3$, $k[f_1, f_2] \neq k[g_1, g_2]$ and $\deg f_j = \deg f_3$ for some $j \in \{1, 2\}$, then there exists $u \in k \setminus \{0\}$ such that (F', G') or (F'_τ, G'') satisfies the weak Shestakov-Umirbaev condition. Here, $G' = (g_1', g_2', ug_3)$ and $G'' = (g_1', g_2', -ug_3)$ with $g_j' = u^{-1}g_j$ and $g_l' = g_l$ for $l \in \{1, 2\} \setminus \{j\}$, and $\tau = (j, 3)$.*

PROOF. Set $f_i' = F'(y_i)$ and $\phi_i = f_i' - f_i$. Then, $\deg f_i' \leq \deg f_i$, since $\deg F' \leq \deg F$ by assumption. Hence, $\deg \phi_i \leq \max\{\deg f_i', \deg f_i\} \leq \deg f_i$. We note that ϕ_i belongs to $k[S_i]$. Besides, $g_1 - f_1$, $g_2 - f_2$ and $g_3 - f_3$ belong to $k[f_2, f_3]$, $[f_3]$ and $k[g_1, g_2]$ by (SU1$'$), respectively, since (F, G) satisfies the weak Shestakov-Umirbaev condition.

(i) First, assume that $i \in \{1, 2\}$, or $i = 3$ and ϕ_3 is contained in k. Since $\deg F' \leq \deg F$, we know by the remark above that (F', G) satisfies (SU2$'$), (SU3$'$), (SU4), (SU5) and (SU6) if $i \in \{1, 2\}$. If $i = 3$, then $(f_3')^{\mathbf{w}} = f_3^{\mathbf{w}}$, since $f_3' - f_3 = \phi_3$ belongs to k by assumption. Hence, (F', G) satisfies the five conditions similarly. We check that (F', G) satisfies (SU1$'$). If $i = 1$, then $g_1 - f_1' = (g_1 - f_1) - \phi_1$ belongs to $k[S_1]$, since so do $g_1 - f_1$ and ϕ_1. If $i = 2$, then ϕ_2 belongs to $k[f_3]$ by (P9), because ϕ_2 is an element of $k[S_2]$ such that $\deg \phi_2 \leq \deg f_2$. Hence, $k[f_2', f_3] = k[f_2, f_3]$, to which $g_1 - f_1$ belongs. Moreover, $g_2 - f_2' = (g_2 - f_2) - \phi_2$ belongs to $k[f_3]$, since so does $g_2 - f_2$. If $i = 3$, then ϕ_3 is contained in k. Hence, $g_1 - f_1$ and $g_2 - f_2$ belong to $k[f_2, f_3'] = k[f_2, f_3]$ and $k[f_3'] = k[f_3]$, respectively. Moreover, $g_3 - f_3' = (g_3 - f_3) - \phi_3$ belongs to $k[g_1, g_2]$, since so does $g_3 - f_3$. Thus, (F', G) satisfies (SU1$'$) in each case. Therefore, (F', G) satisfies the weak Shestakov-Umirbaev condition.

Next, assume that $i = 3$ and ϕ_3 is not contained in k. We show that $(f_3')^{\mathbf{w}} = f_3^{\mathbf{w}} + \alpha(g_2^{\mathbf{w}})^p$ for some $\alpha \in k$ and $p \in \boldsymbol{N}$, which implies that (G', F) satisfies (SU2$'$), (SU3$'$), (SU4), (SU5) and (SU6) by the remark. Since $f_3' = f_3 + \phi_3$, $\deg \phi_3 \leq \deg f_3$, and $f_3^{\mathbf{w}}$ does not belong to $k[g_2^{\mathbf{w}}]$ by (SU4), it suffices to check that $\phi_3^{\mathbf{w}} \approx (g_2^{\mathbf{w}})^p$ for some $p \in \boldsymbol{N}$. We establish that ϕ_3 belongs to $k[f_2]$, and $f_2^{\mathbf{w}} = g_2^{\mathbf{w}}$. Since $\deg f_1 \neq \deg f_3$ by assumption, we have $\deg f_3 < \deg f_1$ by (P7). Hence, $\deg \phi_3 < \deg f_1$. Since $k[f_1, f_2] \neq k[g_1, g_2]$ by assumption, it follows from (P10) that ϕ_3 belongs to $k[f_2]$. Since ϕ_3 is not contained in k, we get $\deg f_2 \leq \deg \phi_3$. Hence, $\deg f_2 \leq \deg f_3$. Since $\deg f_2 \neq \deg f_3$ by assumption, we get $\deg f_2 < \deg f_3$. By (P11), it follows that $b = 0$, where we write $g_2 = f_2 + bf_3 + d$. Hence, $g_2 = f_2 + d$, and so $g_2^{\mathbf{w}} = f_2^{\mathbf{w}}$. Thus, we have proved that $(f_3')^{\mathbf{w}} = f_3^{\mathbf{w}} + \alpha(g_2^{\mathbf{w}})^p$ for some $\alpha \in k$ and $p \in \boldsymbol{N}$, and thereby proved that (G', F) satisfies the five conditions.

As for (SU1$'$), $g_2 - f_2 = d$ clearly belongs to $k[f_3']$. Since ϕ_3 is contained in $k[f_2]$, we know that $g_1 - f_1$ and $g_3 - f_3' = (g_3 - f_3) - \phi_3$ belong to $k[f_2, f_3'] = k[f_2, f_3]$ and $k[g_1, g_2] = k[g_1, g_2, f_2]$, respectively. Thus, (F', G) satisfies (SU1$'$), and therefore satisfies the weak Shestakov-Umirbaev condition.

(ii) By (P7), $\deg f_2 < \deg f_1 = \deg f_3$ if $j = 1$, and $\deg f_3 = \deg f_2 < \deg f_1$ if $j = 2$. In view of (P5), $\deg f_1 = \deg g_1$ in either case. Furthermore, if $j = 1$, then we can write $g_1 = f_1 + cf_3 + \psi$ and $g_2 = f_2 + d$ by (P11), since $a = b = 0$ if $\deg f_2 < \deg f_3$. We claim that $g_j = f_j + \alpha f_3 + \psi^1$ and $\phi_3 = \beta f_j + \psi^2$ for some $\alpha, \beta \in k$, and $\psi^p \in k[f_2]$ for $p = 1, 2$ such that $\deg \psi^p < \deg f_1$ if $j = 1$, and $\deg \psi^p \leq 0$ if $j = 2$. In fact, g_1 has such an expression if $j = 1$ as mentioned, since $\deg \psi \leq (s-1)\delta < s\delta = \deg g_1 = \deg f_1$. If $j = 1$, then $\deg \phi_3 \leq \deg f_3 = \deg f_1$. Hence, it follows from (P10) that ϕ_3 is expressed as claimed. If $j = 2$, then $\deg \phi_3 \leq \deg f_3 < \deg f_1$, and so ϕ_3 belongs to $k[f_2]$ by (P10). Since $\deg f_2 = \deg f_3$ and $\deg \phi_3 \leq \deg f_3$, we have $\phi_3 = \beta f_2 + \psi^2$ for some $\beta, \psi^2 \in k$. The expression of g_2 is due to (P11). Therefore, g_j and ϕ_3 have expressions as claimed. Observe that $\deg \psi^p < \deg f_j$ for $p = 1, 2$. Moreover, $\deg f_j = \deg f_3$, while $f_j^{\mathbf{w}} \not\approx f_3^{\mathbf{w}}$ by (P8). Thus, we have

$$(4.10) \qquad g_j^{\mathbf{w}} = f_j^{\mathbf{w}} + \alpha f_3^{\mathbf{w}}, \quad (f_3')^{\mathbf{w}} = (f_3 + \phi_3)^{\mathbf{w}} = f_3^{\mathbf{w}} + \beta f_j^{\mathbf{w}} = (1 - \alpha\beta) f_3^{\mathbf{w}} + \beta g_j^{\mathbf{w}}.$$

First, assume that $\alpha\beta \neq 1$. We show that (F', G') satisfies the weak Shestakov-Umirbaev condition for $u = 1 - \alpha\beta$. From the second equality of (4.10), we get $(f_3')^{\mathbf{w}} \approx f_3^{\mathbf{w}} + u^{-1}\beta g_j^{\mathbf{w}}$. Hence, (F', G') satisfies (SU2$'$), (SU3$'$), (SU4), (SU5) and (SU6) as remarked. We check (SU1$'$). If $j = 1$, then $g_2' = g_2$, and $g_2' - f_2 = g_2 - f_2 = d$ belongs to $k[f_3']$. If $j = 2$, then $f_3' - f_3 = \phi_3$ is contained in $k[f_2]$ by (P10) as mentioned. Hence, $k[f_2, f_3'] = k[f_2, f_3]$, to which $g_1' - f_1 = g_1 - f_1$ belongs. A direct computation shows that

$$g_j' - f_j = \frac{1}{u} g_j - f_j = \frac{1}{1 - \alpha\beta}(f_j + \alpha f_3 + \psi^1) - f_j = \frac{1}{1 - \alpha\beta}(\alpha f_3' + \psi^1 - \alpha\psi^2),$$
$$u g_3 - f_3' = (1 - \alpha\beta) g_3 - (f_3 + \beta f_j + \psi^2) = (1 - \alpha\beta)(g_3 - f_3) - \beta g_j + \beta\psi^1 - \psi^2.$$

By the first expression, $g_j' - f_j$ belongs to $k[f_2, f_3']$ if $j = 1$, and to $k[f_3']$ if $j = 2$, since ψ^1 and ψ^2 belong to $k[f_2]$ if $j = 1$, and to k if $j = 2$. We show that $u g_3 - f_3'$ belongs to $k[g_1, g_2]$. Since $g_3 - f_3$ and g_j belong to $k[g_1, g_2]$, it suffices to check that ψ^1 and ψ^2 belong to $k[g_1, g_2]$. This is obvious if $j = 2$. If $j = 1$, then $g_2 = f_2 + d$. Hence, $k[g_2] = k[f_2]$, to which ψ^1 and ψ^2 belong. Thus, $u g_3 - f_3'$ belongs to $k[g_1, g_2]$. This proves that (F', G') satisfies (SU1$'$), Therefore, (F', G') satisfies the weak Shestakov-Umirbaev condition.

Next, assume that $\alpha\beta = 1$. We show that (F_τ', G'') satisfies the weak Shestakov-Umirbaev condition for $u = \alpha$. Write $F_\tau' = (h_1, h_2, h_3)$. Then, $\deg h_j = \deg f_3' \leq \deg f_3 = \deg f_j$ and $\deg h_l = \deg f_l$ for $l \in \{1, 2\} \setminus \{j\}$. By the first equality of (4.10), we get $h_3^{\mathbf{w}} = f_j^{\mathbf{w}} = -\alpha f_3^{\mathbf{w}} + g_j^{\mathbf{w}}$ since $\beta^{-1} = \alpha$. Hence, (F_τ', G'') satisfies (SU2$'$), (SU3$'$), (SU4), (SU5) and (SU6) by the remark. We check (SU1$'$). As in the case of $\alpha\beta \neq 1$ above, $g_2'' - h_2 = g_2 - f_2 = d$ belongs to $k[h_3]$ if $j = 1$, and $g_1'' - h_1 = g_1 - f_1$ belongs to $k[h_2, h_3] = k[f_3', f_2] = k[f_2, f_3]$ if $j = 2$. A direct computation shows that

$$g_j'' - h_j = \frac{1}{\alpha} g_j - f_3' = \frac{1}{\alpha}(f_j + \alpha f_3 + \psi^1) - (f_3 + \beta f_j + \psi^2) = \frac{1}{\alpha}\psi^1 - \psi^2\,,$$
$$-u g_3 - h_3 = -\alpha g_3 - f_j = -\alpha(g_3 - f_3) - \alpha f_3 - f_j = -\alpha(g_3 - f_3) - g_j + \psi^1\,.$$

By the first expression, $g_j'' - h_j$ belongs to $k[h_2, h_3] = k[f_2, f_1]$ if $j = 1$, and to $k[h_3]$ if $j = 2$. As in the case of $\alpha\beta \neq 1$ above, $-ug_3 - h_3$ belongs to $k[g_1, g_2]$ by the second expression. Thus, (F', G) satisfies (SU1$'$). Therefore, (F', G) satisfies the weak Shestakov-Umirbaev condition. □

5. Analysis of reductions. In this section, we prove some technical propositions which will be needed in the proof of Theorem 2.1. First, we show a useful lemma.

LEMMA 5.1. *Assume that (F_σ, G) satisfies the weak Shestakov-Umirbaev condition for some $\sigma \in \mathfrak{S}_3$. Then, the following assertions hold*:

(i) *If* $\deg f_i < \deg f_1$ *for* $i = 2, 3$, *then* $\sigma(1) = 1$.

(ii) *If (F_σ, G) satisfies the Shestakov-Umirbaev condition, and if $\sigma(1) = 1$ and* $\deg df_1 \wedge df_2 < \deg f_1$, *then* $\sigma = \mathrm{id}$ *and* $(f_1, f_2) = (g_1, g_2)$.

(iii) *If* $\deg f_3 < \deg f_2 < \deg f_1$ *and* $2 \deg f_1 < 3 \deg f_2$, *then either* $3 \deg f_2 = 4 \deg f_3$, *or* $2 \deg f_1 = s \deg f_3$ *for some odd number* $s \geq 3$.

(iv) *If* $\deg df_2 \wedge df_3 < \deg df_1 \wedge df_3 < \deg df_1 \wedge df_2$, *then one of the following holds*:

(1) $\sigma = \mathrm{id}$ *and* $2 \deg g_1 = 3 \deg f_2$.

(2) $\sigma = (1, 2, 3)$ *and* $2 \deg f_2 = s \deg f_3$ *for some odd number* $s \geq 3$.

PROOF. (i) By (P7), we have $\deg f_{\sigma(i)} \leq \deg f_{\sigma(1)}$ for $i = 2, 3$. Hence, $\sigma(1) = 1$ if $\deg f_i < \deg f_1$ for $i = 2, 3$.

(ii) Since $\sigma(1) = 1$ by assumption, we have

$$\begin{aligned}\deg f_1 &= \deg f_{\sigma(1)} \leq \deg g_1 \\ &= s\delta < \deg df_{\sigma(2)} \wedge df_{\sigma(3)} < \deg df_{\sigma(1)} \wedge df_{\sigma(3)} = \deg df_1 \wedge df_{\sigma(3)}\end{aligned}$$

by (SU2$'$) and the last two conditions of (P12). Since $\deg df_1 \wedge df_2 < \deg f_1$ by assumption, we get $\sigma(3) \neq 2$. Hence, $\sigma(3) = 3$, and so $\sigma = \mathrm{id}$. Because $(F, G) = (F_\sigma, G)$ satisfies the Shestakov-Umirbaev condition by assumption, we may write g_1 and g_2 as in (SU1). It follows from the inequality above that the $\mathbf{w}$-degrees of $df_1 \wedge df_3$ and $df_2 \wedge df_3$ are greater than $\deg f_1$, and hence greater than $\deg df_1 \wedge df_2$. This implies that $a = b = c = 0$ by the first equality of (P12). Therefore, we obtain $(f_1, f_2) = (g_1, g_2)$.

(iii) Since $\deg f_i < \deg f_1$ for $i = 2, 3$ by assumption, we have $\sigma(1) = 1$ by (i). Hence, $\sigma = \mathrm{id}$ or $\sigma = (2, 3)$. First, assume that $\sigma = \mathrm{id}$. Then, $\deg f_2 = \deg g_2 = 2\delta$ by (P3). Since $2 \deg f_1 < 3 \deg f_2$ by assumption, we have $\deg f_1 < (3/2) \deg f_2 = 3\delta \leq s\delta = \deg g_1$. Hence, $\deg f_3 = (3/2)\delta$ by (P5). Therefore, we obtain $3 \deg f_2 = 6\delta = 4 \deg f_3$. Next, assume that $\sigma = (2, 3)$. Then,

$$\frac{3}{2}\delta < 2\delta = \deg f_{\sigma(2)} = \deg f_3 < \deg f_2 = \deg f_{\sigma(3)}\,.$$

Hence, $\deg f_1 = \deg g_1$ in view of (P5). By (P1), we have $2 \deg g_1 = s \deg g_2$ for some odd number $s \geq 3$. By (P3), $\deg g_2 = \deg f_{\sigma(2)} = \deg f_3$. Therefore, $2 \deg f_1 = s \deg f_3$.

(iv) Set $\gamma_i = \deg df_p \wedge df_q$ for each i, where $p, q \in \boldsymbol{N} \setminus \{i\}$ with $1 \leq p < q \leq 3$. In view of the first equality of (P12), we know that there exist four possibilities for $\gamma_{\sigma(3)} = \deg df_{\sigma(1)} \wedge df_{\sigma(2)}$. Since $\gamma_1 < \gamma_2 < \gamma_3$ by assumption, we have $\gamma_{\sigma(3)} \neq \gamma_{\sigma(i)}$ for $i = 1, 2$. Hence, the second and the third cases do not arise. Accordingly, $\gamma_{\sigma(3)}$ must be either $\deg f_{\sigma(3)} + \gamma_{\sigma(1)}$ or $\deg dg_1 \wedge dg_2$, where $a \neq 0$ or $a = b = c = 0$, respectively. In the former case, $\gamma_{\sigma(2)} = (s-2)\delta + \gamma_{\sigma(1)} < \deg f_{\sigma(3)} + \gamma_{\sigma(1)} = \gamma_{\sigma(3)}$ by the second equality of (P12) and (P2). Hence, $\gamma_{\sigma(1)} < \gamma_{\sigma(2)} < \gamma_{\sigma(3)}$. Thus, we get $\sigma = \mathrm{id}$. Since $a \neq 0$, we have $s = 3$ by (P11). Therefore, $2 \deg g_1 = 3 \deg g_2 = 3 \deg f_2$ by (P1) and (P3). In the latter case, $\gamma_{\sigma(3)} = \deg dg_1 \wedge dg_2 < \gamma_{\sigma(1)} < \gamma_{\sigma(2)}$ by the last two conditions of (P12). Hence, we get $\sigma = (1, 2, 3)$. Since $a = 0$, we have $\deg f_2 = \deg f_{\sigma(1)} = \deg g_1$ in view of (P5). By (P3), $\deg f_3 = \deg f_{\sigma(2)} = \deg g_2$. Therefore, $2 \deg f_2 = s \deg f_3$ for some odd number $s \geq 3$ by (P1). □

From Lemma 5.1(i) and (ii), we get the following proposition.

PROPOSITION 5.2. *Assume that*

$$\deg f_i < \deg f_1 \quad (i = 2, 3) \quad \textit{and} \quad \deg df_1 \wedge df_2 < \deg f_1 . \tag{5.1}$$

If (F_σ, G) satisfies the Shestakov-Umirbaev condition for some $\sigma \in \mathfrak{S}_3$ and $G \in \mathcal{T}$, then there exists $E \in \mathcal{E}_3$ such that $F \circ E = G$.

PROOF. Since $\deg f_i < \deg f_1$ for $i = 2, 3$, we have $\sigma(1) = 1$ by Lemma 5.1(i). Since $\deg df_1 \wedge df_2 < \deg f_1$, we get $\sigma = \mathrm{id}$ and $(f_1, f_2) = (g_1, g_2)$ by Lemma 5.1(ii). Then, (SU1) implies that $G = F \circ E$ for some $E \in \mathcal{E}_3$. □

In the rest of this section, we assume that $f_3^{\mathbf{w}}$ does not belong to $k[f_2^{\mathbf{w}}]$, and

$$\deg f_1 = s\delta\,, \quad \deg f_2 = 2\delta\,, \quad (s-2)\delta < \deg f_3 < s\delta \tag{5.2}$$

for some odd number $s \geq 3$ and $\delta \in \Gamma$. Under the assumption, $f_2^{\mathbf{w}}$ does not belong to $k[f_3^{\mathbf{w}}]$, because $f_2^{\mathbf{w}} \not\approx f_3^{\mathbf{w}}$ and $\deg f_2 = 2\delta \leq 2(s-2)\delta < \deg f_3^2$. Furthermore, $f_1^{\mathbf{w}}$ belongs to $k[f_3^{\mathbf{w}}]$ if and only if $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$, in which case $s = 3$. In fact, if $f_1^{\mathbf{w}}$ belongs to $k[f_3^{\mathbf{w}}]$, then $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^l$ for some $l \in \boldsymbol{N}$. Since $\deg f_3 < \deg f_1$ by assumption, we have $l \geq 2$. If $l \geq 3$ or $s \geq 5$, then $\deg f_1 = s\delta \leq l(s-2)\delta < l \deg f_3$, a contradiction. Thus, $l = 2$ and $s = 3$. If $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$, then $f_1^{\mathbf{w}}$ clearly belongs to $k[f_3^{\mathbf{w}}]$.

Under the assumption above, the following two propositions hold.

PROPOSITION 5.3. *Assume that*

$$\deg df_1 \wedge df_2 \leq \deg f_3 - (s-2)\delta + \varepsilon\,, \tag{5.3}$$

where $\varepsilon := \deg df_1 \wedge df_2 \wedge df_3 > 0$. If $f_2^{\mathbf{w}}$ belongs to $k[S_2]^{\mathbf{w}}$, then $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$.

PROOF. By assumption, there exists $\phi_2 \in k[S_2]$ such that $\phi_2^{\mathbf{w}} = f_2^{\mathbf{w}}$. As mentioned after (5.2), $f_2^{\mathbf{w}}$ does not belong to $k[f_3^{\mathbf{w}}]$. Since $\deg f_2 < \deg f_1$ by (5.2), $f_2^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}, f_3^{\mathbf{w}}] \setminus k[f_3^{\mathbf{w}}]$. Thus, $f_2^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}, f_3^{\mathbf{w}}]$, and hence neither does $\phi_2^{\mathbf{w}}$. Therefore, we have $\deg \phi_2 < \deg^{S_2} \phi_2$. By Lemma 3.2(ii), there exist $p, q \in \boldsymbol{N}$ with $\gcd(p, q) = 1$ such that $(f_1^{\mathbf{w}})^p \approx (f_3^{\mathbf{w}})^q$ and

$$\begin{aligned} 2\delta = \deg f_2 > \deg(f_2 - \phi_2) &\geq p \deg f_1 + \varepsilon - \deg df_1 \wedge df_2 - \deg f_3 \\ &\geq p \deg f_1 - (\deg f_3 - (s-2)\delta) - \deg f_3 \\ &= \left(s \left(p + 1 - \frac{2p}{q} \right) - 2 \right) \delta . \end{aligned} \tag{5.4}$$

Here, we use (5.3) for the last inequality, and $\deg f_3 = (p/q) \deg f_1$ and $\deg f_1 = s\delta$ for the last equality. Now, suppose to the contrary that $f_1^{\mathbf{w}} \not\approx (f_3^{\mathbf{w}})^2$. Then, the assumptions of Lemma 3.3(ii) hold for $f = f_3$ and $g = f_1$. In fact, $f_1^{\mathbf{w}}$ does not belong to $k[f_3^{\mathbf{w}}]$ if $f_1^{\mathbf{w}} \not\approx (f_3^{\mathbf{w}})^2$ as remarked after (5.2). By (5.2), it follows that $\deg f_3 < \deg f_1$ and $\deg \phi_2 = \deg f_2 < \deg f_1$. Thus, we may conclude by Lemma 3.3(ii) that $p = 2$, and $q \geq 3$ is an odd number. Consequently, the right-hand side of (5.4) is at least $(3(2 + 1 - 2 \cdot 2/3) - 2)\delta = 3\delta$, a contradiction. Therefore, we must have $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$ if $f_2^{\mathbf{w}}$ belongs to $k[S_2]^{\mathbf{w}}$. □

The following proposition forms the core of the proof of Theorem 2.1.

PROPOSITION 5.4. *Assume that*

$$\deg df_1 \wedge df_2 < \deg f_3 - (s-2)\delta + \min\{\delta, \varepsilon\} . \tag{5.5}$$

If there exists $\phi_1 \in k[S_1]$ *such that* $\deg f_1' < \deg f_1$, *then either* $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$, *or* $(f_2^{\mathbf{w}})^2 \approx (f_3^{\mathbf{w}})^3$ *and* F' *does not admit a Shestakov-Umirbaev reduction, where* $f_1' = f_1 + \phi_1$ *and* $F' = (f_1', f_2, f_3)$. *Assume further that* $(f_1')^{\mathbf{w}}$ *does not belong to* $k[S_1]^{\mathbf{w}}$. *Then, the following assertions hold*:

(1) $f_i^{\mathbf{w}}$ *does not belong to* $k[S_i']^{\mathbf{w}}$ *for* $i = 2, 3$, *where* $S_i' = \{f_1', f_2, f_3\} \setminus \{f_i\}$. *Hence,* F' *does not admit an elementary reduction.*

(2) *If* $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$ *and* (F_σ', G) *satisfies the weak Shestakov-Umirbaev condition for some* $\sigma \in \mathfrak{S}_3$ *and* $G \in \mathcal{T}$, *then* $\sigma = \mathrm{id}$ *and* (F, G) *satisfies the weak Shestakov-Umirbaev condition.*

PROOF. To begin with, we show that $\deg \phi_1 < \deg^{S_1} \phi_1$ if $f_1^{\mathbf{w}} \not\approx (f_3^{\mathbf{w}})^2$. Since ϕ_1 is an element of $k[S_1]$, we check that $\phi_1^{\mathbf{w}}$ does not belong to $k[f_2^{\mathbf{w}}, f_3^{\mathbf{w}}]$. By the assumption that $\deg(f_1 + \phi_1) < \deg f_1$, we have $\phi_1^{\mathbf{w}} \approx f_1^{\mathbf{w}}$. Since $\deg f_1 = (s/2) \deg f_2$ for an odd number s by (5.2), $f_1^{\mathbf{w}}$ does not belong to $k[f_2^{\mathbf{w}}]$. Since $f_1^{\mathbf{w}} \not\approx (f_3^{\mathbf{w}})^2$ by assumption, $f_1^{\mathbf{w}}$ does not belong to $k[f_3^{\mathbf{w}}]$ as mentioned after (5.2). By (5.2), it follows that

$$\deg f_1 = 2\delta + (s-2)\delta < \deg f_2 + \deg f_3 .$$

Hence, $f_1^{\mathbf{w}}$ does not belong to $k[f_2^{\mathbf{w}}, f_3^{\mathbf{w}}] \setminus (k[f_2^{\mathbf{w}}] \cup k[f_3^{\mathbf{w}}])$. Thus, $f_1^{\mathbf{w}}$ does not belong to $k[f_2^{\mathbf{w}}, f_3^{\mathbf{w}}]$, and hence neither does $\phi_3^{\mathbf{w}}$. Therefore, $\deg \phi_1 < \deg^{S_1} \phi_1$ if $f_1^{\mathbf{w}} \not\approx (f_3^{\mathbf{w}})^2$.

First, we show that $(f_2^{\mathbf{w}})^2 \approx (f_3^{\mathbf{w}})^3$ and F' does not admit a Shestakov-Umirbaev reduction in the case where $\deg \phi_1 < \deg^{S_1} \phi_1$. Then, we obtain the first part of the proposition as a consequence, since $\deg\phi_1 < \deg^{S_1}\phi_1$ if $f_1^{\mathbf{w}} \not\approx (f_3^{\mathbf{w}})^2$ as shown above. By Lemma 3.2(ii), there exist $p, q \in \boldsymbol{N}$ with $\gcd(p,q) = 1$ such that $(f_3^{\mathbf{w}})^p \approx (f_2^{\mathbf{w}})^q$ and

$$
\begin{aligned}
s\delta = \deg f_1 > \deg(f_1 + \phi_1) &\geq q \deg f_2 + \varepsilon - \deg df_1 \wedge df_2 - \deg f_3 \\
&> q \deg f_2 - (\deg f_3 - (s-2)\delta) - \deg f_3 \\
&= \left(q\left(2 - \frac{4}{p}\right) + s - 2\right)\delta\,,
\end{aligned}
\tag{5.6}
$$

where we use (5.5) for the last inequality, and $\deg f_3 = (q/p)\deg f_2$ and $\deg f_2 = 2\delta$ for the last equality. Recall that we are assuming that $f_3^{\mathbf{w}}$ does not belong to $k[f_2^{\mathbf{w}}]$, while $f_2^{\mathbf{w}}$ does not belong to $k[f_3^{\mathbf{w}}]$ as mentioned after (5.2). Hence, we have $p \geq 2$ and $q \geq 2$. We show that $p = 3$ and $q = 2$ by contradiction. Supposing that $p = 2$, we have $\deg f_3 = (q/2)\deg f_2 = q\delta$. Hence, $(s-2)\delta < q\delta < s\delta$ by (5.2), and so $q = s - 1$. Since $p = 2$ and s is an odd number, we get $\gcd(p,q) = 2$, a contradiction. If $p \geq 4$, then the right-hand side of (5.6) would be at least $(q + s - 2)\delta \geq s\delta$ since $q \geq 2$. This is a contradiction. Thus, we get $p = 3$. If $q \geq 3$, then the right-hand side of (5.6) would be at least $s\delta$, a contradiction. Hence, we have $q = 2$. Therefore, we obtain $(f_3^{\mathbf{w}})^3 \approx (f_2^{\mathbf{w}})^2$. From this, we know that $\deg f_3 = (2/3)\deg f_2 = (4/3)\delta$. Since $\deg f_3 > (s-2)\delta$ by (5.2), it follows that $s = 3$. Consequently, the right-hand side of (5.6) is equal to $(7/3)\delta$. Thus, we get

$$
\deg f_3 = \frac{4}{3}\delta < \deg f_2 = 2\delta < \frac{7}{3}\delta < \deg f_1' < 3\delta\,. \tag{5.7}
$$

It follows that $2\deg f_1' < 6\delta = 3\deg f_2$. Then, by Lemma 5.1(iii), we can conclude that F' does not admit a weak Shestakov-Umirbaev reduction, since

$$
\begin{aligned}
&3\deg f_2 = 6\delta \neq \frac{16}{3}\delta = 4\deg f_3\,, \\
&3\deg f_3 = 4\delta < \frac{14}{3}\delta < 2\deg f_1' < 6\delta < \frac{20}{3}\delta = 5\deg f_3\,.
\end{aligned}
$$

Therefore, F' does not admit a Shestakov-Umirbaev reduction.

In this situation, assume further that $(f_1')^{\mathbf{w}}$ does not belong to $k[S_1]^{\mathbf{w}}$. We show that $f_i^{\mathbf{w}}$ does not belong to $k[S_i']$ for $i = 2, 3$ by contradiction. Suppose that there exists $\phi_i \in k[S_i']$ such that $\phi_i^{\mathbf{w}} = f_i^{\mathbf{w}}$ for some $i \in \{2,3\}$. Then, the conditions (i) through (iv) after Lemma 3.3 are fulfilled for $f = f_j$, $g = f_1'$, $h = f_i$ and $\phi = \phi_i$, where $j \in \{2,3\} \setminus \{i\}$. Actually, $f_1' = f_1 + \phi_1$, f_2 and f_3 are algebraically independent over k, since so are f_1, f_2 and f_3, and ϕ_1 is an element of $k[S_1]$. Moreover, $\deg f_l < \deg f_1'$ for $l = 2, 3$ by (5.7), and $f_i^{\mathbf{w}}$ does not belong to $k[f_j^{\mathbf{w}}]$ by assumption, since (i,j) is $(2,3)$ or $(3,2)$. By assumption, $(f_1')^{\mathbf{w}}$ does not belong to $k[S_1]^{\mathbf{w}}$, and hence does not belong to $k[f_j^{\mathbf{w}}]$. By the choice of ϕ_i, we have $\deg(f_i - \phi_i) < \deg f_i$. Thus, (i) through (iv) are satisfied. By Lemma 3.3(ii) and the remark following it, we may conclude that $((f_1')^{\mathbf{w}})^2 \approx (f_j^{\mathbf{w}})^q$ for some odd number $q \geq 3$. Hence, $\deg f_1' = (q/2)\deg f_j$ is equal to $(2q/3)\delta$ if $j = 3$, and $q\delta$ if $j = 2$. Since no odd number $q \geq 3$ satisfies $7/3 < 2q/3 < 3$ or $7/3 < q < 3$, we get a contradiction by (5.7). Therefore, $f_i^{\mathbf{w}}$ does not belong to $k[S_i']^{\mathbf{w}}$ for $i = 2, 3$. Since $(f_1')^{\mathbf{w}}$ does not belong to $k[S_1]^{\mathbf{w}}$, it follows that F' does not admit an elementary reduction. This proves (1) in the case where $\deg \phi_1 < \deg^{S_1} \phi_1$. The assumption of (2) does not hold in this situation, since $\deg f_1 = 3\delta \neq (8/3)\delta = \deg f_3^2$ by (5.7).

Next, we show (1) and (2) in the case where $\deg \phi_1 = \deg^{S_1} \phi_1$ and $(f_1')^{\mathbf{w}}$ does not belong to $k[S_1]^{\mathbf{w}}$. By the remark in the first paragraph, we know that $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$ if $\deg \phi_1 = \deg^{S_1} \phi_1$. As mentioned after (5.2), $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$ implies $s = 3$. Hence, $\deg f_1 = s\delta = 3\delta$, and so $\deg f_3 = (1/2) \deg f_1 = (3/2)\delta$. Since $\deg^{S_1} \phi_1 = \deg \phi_1$ and $\deg \phi_1 = \deg f_1$, we have $\deg^{S_1} \phi_1 = 3\delta$. By Lemma 4.1(i), we may write $\phi_1 = af_3^2 + cf_3 + \psi$, where $a, c \in k$ and $\psi \in k[f_2]$ with $\deg \psi \leq (3-1)\delta = 2\delta$. Since $\deg f_2 = 2\delta$, we get $\psi = ef_2 + e'$ for some $e, e' \in k$. Note that a is not zero, for otherwise $\deg \phi_1 \leq \max\{\deg f_3, \deg \psi\} < \deg f_1$, a contradiction. We claim that the conditions (a) through (d) before Lemma 3.5 hold for $k_i = f_i$ for $i = 1, 2, 3$ and $k_i' = k_i$ for $i = 1, 2$. In fact, (a), (b), (c) follow from $\deg k_1 = \deg k_1' = 3\delta$, $\deg k_2 = \deg k_2' = 2\delta$ and $\deg k_3 = (3/2)\delta$. The left-hand side of (d) is less than 3δ, since $\deg df_1 \wedge df_2 < \deg f_3 = (3/2)\delta$ by (5.5) with $s = 3$. Because the right-hand side of (d) is greater than $\deg k_1 = 3\delta$, we know that (d) holds true. Therefore, by (3.4), we obtain

$$\deg df_1 \wedge df_3 = \deg f_1 - \deg f_2 + \deg df_2 \wedge df_3 = \delta + \deg df_2 \wedge df_3 \,. \tag{5.8}$$

Hence, $\deg df_2 \wedge df_3 < \deg df_1 \wedge df_3$. Since $d\phi_1 \wedge df_3 = d\psi \wedge df_3 = edf_2 \wedge df_3$, we have $df_1' \wedge df_3 = df_1 \wedge df_3 + edf_2 \wedge df_3$. Thus, $\deg df_1' \wedge df_3 = \deg df_1 \wedge df_3$, and so

$$\deg df_1' \wedge df_3 = \delta + \deg df_2 \wedge df_3 \tag{5.9}$$

by (5.8). For the same reason as above, the conditions (a) through (d) before Lemma 3.5 hold for $k_1 = f_1'$, $k_i = f_i$ for $i = 2, 3$, $k_1' = f_1 = k_1 - ak_3^2 - ck_3 - \psi$ and $k_2' = k_2$, for k_1 is not involved in the conditions. Since $a \neq 0$ and $\deg df_1 \wedge df_2 < \deg f_3$, we know by Lemma 3.5(i) that

$$\deg df_1' \wedge df_2 = \deg f_3 + \deg df_2 \wedge df_3 = \frac{3}{2}\delta + \deg df_2 \wedge df_3 \,. \tag{5.10}$$

Set $\Phi = f_1 + ay^2 + cy + ef_2 + e'$. Then, $\deg_{\mathbf{w}}^{f_3} \Phi = \deg f_1$, while $\deg \Phi(f_3) = \deg f_1' < \deg f_1$. Since $\Phi^{(1)} = 2ay + c$ and $a \neq 0$, we have $\deg_{\mathbf{w}}^{f_3} \Phi^{(1)} = \deg f_3 = \deg \Phi^{(1)}(f_3)$. Hence, $m_{\mathbf{w}}^{f_3}(\Phi) = 1$. By Theorem 3.1, it follows that

$$\begin{aligned} \deg f_1' = \deg \Phi(f_3) &\geq \deg_{\mathbf{w}}^{f_3} \Phi + m_{\mathbf{w}}^{f_3}(\Phi)(\varepsilon - \deg df_1 \wedge df_2 - \deg f_3) \\ &= \deg f_1 + \varepsilon - \deg df_1 \wedge df_2 - \deg f_3 \\ &> \deg f_1 - 2\deg f_3 + (s-2)\delta = \delta \,, \end{aligned} \tag{5.11}$$

where the last inequality is due to (5.5). With the aid of (5.11), we show the following:

(i) $(f_1')^{\mathbf{w}} \notin k[f_2^{\mathbf{w}}, f_3^{\mathbf{w}}]$. (ii) $f_2^{\mathbf{w}} \notin k[(f_1')^{\mathbf{w}}, f_3^{\mathbf{w}}]$. (iii) $f_3^{\mathbf{w}} \notin k[(f_1')^{\mathbf{w}}, f_2^{\mathbf{w}}]$.

Since $k[f_2^{\mathbf{w}}, f_3^{\mathbf{w}}]$ is contained in $k[S_1]^{\mathbf{w}}$, (i) follows from the assumption that $(f_1')^{\mathbf{w}}$ does not belong to $k[S_1]^{\mathbf{w}}$. In particular, $f_2^{\mathbf{w}} \not\approx (f_1')^{\mathbf{w}}$. By (5.11), $\deg f_2 = 2\delta < \deg (f_1')^2$. Hence, $f_2^{\mathbf{w}}$ does not belong to $k[(f_1')^{\mathbf{w}}]$. Since $\deg f_3 = (3/2)\delta < \deg f_2 < 3\delta = \deg f_3^2$, it follows that $f_2^{\mathbf{w}}$ does not belong to $k[f_3^{\mathbf{w}}]$. By (5.11), $\deg f_2 < \delta + (3/2)\delta < \deg f_1' f_3$, and so $f_2^{\mathbf{w}}$ does not belong to $k[(f_1')^{\mathbf{w}}, f_3^{\mathbf{w}}] \setminus (k[(f_1')^{\mathbf{w}}] \cup k[f_3^{\mathbf{w}}])$. Thus, $f_2^{\mathbf{w}}$ does not belong to $k[(f_1')^{\mathbf{w}}, f_3^{\mathbf{w}}]$, proving (ii). It follows that $f_3^{\mathbf{w}} \not\approx (f_1')^{\mathbf{w}}$ by (i), and $\deg f_3 < 2\delta < \deg(f_1')^2$ by (5.11). Hence, $f_3^{\mathbf{w}}$ does not belong to $k[(f_1')^{\mathbf{w}}]$. Since $\deg f_3 < \deg f_2$, we get that $f_3^{\mathbf{w}}$ does not belong to $k[(f_1')^{\mathbf{w}}, f_2^{\mathbf{w}}] \setminus k[(f_1')^{\mathbf{w}}]$. This proves (iii).

Now, we show that $f_2^{\mathbf{w}}$ does not belong to $k[S_2']^{\mathbf{w}}$ by contradiction. Supposing the contrary, there exists $\phi_2 \in k[S_2']$ such that $\phi_2^{\mathbf{w}} = f_2^{\mathbf{w}}$. Then, $\phi_2^{\mathbf{w}}$ does not belong to $k[(f_1')^{\mathbf{w}}, f_3^{\mathbf{w}}]$ by (ii). Hence, $\deg \phi_2 < \deg^{S_2'} \phi_2$. By Lemma 3.2(i), there exist $p, q \in \boldsymbol{N}$ with $\gcd(p, q) = 1$ for which $((f_1')^{\mathbf{w}})^q \approx (f_3^{\mathbf{w}})^p$ and

$$
\begin{aligned}
2\delta = \deg f_2 = \deg \phi_2 &\geq pq\gamma + \deg df_1' \wedge df_3 - p\gamma - q\gamma \\
&= pq\gamma + \delta + \deg df_2 \wedge df_3 - p\gamma - q\gamma\,,
\end{aligned}
\tag{5.12}
$$

where $\gamma \in \Gamma$ such that $\deg f_1' = p\gamma$ and $\deg f_3 = q\gamma$, and the last equality is due to (5.9). From (5.12), it follows that $(pq - p - q)\gamma < \delta$. Since $\deg f_1' > \delta$ by (5.11), and since $\deg f_3 = (3/2)\delta > \delta$, we have $\delta < \min\{\deg f_1', \deg f_3\} = \min\{p, q\}\gamma$. Hence, $pq - p - q < \min\{p, q\}$. By (iii) and (i), $f_3^{\mathbf{w}}$ and $(f_1')^{\mathbf{w}}$ do not belong to $k[(f_1')^{\mathbf{w}}]$ and $k[f_3^{\mathbf{w}}]$, respectively. As $\gcd(p, q) = 1$, we get $2 \leq p < q$ or $2 \leq q < p$. It follows from the claim before Lemma 3.3 that $(p, q) = (2, 3)$ or $(p, q) = (3, 2)$. If $(p, q) = (2, 3)$, then $3\delta < 3\deg f_1' = 2\deg f_3 = 3\delta$ by (5.11), a contradiction. Thus, $(p, q) = (3, 2)$. Then, $\deg f_1' = (3/2)\deg f_3 = (9/4)\delta$ and $\gamma = (1/2)\deg f_3 = (3/4)\delta$, and so

$$
\deg df_2 \wedge df_3 \leq 2\delta - pq\gamma - \delta + p\gamma + q\gamma = 2\delta - 6\gamma - \delta + 3\gamma + 2\gamma = \frac{1}{4}\delta
\tag{5.13}
$$

by (5.12). By Lemma 3.2(ii) and (5.13), we get

$$
\deg(f_2 - \phi_2) \geq 3\deg f_3 + \varepsilon - \deg df_2 \wedge df_3 - \deg f_1' > \frac{9}{2}\delta - \frac{1}{4}\delta - \frac{9}{4}\delta = 2\delta\,.
$$

However, since $\phi_2^{\mathbf{w}} = f_2^{\mathbf{w}}$, we have $\deg(f_2 - \phi_2) < \deg f_2 = 2\delta$, a contradiction. Therefore, $f_2^{\mathbf{w}}$ does not belong to $k[S_2']^{\mathbf{w}}$.

Similarly, suppose to the contrary that there exists $\phi_3 \in k[S_3']$ such that $\phi_3^{\mathbf{w}} \approx f_3^{\mathbf{w}}$. Then, $\phi_3^{\mathbf{w}}$ does not belong to $k[(f_1')^{\mathbf{w}}, f_2^{\mathbf{w}}]$ by (iii). Hence, $\deg \phi_3 < \deg^{S_3'} \phi_3$. By (i) and (ii), $(f_1')^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ do not belong to $k[f_2^{\mathbf{w}}]$ and $k[(f_1')^{\mathbf{w}}]$, respectively. Thus,

$$
\deg df_1' \wedge df_2 < \deg \phi_3 = \deg f_3 = \frac{3}{2}\delta
$$

by Lemma 3.3(i). This contradicts (5.10). Therefore, $f_3^{\mathbf{w}}$ does not belong to $k[S_3']^{\mathbf{w}}$. This completes the proof of (1).

Finally, we show (2). Assume that (F_σ', G) satisfies the weak Shestakov-Umirbaev condition for some $\sigma \in \mathfrak{S}_3$ and $G \in \mathcal{T}$. By (5.9) and (5.10), we have

$$
\deg df_2 \wedge df_3 < \deg df_1' \wedge df_3 < \deg df_1' \wedge df_2\,.
$$

In addition, $2\deg f_2 = 4\delta \neq (3/2)r\delta = r\deg f_3$ for any odd number $r \geq 3$. Hence, we get $\sigma = \mathrm{id}$ and $2\deg g_1 = 3\deg f_2$ by Lemma 5.1(iv). Thus, (F', G) satisfies the weak Shestakov-Umirbaev condition, and $\deg g_1 = (3/2)\deg f_2 = \deg f_1$. Then, it is immediate that (F, G) satisfies (SU2′), (SU3′), (SU4), (SU5) and (SU6). As for (SU1′), we have only to check that $g_1 - f_1$ belongs to $k[f_2, f_3]$. Since (F', G) satisfies the weak Shestakov-Umirbaev condition, $g_1 - f_1'$ belongs to $k[f_2, f_3]$ by (SU1′). Hence, $g_1 - f_1 = (g_1 - f_1') + \phi_1$ belongs to $k[f_2, f_3]$, since so does ϕ_1. Thus, (F, G) satisfies (SU1′). Therefore, (F, G) satisfies the weak Shestakov-Umirbaev condition. This completes the proof of (2). □

We note that (5.11) is the key estimation which guarantees that no tame automorphism admits a reduction of type IV.

6. Proof of Theorem 2.1. We begin with the following lemma.

LEMMA 6.1. (i) *If* $\deg F = |\mathbf{w}|$ *for* $F \in \mathrm{Aut}_k\, k[\mathbf{x}]$, *then* F *is tame.*
(ii) $\Sigma := \{a_1 w_1 + \cdots + a_n w_n;\, a_1, \ldots, a_n \in \mathbf{Z}_{\geq 0}\}$ *is a well-ordered subset of* Γ.

PROOF. (i) We may assume that $w_1 \leq \cdots \leq w_n$ and $\deg f_1 \leq \cdots \leq \deg f_n$ by changing the indices of $w_1, \ldots, w_n$ and $f_1, \ldots, f_n$ if necessary. Write $f_i = b_i + \sum_{j=1}^{n} a_{i,j} x_j + f_i'$ for each i, where $b_i, a_{i,j} \in k$ for each j, and f_i' is an element of the ideal Q of $k[\mathbf{x}]$ generated by all the quadratic monomials. Clearly, F is tame if and only if so is $F \circ G'$ or $G' \circ F$ for some $G' \in \mathrm{T}_k\, k[\mathbf{x}]$. Since $\deg F \circ G = \deg F$ for $G = (x_1 - b_1, \ldots, x_n - b_n)$, we may assume that $b_i = 0$ for each i by replacing F by $F \circ G$. Note that $\det(a_{i,j})_{i,j}$ is equal to the Jacobian of F, so $(a_{i,j})_{i,j}$ is invertible. Let H be an affine automorphism of $k[\mathbf{x}]$ defined by $H(x_i) = \sum_{j=1}^{n} a_{i,j} x_j$ for each i. Then, $\deg H(x_i) \leq \deg f_i$ for each i since $f_i = H(x_i) + f_i'$. We claim that $\deg f_i = w_i$ for each i. In fact, if not, we can find i such that $\deg f_i < w_i$ since $\deg F = |\mathbf{w}|$ by assumption. Then, $\deg H(x_j) \leq \deg f_j \leq \deg f_i < w_i$ for $j \leq i$, while $\deg x_l = w_l \geq w_i$ for $l \geq i$ by assumption. Hence, $H(x_j)$ is contained in the $(i-1)$-dimensional k-vector space $\bigoplus_{l=1}^{i-1} k x_l$ for $j = 1, \ldots, i$. This contradicts that $H(x_1), \ldots, H(x_i)$ are linearly independent over k. Thus, we get $\deg f_i = w_i$, and hence $\deg H(x_i) \leq w_i$ for each i. We show that $\deg H^{-1}(x_i) \leq w_i$ for each i. Let m be the maximal number for which $w_m = w_i$. Then, $H(x_j)$ belongs to $\bigoplus_{l=1}^{m} k x_l$ for $j = 1, \ldots, m$. Hence, H induces an automorphism of $\bigoplus_{l=1}^{m} k x_l$. Thus, $H^{-1}(x_i)$ belongs to $\bigoplus_{l=1}^{m} k x_l$. Therefore, $\deg H^{-1}(x_i) \leq w_m = w_i = \deg x_i$. This implies that $\deg H^{-1}(g) \leq \deg g$ holds for each $g \in k[\mathbf{x}]$. Consequently,

$$|\mathbf{w}| \leq \deg H^{-1} \circ F = \sum_{i=1}^{n} \deg H^{-1}(f_i) \leq \sum_{i=1}^{n} \deg f_i = \deg F = |\mathbf{w}| .$$

Therefore, $\deg H^{-1} \circ F = |\mathbf{w}|$, and so we may replace F by $H^{-1} \circ F$. It follows that $f_i = x_i + f_i''$ for each i, where $f_i'' = H^{-1}(f_i') \in Q$. We show that f_i'' belongs to $k[x_1, \ldots, x_{i-1}]$ for every i by contradiction. Suppose that there appears in f_i'' a monomial $x_{a_1} \cdots x_{a_n}$, where $a_1, \ldots, a_n \in \{1, \ldots, n\}$ with $a_1 \geq i$. Since $x_{a_1} \cdots x_{a_n}$ belongs to Q, we have $n \geq 2$. Hence,

$$w_i = \deg f_i \geq \deg f_i'' \geq \deg x_{a_1} \cdots x_{a_n} = \sum_{i=1}^{n} w_{a_i} > w_{a_1} \geq w_i ,$$

a contradiction. Thus, f_i'' belongs to $k[x_1, \dots, x_{i-1}]$ for each i. This means that F is triangular. Here, we say that $(h_1, \dots, h_n) \in \mathrm{Aut}_k\, k[\mathbf{x}]$ is *triangular* if there exists $\sigma \in \mathfrak{S}_n$ such that $h_{\sigma(i)} = x_{\sigma(i)} + \phi_i$ for some $\phi_i \in k[x_{\sigma(1)}, \dots, x_{\sigma(i-1)}]$ for $i = 1, \dots, n$. Since a triangular automorphism is tame, we conclude that F is tame.

(ii) We show that each nonempty subset S of Σ has the minimum element. As mentioned, we may regard $\Gamma = \mathbf{Z}^r$ for some $r \in \mathbf{N}$. Let $k[\mathbf{y}, \mathbf{y}^{-1}]$ be the Laurent polynomial ring in $y_1, \dots, y_r$ over k, and R the k-subalgebra of $k[\mathbf{y}, \mathbf{y}^{-1}]$ generated by $\mathbf{y}^{w_i}$ for $i = 1, \dots, n$, where $\mathbf{y}^\alpha = y_1^{\alpha_1} \cdots y_r^{\alpha_r}$ for each $\alpha = (\alpha_1, \dots, \alpha_r)$. Then, R is Noetherian, and contains $\mathbf{y}^\alpha$ for each $\alpha \in \Sigma$. Consider the ideal I of R generated by $\{\mathbf{y}^\alpha; \alpha \in S\}$. Since R is Noetherian, there exists a finite subset S' of S with minimum element μ such that I is generated by $\{\mathbf{y}^\alpha; \alpha \in S'\}$. Then, μ becomes the minimum element of S. In fact, for each $\alpha \in S$, there exist $\beta \in S'$ and $\gamma \in \Sigma$ such that $\mathbf{y}^\alpha = \mathbf{y}^\beta \mathbf{y}^\gamma$. Then, $\beta \geq \mu$, $\gamma \geq 0$ and $\alpha = \beta + \gamma$. Hence, $\alpha \geq \beta \geq \mu$. Thus, μ is the minimum element of S. Therefore, Σ is a well-ordered subset of Γ. □

In the rest of the paper, we assume that $n = 3$, and identify $k[\mathbf{y}]$ with $k[\mathbf{x}]$. Let $\mathcal{A}$ be the set of $F \in \mathrm{Aut}_k\, k[\mathbf{x}]$ for which there exists $G_i \in \mathrm{Aut}_k\, k[\mathbf{x}]$ for $i = 1, \dots, l$ with $G_1 = F$ and $\deg G_l = |\mathbf{w}|$ such that G_{i+1} is an elementary reduction or a weak Shestakov-Umirbaev reduction of G_i for $i = 1, \dots, l-1$, where $l \in \mathbf{N}$. Then, each element of $\mathcal{A}$ is tame, since G_l is tame if $\deg G_l = |\mathbf{w}|$ by Lemma 6.1(i), and G_i is tame if and only if so is G_{i+1} for each i. Hence, $\mathcal{A}$ is contained in $\mathrm{T}_k\, k[\mathbf{x}]$. By definition, if $\deg F > |\mathbf{w}|$ for $F \in \mathcal{A}$, then F admits an elementary reduction or a weak Shestakov-Umirbaev reduction. By Proposition 4.3(ii), F admits a weak Shestakov-Umirbaev reduction if and only if F admits a Shestakov-Umirbaev reduction. Thus, if $\deg F > |\mathbf{w}|$ for $F \in \mathcal{A}$, then F admits an elementary reduction or a Shestakov-Umirbaev reduction. The goal of this section is to establish that $\mathcal{A} = \mathrm{T}_k\, k[\mathbf{x}]$, which implies Theorem 2.1 immediately.

We remark that, if F belongs to $\mathcal{A}$, then so do F_σ and $F \circ H$, where $\sigma \in \mathfrak{S}_3$ and $H = (c_1x_1, c_2x_2, c_3x_3)$ with $c_1, c_2, c_3 \in k \setminus \{0\}$. If $\deg F = |\mathbf{w}|$ or if there exists $G \in \mathcal{A}$ such that G is an elementary reduction or a weak Shestakov-Umirbaev reduction of F, then F belongs to $\mathcal{A}$.

The following is a key proposition.

PROPOSITION 6.2. *If* $\deg F \circ E \leq \deg F$ *for* $F \in \mathcal{A}$ *and* $E \in \mathcal{E}$, *then* $F \circ E$ *belongs to* $\mathcal{A}$.

Note that, if $\deg F \circ E > \deg F$ for $F \in \mathcal{A}$ and $E \in \mathcal{E}$, then $F \circ E$ belongs to $\mathcal{A}$. Actually, $(F \circ E) \circ E^{-1} = F$ is an elementary reduction of $F \circ E$.

We deduce from Proposition 6.2 that $\mathrm{T}_k\, k[\mathbf{x}]$ is contained in $\mathcal{A}$. Take any $F \in \mathrm{T}_k\, k[\mathbf{x}]$. Then, we can express $F = H \circ E_1 \circ \cdots \circ E_l$, where $H = (c_1x_1, c_2x_2, c_3x_3)$ with $c_1, c_2, c_3 \in k \setminus \{0\}$, $l \in \mathbf{Z}_{\geq 0}$, and $E_i \in \mathcal{E}$ for $i = 1, \dots, l$. We show that F belongs to $\mathcal{A}$ by induction on l. The assertion is true if $l = 0$, i.e., $F = H$, since $\deg H = |\mathbf{w}|$. Assume that $l > 0$. By induction assumption, $F' := H \circ E_1 \circ \cdots \circ E_{l-1}$ belongs to $\mathcal{A}$. Then, $F = F' \circ E_l$ belongs to $\mathcal{A}$ by Proposition 6.2 and the note following it. Therefore, $\mathrm{T}_k\, k[\mathbf{x}]$ is contained in $\mathcal{A}$ on the assumption that Proposition 6.2 is true.

The following proposition is necessary to prove Proposition 6.2.

PROPOSITION 6.3. *Assume that* $F = (f_1, f_2, f_3) \in \mathcal{A}$ *satisfies*

$$\deg f_1 = s\delta\,, \quad \deg f_2 = 2\delta\,, \quad (s-2)\delta + \deg df_1 \wedge df_2 \leq \deg f_3 < s\delta \tag{6.1}$$

for some odd number $s \geq 3$ *and* $\delta \in \Gamma$, *and that* $f_3^{\mathbf{w}}$ *does not belong to* $k[f_2^{\mathbf{w}}]$. *Then, there exists* $E \in \mathcal{E}_3$ *such that* $\deg F \circ E < \deg F$ *and* $F \circ E$ *belongs to* $\mathcal{A}$.

We note that (6.1) implies (5.1), (5.2), (5.3) and (5.5). Furthermore, $f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ are algebraically dependent over k in this situation, for otherwise

$$\deg df_1 \wedge df_2 = \deg f_1 + \deg f_2 = (s+2)\delta$$

as mentioned after (2.3), which contradicts the last inequality of (6.1).

We establish Propositions 6.2 and 6.3 simultaneously by induction on $\deg F$. Since Σ is well-ordered by Lemma 6.1(ii), so is the subset $\Delta := \{\deg H;\ H \in \mathcal{A}\}$, where $\min \Delta = |\mathbf{w}|$. Assume that $F \in \mathcal{A}$ satisfies $\deg F = |\mathbf{w}|$. If $\deg F \circ E \leq \deg F$ for $E \in \mathcal{E}$, then $\deg F \circ E = |\mathbf{w}|$, since $\deg F \circ E \geq |\mathbf{w}|$ by (2.4). Hence, $F \circ E$ belongs to $\mathcal{A}$. Thus, the statement of Proposition 6.2 holds for $F \in \mathcal{A}$ with $\deg F = |\mathbf{w}|$. Note that $f_1^{\mathbf{w}}$, $f_2^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ are algebraically independent over k if $\deg F = |\mathbf{w}|$, for otherwise $\deg df_1 \wedge df_2 \wedge df_3 < \sum_{i=1}^{3} \deg f_i = |\mathbf{w}|$, a contradiction. Therefore, the assumption of Proposition 6.3 is not fulfilled.

Let μ be an element of Δ such that $\mu > |\mathbf{w}|$, and assume that the statement of Proposition 6.2 holds for each $F \in \mathcal{A}$ with $\deg F < \mu$. For $F \in \mathrm{Aut}_k\, k[\mathbf{x}]$, we define I_F to be the set of $i \in \{1, 2, 3\}$ for which there exists $E \in \mathcal{E}_i$ such that $\deg F \circ E < \deg F$ and $F \circ E$ belongs to $\mathcal{A}$. Note that, if $\deg F > |\mathbf{w}|$ for $F \in \mathcal{A}$, then either $I_F \neq \emptyset$, or (F_σ, G) satisfies the weak Shestakov-Umirbaev condition for some $\sigma \in \mathfrak{S}_3$ and $G \in \mathcal{A}$.

CLAIM 1. *Let* F *be an element of* $\mathcal{A}$ *such that* $\deg F = \mu$.

(i) *If* E *is an element of* $\mathcal{E}_i$ *for some* $i \in I_F$, *then* $F \circ E$ *belongs to* $\mathcal{A}$.

(ii) *If there exist* $E', E'' \in \mathcal{E}$ *and* $E_i \in \mathcal{E}_i$ *with* $\deg F \circ E_i < \deg F$ *for some* $i \in I_F$ *such that* $E \circ E' = E_i \circ E''$ *for* $E \in \mathcal{E}$, *then* $F \circ E$ *belongs to* $\mathcal{A}$.

(iii) *For a triangular automorphism* H *of* $k[\mathbf{x}]$, *we define* $E_i \in \mathcal{E}_i$ *by* $E_i(x_i) = H(x_i)$ *for each* i. *If* $\deg(F \circ H)(x_i) < \deg f_i$, *or equivalently* $\deg F \circ E_i < \deg F$, *for some* $i \in I_F$, *then* $F \circ E_j$ *belongs to* $\mathcal{A}$ *for* $j = 1, 2, 3$.

(iv) *If* $I_F \setminus \{i\} \neq \emptyset$ *and* $f_j^{\mathbf{w}}$ *belongs to* $k[f_i^{\mathbf{w}}]$ *for some* $i, j \in \{1, 2, 3\}$ *with* $i \neq j$, *then* j *belongs to* I_F.

(v) *If* (F, G) *satisfies the weak Shestakov-Umirbaev condition for some* $G \in \mathcal{A}$, *then there exists* $G' \in \mathcal{A}$ *such that* (F, G') *satisfies the Shestakov-Umirbaev condition.*

PROOF. (i) Since i is an element of I_F, there exists $E_i \in \mathcal{E}_i$ such that $\deg F \circ E_i < \deg F$ and $F \circ E_i$ belongs to $\mathcal{A}$. Then, we have $\deg F \circ E_i < \mu$, since $\deg F = \mu$ by assumption. For each $E \in \mathcal{E}_i$, it follows that $E' := E_i^{-1} \circ E$ is an element of $\mathcal{E}_i$. Hence, $F \circ E = (F \circ E_i) \circ E'$ belongs to $\mathcal{A}$ by the induction assumption of Proposition 6.2.

(ii) We may assume that E is contained in $\mathcal{E}_j$ for some $j \neq i$ by (i), and $\deg F \circ E \leq \deg F$ by the note after Proposition 6.2. Then, E' and E'' belong to $\mathcal{E}_i$ and $\mathcal{E}_j$, respectively, since $E \circ E' = E_i \circ E''$ by assumption. Hence, $(E_i \circ E'')(x_j) = (E \circ E')(x_j) = E(x_j)$, and $(E_i \circ E'')(x_l) = E_i(x_l)$ for $l \neq j$. Since $\deg F \circ E_i < \deg F$ and $\deg F \circ E \leq \deg F$, we have

$$\deg(F \circ E_i \circ E'')(x_l) = \begin{cases} \deg(F \circ E_i)(x_i) < \deg f_i & \text{if } l = i \\ \deg(F \circ E)(x_j) \leq \deg f_j & \text{if } l = j \\ \deg(F \circ E_i)(x_l) = \deg f_l & \text{otherwise}. \end{cases}$$

Thus, $\deg F \circ E_i \circ E'' < \deg F$. Note that $F \circ E_i \circ E''$ belongs to $\mathcal{A}$ by the induction assumption of Proposition 6.2, since $\deg F \circ E_i < \deg F = \mu$, and $F \circ E_i$ belongs to $\mathcal{A}$ by (i). Therefore, $(F \circ E_i \circ E'') \circ (E')^{-1}$ belongs to $\mathcal{A}$ for the same reason. This shows that $F \circ E$ belongs to $\mathcal{A}$, since $F \circ E_i \circ E'' \circ (E')^{-1} = F \circ E \circ E' \circ (E')^{-1} = F \circ E$.

(iii) Without loss of generality, we may assume that $i \neq j$ by (i). We may also assume that $H(x_l) = x_l + \phi_l$ for each l, where $\phi_l \in k[x_1, \ldots, x_{l-1}]$. Then, $E_p \circ E' = E_q \circ E_p$ holds for each $p < q$, where $E' \in \mathcal{E}_q$ such that $E'(x_q) = x_q + E_p^{-1}(\phi_q)$. In view of this, we can find $E', E'' \in \mathcal{E}$ such that $E_j \circ E' = E_i \circ E''$. By assumption, we have $\deg F \circ E_i < \deg F$, and i is an element of I_F. Hence, we conclude that $F \circ E_j$ belongs to $\mathcal{A}$ by (ii).

(iv) Since $I_F \setminus \{i\} \neq \emptyset$ by assumption, we can find $l \in I_F \setminus \{i\}$ and $E_l \in \mathcal{E}_l$ such that $\deg F \circ E_l < \deg F$. Clearly, we may assume that $j \neq l$. Since $f_j^{\mathbf{w}}$ belongs to $k[f_i^{\mathbf{w}}]$ by assumption, there exist $c \in k \setminus \{0\}$ and $r \in N$ such that $f_j^{\mathbf{w}} = c(f_i^{\mathbf{w}})^r$. Then, we can define a triangular automorphism H of $k[\mathbf{x}]$ by $H(x_i) = x_i$, $H(x_j) = x_j - cx_i^r$ and $H(x_l) = E_l(x_l)$. Define $E_j \in \mathcal{E}_j$ by $E_j(x_j) = H(x_j)$. Since $\deg F \circ E_l < \deg F$ for $l \in I_F$, it follows from (iii) that $F \circ E_j$ belongs to $\mathcal{A}$. Moreover, since $\deg(f_j - cf_i^r) < \deg f_j$, we have $\deg F \circ E_j < \deg F$. Therefore, j belongs to I_F.

(v) Since (F, G) satisfies the Shestakov-Umirbaev condition by assumption, there exists $E_i \in \mathcal{E}_i$ for $i = 1, 2$ such that $\deg G \circ E_1 = \deg G$, and (F, G') satisfies the Shestakov-Umirbaev condition by Proposition 4.3(i), where $G' = G \circ E_1 \circ E_2$. We show that G' belongs to $\mathcal{A}$. Since G is an element of $\mathcal{A}$, and since $\deg G < \deg F = \mu$ by (P6), it follows that $G \circ E_1$ belongs to $\mathcal{A}$ by the induction assumption of Proposition 6.2. Then, $(G \circ E_1) \circ E_2$ belongs to $\mathcal{A}$ for the same reason, since $\deg G \circ E_1 = \deg G < \mu$. Therefore, the assertion holds for $G' = G \circ E_1 \circ E_2$. □

Now, we show that the statement of Proposition 6.3 holds for each $F \in \mathcal{A}$ with $\deg F = \mu$. Since $\mu > |\mathbf{w}|$, we have $\deg F > |\mathbf{w}|$. Hence, $I_F \neq \emptyset$ or (F_σ, G) satisfies the weak Shestakov-Umirbaev condition for some $\sigma \in \mathfrak{S}_3$ and $G \in \mathcal{A}$ as noted. The conclusion of Proposition 6.3 is obvious if I_F contains 3. If I_F contains 2, then $\deg F \circ E_2 < \deg F$ for some $E_2 \in \mathcal{E}_2$. Hence, $f_2^{\mathbf{w}}$ belongs to $k[S_2]^{\mathbf{w}}$. Then, we get $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$ by Proposition 5.3. Here, we remind that the assumption of Proposition 6.3 implies (5.1), (5.2), (5.3) and (5.5). Thus, $f_1^{\mathbf{w}}$ belongs to $k[f_3^{\mathbf{w}}]$. Since $I_F \setminus \{3\} \neq \emptyset$, this implies that I_F contains 1 by Claim 1(iv). So, assume that I_F contains 1. Then, there exists $E_1 \in \mathcal{E}_1$ such that $\deg F' < \deg F$ and F' belongs to $\mathcal{A}$, where $F' = F \circ E_1$. Clearly, $F'(x_1) = f_1 + \phi_1$ for some $\phi_1 \in k[S_1]$ and $\deg F'(x_1) < \deg f_1$. On account of Claim 1(i), we may assume that $F'(x_1)^{\mathbf{w}}$ does not belong to $k[S_1]^{\mathbf{w}}$ by replacing E_1 if necessary. Then, F and F' satisfy all the assumptions of Proposition 5.4. By the first part of this proposition, we may conclude that either $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$, or $(f_2^{\mathbf{w}})^2 \approx (f_3^{\mathbf{w}})^3$ and F' does not admit a Shestakov-Umirbaev reduction. We show that F' admits a Shestakov-Umirbaev reduction, and hence the latter case is impossible. Observe that $f_2^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ are algebraically dependent over k in either case, since so are $f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ due to (6.1). This implies that $\deg F' > |\mathbf{w}|$ by (2.4). Since F' is an element of $\mathcal{A}$, it follows that $I_{F'} \neq \emptyset$ or $(F'_{\sigma'}, G')$ satisfies the weak Shestakov-Umirbaev condition for some $\sigma' \in \mathfrak{S}_3$ and $G' \in \mathcal{A}$. By Proposition 5.4(1), F' does not admit an elementary reduction. Hence, $I_{F'} = \emptyset$. Thus, $(F'_{\sigma'}, G')$ satisfies the weak Shestakov-Umirbaev condition for some $\sigma' \in \mathfrak{S}_3$ and $G' \in \mathcal{A}$. Accordingly, F' admits a weak Shestakov-Umirbaev reduction. Therefore, F' admits a Shestakov-Umirbaev reduction by Proposition 4.3(ii). As a result, we get $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$. Then, it follows from Proposition 5.4(2) that $\sigma' = \mathrm{id}$ and (F, G') satisfies the weak Shestakov-Umirbaev condition. So, we are reduced to the case where (F_σ, G) satisfies the weak Shestakov-Umirbaev condition for some $\sigma \in \mathfrak{S}_3$ and $G \in \mathcal{A}$. By Claim 1(iv), we may assume that (F_σ, G) satisfies the Shestakov-Umirbaev condition by replacing G if necessary. Then, there exists $E \in \mathcal{E}_3$ such that $F \circ E = G$ by Proposition 5.2. Since $\deg G < \deg F$ by (P6), and since G is an element of $\mathcal{A}$, it follows that $\deg F \circ E < \deg F$, and $F \circ E$ belongs to $\mathcal{A}$. Thus, we arrive at the conclusion of Proposition 6.3. Therefore, we have proved the assertion of Proposition 6.3 in the case where $\deg F = \mu$ on the assumption that the assertion of Proposition 6.2 is true if $\deg F < \mu$.

To complete the induction, we next show the assertion of Proposition 6.2 in the case where $\deg F = \mu$ on the assumption that the assertions of Propositions 6.2 and 6.3 are true if $\deg F < \mu$ and $\deg F \leq \mu$, respectively. First, assume that $I_F = \emptyset$. Then, (F_σ, G) satisfies the weak Shestakov-Umirbaev condition for some $\sigma \in \mathfrak{S}_3$ and $G \in \mathcal{A}$. Without loss of generality, we may assume that $\sigma = \mathrm{id}$. By Claim 1(iv), we may also assume that (F, G) satisfies the Shestakov-Umirbaev condition by replacing G if necessary. Since $I_F = \emptyset$, it follows that F does not admit an elementary reduction. In view of (SU1), this implies that $(f_1, f_2) \neq (g_1, g_2)$ and $k[f_1, f_2] \neq k[g_1, g_2]$. Then, we know by the following claim that $F \circ E$ belongs to $\mathcal{A}$ for $E \in \mathcal{E}$ if $\deg F \circ E \leq \deg F$.

CLAIM 2. *Assume that* (F, G) *satisfies the weak Shestakov-Umirbaev condition for some* $G \in \mathcal{A}$, *and* $E \in \mathcal{E}_i$ *satisfies* $\deg F \circ E \leq \deg F$, *where* $i \in \{1, 2, 3\}$. *If* $i = 1$ *or* $i = 2$, *or if* $i = 3$ *and* $k[f_1, f_2] \neq k[g_1, g_2]$, *then* $F \circ E$ *belongs to* $\mathcal{A}$.

PROOF. In the notation of Proposition 4.4, one of the pairs $(F \circ E, G)$, $(F \circ E, G')$ and $((F \circ E)_\tau, G'')$ satisfies the weak Shestakov-Umirbaev condition. Since G belongs to $\mathcal{A}$, so do G' and G''. Hence, in each case, $F \circ E$ admits a weak Shestakov-Umirbaev reduction to an element of $\mathcal{A}$. Therefore, $F \circ E$ belongs to $\mathcal{A}$. $\square$

Therefore, the assertion of Proposition 6.2 is true if $\deg F = \mu$ and $I_F = \emptyset$.

Next, assume that $I_F \neq \emptyset$, say I_F contains 3. We have to check that $F \circ E_i$ belongs to $\mathcal{A}$ for any $E_i \in \mathcal{E}_i$ with $\deg F \circ E_i \leq \deg F$ for each $i \in \{1, 2, 3\}$. By Claim 1(i), this is clear if $i = 3$. Since the cases $i = 1$ and $i = 2$ are similar, we only consider the case where $i = 1$.

Since we assume that I_F contains 3, there exists $E_3 \in \mathcal{E}_3$ such that $G := F \circ E_3$ belongs to $\mathcal{A}$ and $\deg G < \deg F$. By Claim 1(i), we may assume that $g_3^{\mathbf{w}}$ does not belong to $k[S_3]^{\mathbf{w}}$ by replacing E_3 if necessary. Set $\phi_i = F(E_i(x_i) - x_i)$ for $i = 1, 3$. Then, ϕ_i belongs to $k[S_i]$ for $i = 1, 3$, and $g_3 = f_3 + \phi_3$. Since $\deg F \circ E_1 \le \deg F$ and $\deg G < \deg F$, we have $\deg \phi_1 \le \deg f_1$, $\phi_3^{\mathbf{w}} = -f_3^{\mathbf{w}}$ and $\deg g_3 < \deg f_3$.

CLAIM 3. *$F \circ E_1$ belongs to $\mathcal{A}$ if one of the following conditions holds*:
(i) *$E_1(x_1) - x_1$ belongs to $k[x_2]$, or equivalently, ϕ_1 belongs to $k[f_2]$.*
(ii) *$f_1^{\mathbf{w}}$ or $f_3^{\mathbf{w}}$ belongs to $k[f_2^{\mathbf{w}}]$.*
(iii) *$f_3^{\mathbf{w}} \approx f_1^{\mathbf{w}} + c(f_2^{\mathbf{w}})^p$ for some $c \in k$ and $p \in \boldsymbol{N}$.*

PROOF. (i) If $E_1(x_1) - x_1$ belongs to $k[x_2]$, then we can define a triangular automorphism H of $k[\mathbf{x}]$ by $H(x_2) = x_2$ and $H(x_i) = E_i(x_i)$ for $i = 1, 3$. Since $\deg F \circ E_3 < \deg F$ and 3 is contained in I_F, it follows from Claim 1(iii) that $F \circ E_1$ belongs to $\mathcal{A}$.

(ii) If $f_3^{\mathbf{w}}$ belongs to $k[f_2^{\mathbf{w}}]$, then $\deg(f_3 - cf_2^r) < \deg f_3$ for some $c \in k \setminus \{0\}$ and $r \in \boldsymbol{N}$. Define a triangular automorphism H of $k[\mathbf{x}]$ by $H(x_2) = x_2$, $H(x_3) = x_3 - cx_2^r$ and $H(x_1) = E_1(x_1)$. Since $\deg(F \circ H)(x_3) < \deg f_3$ and 3 is contained in I_F, it follows from Claim 1(iii) that $F \circ E_1$ belongs to $\mathcal{A}$. If $f_1^{\mathbf{w}}$ belongs to $k[f_2^{\mathbf{w}}]$, then I_F contains 1 by Claim 1(iv), since $I_F \setminus \{2\} \ne \emptyset$. Therefore, $F \circ E_1$ belongs to $\mathcal{A}$ by Claim 1(i).

(iii) By assumption, there exists $c' \in k \setminus \{0\}$ such that $\deg f' < \deg f_3$, where $f' = f_3 + c'(f_1 + cf_2^p)$. Define $E_1', E_1'' \in \mathcal{E}_1$ and $E_3' \in \mathcal{E}_3$ by $E_1'(x_1) = x_1 + cx_2^p - (1/c')x_3$, $E_1''(x_1) = (c')^{-1}(x_3 + c'(x_1 + cx_2^p))$ and $E_3'(x_3) = x_3 + c'(x_1 + cx_2^p)$. Then, we have $\deg F \circ E_3' < \deg F$, because $(F \circ E_3')(x_3) = f'$. Since 3 is contained in I_F by assumption, $F \circ E_3'$ belongs to $\mathcal{A}$ by Claim 1(i). Hence, $F' := (F \circ E_3') \circ E_1'$ belongs to $\mathcal{A}$ by the induction assumption of Proposition 6.2. Since $F' = (-(1/c')f_3, f_2, f')$, this implies that $F \circ E_1'' = ((1/c')f', f_2, f_3)$ belongs to $\mathcal{A}$. By assumption, it follows that $\deg f_3 = \deg f_1$. Hence, $\deg F \circ E_1'' < \deg F$. Thus, 1 belongs to I_F. Therefore, $F \circ E_1$ belongs to $\mathcal{A}$ by Claim 1(i). □

In the case where 2 belongs to I_F besides 3, the statement of Claim 3 is true if we interchange f_2 and f_3. Hence, we obtain the following claim.

CLAIM 4. *Assume that 2 is contained in I_F. If ϕ_1 belongs to $k[f_3]$, or if $f_1^{\mathbf{w}}$ or $f_2^{\mathbf{w}}$ belongs to $k[f_3^{\mathbf{w}}]$, then $F \circ E_1$ belongs to $\mathcal{A}$.*

Now, there exist five cases to be considered as follows:

$$\begin{aligned}
&(1)\ \deg f_1 = \deg f_2 = \deg f_3\,, \qquad (2)\ \deg f_1 < \deg f_2 = \deg f_3\,,\\
&(3)\ \deg f_3 < \deg f_1 = \deg f_2\,, \qquad (4)\ \deg f_2 < \deg f_3 = \deg f_1\,,\\
&(5)\ \deg f_l < \deg f_m \text{ for each } l \in \{1,2,3\} \setminus \{m\} \text{ for some } m \in \{1,2,3\}\,.
\end{aligned}$$

Here, we remark that the cases (1) through (4) can be excluded from consideration in the case where $\operatorname{rank} \mathbf{w} = 3$. In fact, $\deg f_i = \deg f_j$ implies $f_i^{\mathbf{w}} \approx f_j^{\mathbf{w}}$ for each i and j if $\operatorname{rank} \mathbf{w} = 3$. Hence, it immediately follows from Claim 3(ii) and (iii) that $F \circ E_1$ belongs to $\mathcal{A}$ in the cases (1) through (4). For this reason, Claim 5 and the statement (I) of Claim 6 below are not necessary when considering $\mathbf{w}$ with $\operatorname{rank}\mathbf{w} = 3$.

CLAIM 5. *$F \circ E_1$ belongs to $\mathcal{A}$ if one of the following holds*:
(i) *$f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ are algebraically independent over k.*
(ii) *F satisfies one of* (1), (2) *and* (3).

PROOF. By Claim 3(i), we may assume that ϕ_1 belongs to $k[f_2, f_3] \setminus k[f_2]$. Then, it follows that, if $\deg f_1 < \deg f_3$, then $f_2^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ are algebraically dependent over k. In fact, since $\deg \phi_1 \leq \deg f_1 < \deg f_3$, and since ϕ_1 belongs to $k[f_2, f_3] \setminus k[f_2]$, we have $\deg \phi_1 < \deg^{S_1} \phi_1$. Hence, $(f_2^{\mathbf{w}})^p \approx (f_3^{\mathbf{w}})^q$ for some $p, q \in \boldsymbol{N}$ by Lemma 3.2.

(i) Recall that $f_3^{\mathbf{w}} \approx \phi_3^{\mathbf{w}}$ and ϕ_3 is an element of $k[S_3]$. Hence, $f_3^{\mathbf{w}}$ belongs to $k[S_3]^{\mathbf{w}}$. Since $f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ are algebraically independent over k, we have $k[S_3]^{\mathbf{w}} = k[f_1^{\mathbf{w}}, f_2^{\mathbf{w}}]$. Thus, $f_3^{\mathbf{w}}$ is a polynomial in $f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ over k. By Claim 3(ii), we may assume that $f_3^{\mathbf{w}}$ does not belong to $k[f_2^{\mathbf{w}}]$. Then, it follows that $\deg f_1 \leq \deg f_3$. We show that $\deg f_1 = \deg f_3$ by contradiction. Supposing $\deg f_1 < \deg f_3$, we get that $f_2^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ are algebraically dependent over k as remarked above. Since $f_3^{\mathbf{w}}$ is an element of $k[f_1^{\mathbf{w}}, f_2^{\mathbf{w}}] \setminus k[f_2^{\mathbf{w}}]$, it follows that $f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ are algebraically dependent over k, a contradiction. Thus, $\deg f_1 = \deg f_3$. This implies that $f_3^{\mathbf{w}} \approx f_1^{\mathbf{w}} + c(f_2^{\mathbf{w}})^p$ for some $c \in k$ and $p \in \boldsymbol{N}$. Therefore, $F \circ E_1$ belongs to $\mathcal{A}$ by Claim 3(iii).

(ii) By (i), we may assume that $f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ are algebraically dependent over k. Then, $f_1^{\mathbf{w}} \approx f_2^{\mathbf{w}}$ follows from $\deg f_1 = \deg f_2$ in the cases (1) and (3). If (2), it follows from $\deg f_1 < \deg f_3$ that $f_2^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ are algebraically dependent over k as remarked above. Then, $f_2^{\mathbf{w}} \approx f_3^{\mathbf{w}}$ follows from $\deg f_3 = \deg f_2$. By Claim 3(ii), $F \circ E_1$ belongs to $\mathcal{A}$ in every case. □

Let us complete the proof of Proposition 6.2 by contradiction. Suppose to the contrary that $F \circ E_1$ does not belong to $\mathcal{A}$. Then, the conditions (i), (ii) and (iii) of Claim 3 and (i) and (ii) of Claim 5 cannot be satisfied. In particular, F satisfies (4) or (5). Furthermore, $f_1^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ must be algebraically independent over k if (4). We show that, if F satisfies (5) for $m = 2$, and if $f_2^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}]$, then $f_3^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}]$. Supposing the contrary, we have $f_3^{\mathbf{w}} \approx (f_1^{\mathbf{w}})^p$ for some $p \in \boldsymbol{N}$. Then, we have $p \geq 2$ in view of Claim 3(iii). Hence, $\deg f_1 < \deg f_3$. We verify that $f = f_3$, $g = f_2$ and $\phi = \phi_1$ satisfy the assumptions of Lemma 3.3(ii) with $\deg \phi < \deg f$. Recall that ϕ_1 is an element of $k[f_2, f_3]$ such that $\deg \phi_1 \leq \deg f_1$. Since $\deg f_1 < \deg f_3$, we have $\deg \phi_1 < \deg f_3$. On account of Claim 3(i), ϕ_1 cannot belong to $k[f_2]$. Thus, it follows that $\deg \phi_1 < \deg^{S_1} \phi_1$. By assumption, $f_2^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}]$. Since $f_3^{\mathbf{w}} \approx (f_1^{\mathbf{w}})^p$, it follows that $f_2^{\mathbf{w}}$ does not belong to $k[f_3^{\mathbf{w}}]$. By the condition (5) for $m = 2$, we have $\deg f_3 < \deg f_2$. Thus, the assumptions of Lemma 3.3(ii) are satisfied, and so we conclude that

$$\deg \phi_1 \geq (3-2)\frac{1}{2} \deg f_3 + \deg df_2 \wedge df_3 > \frac{1}{2} \deg f_3 = \frac{p}{2} \deg f_1 \geq \deg f_1 .$$

This contradicts that $\deg \phi_1 \leq \deg f_1$. Therefore, $f_3^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}]$ if F satisfies (5) for $m = 2$, and $f_2^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}]$.

CLAIM 6. *If $F \circ E_1$ does not belong to $\mathcal{A}$, then one of the following holds*:
(I) $\deg f_2 < \deg f_1$, $\deg f_1 = \deg f_3$, $f_1^{\mathbf{w}} \not\approx f_3^{\mathbf{w}}$, *and $f_1^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ do not belong to $k[f_2^{\mathbf{w}}]$ and $k[f_1^{\mathbf{w}}, f_2^{\mathbf{w}}]$, respectively.*
(II) $\deg f_i < \deg f_j$, $\deg f_3 < \deg f_j$, *and $f_j^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ do not belong to $k[f_i^{\mathbf{w}}]$ for some $(i, j) \in \{(1, 2), (2, 1)\}$.*
(III) $\deg f_1 < \deg f_j$, $\deg f_i < \deg f_j$, *$f_1^{\mathbf{w}}$ and $f_j^{\mathbf{w}}$ do not belong to $k[f_i^{\mathbf{w}}]$, and ϕ_1 belongs to $k[S_1] \setminus k[f_i]$ for some $(i, j) \in \{(2, 3), (3, 2)\}$.*

PROOF. We show that F satisfies (I) in the case (4), where $\deg f_2 < \deg f_1$ and $\deg f_1 = \deg f_3$. On account of Claim 3(ii) and (iii), $f_l^{\mathbf{w}}$ does not belong to $k[f_2^{\mathbf{w}}]$ for $l = 1, 3$, and $f_3^{\mathbf{w}} \not\approx f_1^{\mathbf{w}}$. We show that $f_3^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}, f_2^{\mathbf{w}}]$ by contradiction. Supposing the contrary, we have $f_3^{\mathbf{w}} = af_1^{\mathbf{w}} + b(f_2^{\mathbf{w}})^p$ for some $a, b \in k$ with $(a, b) \neq (0, 0)$ and $p \geq 2$, since $\deg f_3 = \deg f_1$ and $\deg f_1 > \deg f_2$. If $a = 0$ or $b = 0$, then $f_3^{\mathbf{w}}$ belongs to $k[f_2^{\mathbf{w}}]$ or $f_3^{\mathbf{w}} \approx f_1^{\mathbf{w}}$, a contradiction. Hence, $a \neq 0$ and $b \neq 0$. It follows that $\deg f_1^{\mathbf{w}} = \deg (f_2^{\mathbf{w}})^p$. Owing to Claim 5(i), $f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ must be algebraically dependent over k. Thus, $f_1^{\mathbf{w}} \approx (f_2^{\mathbf{w}})^p$, and so $f_1^{\mathbf{w}}$ belongs to $k[f_2^{\mathbf{w}}]$, a contradiction. Therefore, $f_3^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}, f_2^{\mathbf{w}}]$. This proves that F satisfies (I) in the case (4).

We show that F satisfies (II) or (III) in the case (5). Since the conditions (i), (ii) and (iii) of Claim 3 are not satisfied by supposition, (II) holds for $(i, j) = (2, 1)$ if $m = 1$, and (III) holds for $(i, j) = (2, 3)$ if $m = 3$. Assume that $m = 2$. As shown before this claim, if $f_2^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}]$, then neither does $f_3^{\mathbf{w}}$. Hence, (II) holds for $(i, j) = (1, 2)$. If $f_2^{\mathbf{w}}$ belongs to $k[f_1^{\mathbf{w}}]$, then I_F contains 2 by Claim 1(iv) since $I_F \setminus \{1\} \neq \emptyset$. By Claim 4, we know that ϕ_1 belongs to $k[S_1] \setminus k[f_3]$, and $f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ do not belong to $k[f_3^{\mathbf{w}}]$. Therefore, (III) holds for $(i, j) = (3, 2)$. □

We consider the cases (I) and (II) together. Recall that $\phi_3^{\mathbf{w}} \approx f_3^{\mathbf{w}}$, $\deg g_3 < \deg f_3$, $g_3^{\mathbf{w}}$ does not belong to $k[S_3]^{\mathbf{w}}$, and $G = (f_1, f_2, g_3)$ belongs to $\mathcal{A}$. We establish the inequality

$$\deg g_3 < \deg f_j - \deg f_i + \deg df_1 \wedge df_2 \tag{6.2}$$

by contradiction, where we set $(i, j) = (2, 1)$ in the case (I). When (I), $f_3^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}, f_2^{\mathbf{w}}]$, and hence neither does $\phi_3^{\mathbf{w}}$. The same holds true in the case (II) because $k[f_1^{\mathbf{w}}, f_2^{\mathbf{w}}] = k[f_i^{\mathbf{w}}, f_j^{\mathbf{w}}]$, $\deg f_3 < \deg f_j$ and $f_3^{\mathbf{w}}$ does not belong to $k[f_i^{\mathbf{w}}]$. Since ϕ_3 is an element of $k[S_3]$, it follows that $\deg \phi_3 < \deg^{S_3} \phi_3$ in both cases. We show that $G' := (f_j, f_i, g_3)$ satisfies the assumptions of Proposition 6.3. Clearly, G' is an element of $\mathcal{A}$, since so is G by assumption. By the conditions in (I) and (II), we know $\deg f_i < \deg f_j$, $\deg \phi_3 = \deg f_3 \leq \deg f_j$, and that $f_j^{\mathbf{w}}$ does not belong to $k[f_i^{\mathbf{w}}]$. Hence, it follows from Lemma 3.3(ii) that $\deg f_i = 2\delta$ and $\deg f_j = s\delta$ for some $\delta \in \Gamma$ and an odd number $s \geq 3$. Since (6.2) is supposed to be false, we get

$$(s - 2)\delta + \deg df_1 \wedge df_2 = \deg f_j - \deg f_i + \deg df_1 \wedge df_2$$

$$\leq \deg g_3 < \deg f_3 \leq \deg f_j = s\delta .$$

Since $k[S_3]^{\mathbf{w}}$ does not contain $g_3^{\mathbf{w}}$, neither does $k[f_i]^{\mathbf{w}}$. Thus, G' satisfies the assumptions of Proposition 6.3. Because $\deg G' < \deg F = \mu$, we may conclude that there exists $E_3' \in \mathcal{E}_3$ such that $\deg G' \circ E_3' < \deg G'$ by induction assumption. This contradicts that $g_3^{\mathbf{w}}$ does not belong to $k[S_3]^{\mathbf{w}}$, thereby proves that (6.2) is true. We show that (F', G') satisfies the weak Shestakov-Umirbaev condition, where $F' = (f_j, f_i, f_3)$. The first two conditions of (SU1$'$), and (SU2$'$) are obvious. The last condition of (SU1$'$), and (SU5) follow from the construction of g_3. (SU3$'$) and the first condition of (SU4) are included in (I) and (II). As mentioned after (6.2), $f_3^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}, f_2^{\mathbf{w}}]$, which is the last condition of (SU4). (SU6) is due to (6.2). Thus, (F', G') satisfies the weak Shestakov-Umirbaev condition. It follows from Claim 2 that $F' \circ E$ belongs to $\mathcal{A}$ for each $E \in \mathcal{E}_l$ for $l = 1, 2$ if $\deg F' \circ E \leq \deg F'$. In particular, $(F \circ E_1) \circ H = F' \circ (H \circ E_1 \circ H)$ belongs to $\mathcal{A}$, where $H = (x_j, x_i, x_3)$. Actually, $H \circ E_1 \circ H$ belongs to $\mathcal{E}_j$, and

$$\deg F' \circ H \circ E_1 \circ H = \deg F \circ E_1 \circ H = \deg F \circ E_1 \leq \deg F = \deg F' .$$

This implies that $F \circ E_1$ belongs to $\mathcal{A}$. Therefore, we are led to a contradiction.

Finally, we derive a contradiction in the case (III). It follows that $\deg \phi_1 < \deg^{S_1} \phi_1$, since ϕ_1 is an element of $k[f_i, f_j] \setminus k[f_i]$ with $\deg \phi_1 \leq \deg f_1 < \deg f_j$. Since $\deg f_i < \deg f_j$, and since $f_j^{\mathbf{w}}$ does not belong to $k[f_i^{\mathbf{w}}]$, we know that f_i, f_j and ϕ_1 satisfy the assumptions of Lemma 3.3(ii). Hence, there exist $\delta \in \Gamma$ and an odd number $s \geq 3$ such that $\deg f_i = 2\delta$, $\deg f_j = s\delta$ and

$$(s-2)\delta + \deg df_2 \wedge df_3 = (s-2)\delta + \deg df_i \wedge df_j \leq \deg \phi_1 \leq \deg f_1 < \deg f_j = s\delta .$$

Thus, F_τ satisfies (6.1) for $\tau \in \mathfrak{S}_3$ with $\tau(1) = j$, $\tau(2) = i$ and $\tau(3) = 1$. Note that F_τ is an element of $\mathcal{A}$ with $\deg F_\tau = \mu$ since so is F. As $f_1^{\mathbf{w}}$ does not belong to $k[f_i^{\mathbf{w}}]$, the assumptions of Proposition 6.3 are fulfilled for F_τ. Hence, by induction assumption, we conclude that $\deg F_\tau \circ E_3' < \deg F_\tau$ and $F_\tau \circ E_3'$ belongs to $\mathcal{A}$ for some $E_3' \in \mathcal{E}_3$. Thus, I_{F_τ} contains 3, and so I_F contains 1. Therefore, $F \circ E_1$ belongs to $\mathcal{A}$ by Claim 1(i), a contradiction.

This proves that the statement of Proposition 6.2 holds for each $F \in \mathcal{A}$ with $\deg F = \mu$. Thus, the proofs of Propositions 6.2 and 6.3 are completed by induction. Thereby, we have completed the proof Theorem 2.1.

7. Relations with the theory of Shestakov-Umirbaev. In this section, we discuss relations with the original theory of Shestakov-Umirbaev. Throughout this section, we assume that $\Gamma = \mathbf{Z}$ and $\mathbf{w} = (1, 1, 1)$. Hence, $\deg F \geq |\mathbf{w}| = 3$ for each $F \in \mathrm{Aut}_k\, k[\mathbf{x}]$. First, we recall the notions of reductions of types I, II, III and IV defined by Shestakov-Umirbaev [10, Definitions 1, 2, 3 and 4].

Let $F = (f_1, f_2, f_3)$ be an element of $\mathrm{Aut}_k\, k[\mathbf{x}]$ such that $\deg f_1 = 2l$ and $\deg f_2 = sl$ for some $l \in \mathbf{N}$ and an odd number $s \geq 3$.

(1) F is said to *admit a reduction of type I* if $2l < \deg f_3 \leq sl$, $f_3^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}, f_2^{\mathbf{w}}]$, and there exists $\alpha \in k \setminus \{0\}$ for which $g_1 := f_1$ and $g_2 := f_2 - \alpha f_3$ satisfy the following conditions:

(i) $\deg g_2 = sl$, and $g_1^{\mathbf{w}}$ and $g_2^{\mathbf{w}}$ are algebraically dependent over k.

(ii) $\deg g_3 < \deg f_3$ and $\deg dg_1 \wedge dg_3 < sl + \deg dg_1 \wedge dg_2$ for some $\phi \in k[g_1, g_2]$, where $g_3 = f_3 + \phi$.

(2) F is said to *admit a reduction of type II* if $s = 3$, $(3/2)l < \deg f_3 \leq 2l$, $f_1^{\mathbf{w}} \not\approx f_3^{\mathbf{w}}$, and there exist $\alpha, \beta \in k$ with $(\alpha, \beta) \neq (0, 0)$ for which $g_1 := f_1 - \alpha f_3$ and $g_2 := f_2 - \beta f_3$ satisfy the following conditions:

(iii) $\deg g_1 = 2l$, $\deg g_2 = 3l$, and $g_1^{\mathbf{w}}$ and $g_2^{\mathbf{w}}$ are algebraically dependent over k.

(iv) $\deg g_3 < \deg f_3$ and $\deg dg_1 \wedge dg_3 < 3l + \deg dg_1 \wedge dg_2$ for some $\phi \in k[g_1, g_2]$, where $g_3 = f_3 + \phi$.

Next, let $F = (f_1, f_2, f_3)$ be an element of $\mathrm{Aut}_k\, k[\mathbf{x}]$ such that $\deg f_1 = 2l$, and either $\deg f_2 = 3l$ and $l < \deg f_3 \leq (3/2)l$, or $(5/2)l < \deg f_2 \leq 3l$ and $\deg f_3 = (3/2)l$ for some $l \in \boldsymbol{N}$. Assume that there exist $\alpha, \beta, \gamma \in k$ such that $g_1 := f_1 - \beta f_3$ and $g_2 := f_2 - \gamma f_3 - \alpha f_3^2$ satisfy the following conditions:

(v) $\deg g_1 = 2l$, $\deg g_2 = 3l$, and $g_1^{\mathbf{w}}$ and $g_2^{\mathbf{w}}$ are algebraically dependent over k.

(vi) $\deg g_3 \leq (3/2)l$ and $\deg dg_1 \wedge dg_3 < 3l + \deg dg_1 \wedge dg_2$ for some $\sigma \in k \setminus \{0\}$ and $g \in k[g_1, g_2] \setminus k$, where $g_3 = \sigma f_3 + g$.

(3) F is said to *admit a reduction of type III* if we can choose $\alpha, \beta, \gamma, \sigma$ and g so that $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ and $\deg g_3 < l + \deg dg_1 \wedge dg_2$.

(4) F is said to *admit a reduction of type IV* if we can choose $\alpha, \beta, \gamma, \sigma$ and g so that $\deg(g_2 - \mu g_3^2) \leq 2l$ for some $\mu \in k \setminus \{0\}$.

We also say that F admits a reduction of type I (resp. II, III and IV) if F_σ satisfies (1) (resp. (2), (3) and (4)) for some $\sigma \in \mathfrak{S}_3$.

Here, we note that the conditions (i), (iii) and (v) are equivalent to the condition that g_1, g_2 is a "two-reduced pair", since the conditions on $\deg g_1$ and $\deg g_2$ imply $g_1^{\mathbf{w}} \notin k[g_2^{\mathbf{w}}]$ and $g_2^{\mathbf{w}} \notin k[g_1^{\mathbf{w}}]$. Although Shestakov-Umirbaev [10] considered the "Poisson bracket" $[f, g]$ instead of $df \wedge dg$ for $f, g \in k[\mathbf{x}]$, the degrees of $[f, g]$ and $df \wedge dg$ are defined in the same way.

The following theorem is a consequence of Theorem 2.1 and Proposition 5.4.

THEOREM 7.1. *No tame automorphism of $k[\mathbf{x}]$ admits a reduction of type IV.*

PROOF. Suppose to the contrary that F satisfies (4) for some $F \in \mathrm{T}_k\, k[\mathbf{x}]$. Then, g_1 and g_2 appearing in the condition satisfy $\deg g_1 = 2l$ and $\deg g_2 = 3l$. Moreover, since $\deg(g_2 - \mu g_3^2) \leq 2l < (5/2)l < \deg g_2$ for some $\mu \in k \setminus \{0\}$, we have $g_2^{\mathbf{w}} \approx (g_3^{\mathbf{w}})^2$. Hence, $\deg g_3 = (3/2)l$. Since F belongs to $\mathrm{T}_k\, k[\mathbf{x}]$, so does $H := (g_2, g_1, g_3)$. We show that H satisfies the assumptions of Proposition 5.4 for $s = 3$ and $\delta = l$. The degrees of g_2, g_1 and g_3 satisfy (5.2), and $g_3^{\mathbf{w}}$ does not belong to $k[g_1^{\mathbf{w}}]$ since $\deg g_3 < \deg g_1$. We verify that $\deg dg_1 \wedge dg_2 \leq (1/2)l$, which gives (5.5) that

$$\deg dg_1 \wedge dg_2 \leq \frac{1}{2}l < \frac{3}{2}l - l + 1 \leq \deg g_3 - (3-2)l + \min\{l, \varepsilon\},$$

since $\varepsilon = \deg dg_1 \wedge dg_2 \wedge dg_3 = 3$ and $l \geq 1$. By definition, g is an element of $k[g_1, g_2] \setminus k$ such that $\deg g \leq \max\{\deg f_3, \deg g_3\} = (3/2)l < \deg g_i$ for $i = 1, 2$. Hence, $g^{\mathbf{w}}$ does not belong to $k[g_1^{\mathbf{w}}, g_2^{\mathbf{w}}]$, and so $\deg g < \deg^U g$, where $U = \{g_1, g_2\}$. Since $\deg g_1 = 2l$ and $\deg g_2 = 3l$, it follows that $\deg g_1 < \deg g_2$ and $g_2^{\mathbf{w}}$ does not belong to $k[g_1^{\mathbf{w}}]$. Thus,

$$\deg g \geq (3-2)l + \deg dg_1 \wedge dg_2 = l + \deg dg_1 \wedge dg_2$$

by Lemma 3.3(ii). Since $\deg g \leq (3/2)l$, we conclude that $\deg dg_1 \wedge dg_2 \leq (1/2)l$. Therefore, H satisfies the assumptions of Proposition 5.4. Take $\phi_2 \in k[g_1, g_3]$ so that $(g_2')^{\mathbf{w}}$ does not belong to $k[g_1, g_3]^{\mathbf{w}}$, where $g_2' = g_2 + \phi_2$. Then, $\deg g_2' \leq 2l$ since $\deg(g_2 - \mu g_3^2) \leq 2l$. By Proposition 5.4(1), $H' := (g_2', g_1, g_3)$ does not admit an elementary reduction. Since H belongs to $\mathrm{T}_k\, k[\mathbf{x}]$, so does H'. Furthermore, $\deg H' > 3$, because $\deg g_i > l \geq 1$ for $i = 1, 3$. Thus, H' admits a Shestakov-Umirbaev reduction by Theorem 2.1. Hence, there exist $\sigma \in \mathfrak{S}_3$ and $K \in \mathrm{Aut}_k\, k[\mathbf{x}]$ such that (H'_σ, K) satisfies the Shestakov-Umirbaev condition. Since $g_2^{\mathbf{w}} \approx (g_3^{\mathbf{w}})^2$ as mentioned, we know that $\sigma = \mathrm{id}$ by Proposition 5.4(2). Hence, (H', K) satisfies the Shestakov-Umirbaev condition. Consequently, we get $\deg g_1 < \deg g_2'$ by (P7). This contradicts that $\deg g_1 = 2l$ and $\deg g_2' \leq 2l$. Therefore, F does not admit a reduction of type IV. □

To conclude that Nagata's automorphism is not tame, Shestakov-Umirbaev [10, Theorem 1] showed that, if $\deg F > 3$ for $F \in \mathrm{T}_k\, k[\mathbf{x}]$, then F admits an elementary reduction or a reduction of one of the types I, II, III and IV. With the aid of the following proposition, the criterion of Shestakov-Umirbaev is derived from Theorem 2.1.

PROPOSITION 7.2. *Assume that (F, G) satisfies the Shestakov-Umirbaev condition for $F, G \in \mathrm{Aut}_k\, k[\mathbf{x}]$. If $(f_1, f_2) = (g_1, g_2)$, then F admits an elementary reduction. If $(f_1, f_2) \neq (g_1, g_2)$, then F admits a reduction of one of the types I, II and III.*

PROOF. The first assertion follows from (SU1) and (SU5). We show the last assertion. By (SU1), we may write $g_1 = f_1 + af_3^2 + cf_3$, $g_2 = f_2 + bf_3$ and $g_3 = f_3 + \phi$, where $a, b, c \in k$ and $\phi \in k[g_1, g_2]$. Since $(f_1, f_2) \neq (g_1, g_2)$ by assumption, we have $(a, b, c) \neq (0, 0, 0)$. By (SU3), there exist $l \in \mathbf{N}$ and an odd number $s \geq 3$ such that $\deg g_1 = sl$ and $\deg g_2 = 2l$. Then, it follows that $l < \deg f_3 \leq sl$ by (P7). Put $\tau = (1, 2)$. We show that F_τ satisfies (1) for $\alpha = -c$ if $2l < \deg f_3 \leq sl$, (2) for $(\alpha, \beta) = (-b, -c)$ if $(3/2)l < \deg f_3 \leq 2l$, and (3) for $(\alpha, \beta, \gamma) = (-a, -b, -c)$, $\sigma = 1$ and $g = \phi$ if $l < \deg f_3 \leq (3/2)l$.

Note that $\deg f_2 = 2l$ by (SU2), and that $\deg f_1 = sl$ if $\deg f_3 \neq (3/2)l$, and $(5/2)l < \deg f_1 \leq 3l$ otherwise by (P5). Moreover, $s = 3$ if $\deg f_3 \leq 2l$ by (P11). From this, we see that the conditions on the degrees of f_1 and f_2 are satisfied in every case. It follows that $a = b = 0$ if $2l < \deg f_3 \leq sl$ by (P11), and $a = 0$ if $(3/2)l < \deg f_3 \leq 2l$, since $\deg f_3^2 > 3l = \deg g_1$. Hence, $g_2 = f_2$ and $g_1 = f_1 - \alpha f_3$ for $\alpha = -c$ if $2l < \deg f_3 \leq sl$, $g_2 = f_2 - \alpha f_3$ and $g_1 = f_1 - \beta f_3$ for $(\alpha, \beta) = (-b, -c)$ if $(3/2)l < \deg f_3 \leq 2l$, and $g_2 = f_2 - \beta f_3$ and $g_1 = f_1 - \gamma f_3 - \alpha f_3^2$ for $(\alpha, \beta, \gamma) = (-a, -b, -c)$ if $l < \deg f_3 \leq (3/2)l$, in which $\alpha \neq 0$, $(\alpha, \beta) \neq (0, 0)$, and $(\alpha, \beta, \gamma) \neq (0, 0, 0)$, respectively. Besides, $g = \phi$ in (iv) cannot be an element of k, since $\deg g_3 < \deg f_3$ by (SU5). So, we verify that (i) through (vi) are satisfied for g_2, g_1 and g_3. As mentioned, we have $\deg g_2 = 2l$ and $\deg g_1 = sl$, where $s = 3$ if $\deg f_3 \leq 2l$. By (SU3), $g_2^{\mathbf{w}}$ and $g_1^{\mathbf{w}}$ are algebraically dependent over k. Thus, (i), (iii) and (v) are satisfied. The first conditions in (ii) and (iv) are the same as (SU5). If $\deg f_3 \leq (3/2)l$, then $\deg g_3 < \deg f_3 \leq (3/2)l$ by (SU5), the first condition in (vi). The second conditions in (ii), (iv) and (vi) follow from (SU6), since

$$\begin{aligned}\deg dg_2 \wedge dg_3 &\leq \deg g_2 + \deg g_3 < \deg g_2 + (\deg g_1 - \deg g_2 + \deg dg_1 \wedge dg_2) \\ &= sl + \deg dg_1 \wedge dg_2 .\end{aligned}$$

Therefore, (i) through (vi) are satisfied for g_2, g_1 and g_3.

Let us check the other conditions. It follows from (P8) that $f_2^{\mathbf{w}} \not\approx f_3^{\mathbf{w}}$. Hence, F_τ satisfies (2) if $(3/2)l < \deg f_3 \leq 2l$. We have already shown that F_τ satisfies the assumption of (3) if $l < \deg f_3 \leq (3/2)l$. Since the last condition in (3) is the same as (SU6), F_τ satisfies (3) in this case. We show that $f_3^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}, f_2^{\mathbf{w}}]$ as required in (1). By (P8), $f_3^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}]$ nor $k[f_2^{\mathbf{w}}]$. Since $\deg f_3 \leq \deg f_1$ by (P7), we have $\deg f_3 < \deg f_1 + \deg f_2 = \deg f_1 f_2$. Hence, $f_3^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}, f_2^{\mathbf{w}}]$. This proves that F_τ satisfies (1) if $2l < \deg f_3 \leq sl$. Therefore, F admits a reduction of one of the types I, II and III if $(f_1, f_2) \neq (g_1, g_2)$. □

8. Remarks. In closing, we make some remarks on Shestakov-Umirbaev reductions. As established in Section 6, for each $F \in \mathrm{T}_k\, k[\mathbf{x}]$ with $\deg_{\mathbf{w}} F > |\mathbf{w}|$, there exists a sequence $(G_i)_{i=0}^r$ of elements of $\mathrm{T}_k\, k[\mathbf{x}]$ for some $r \in \boldsymbol{N}$ such that $G_0 = F$, $\deg G_r = |\mathbf{w}|$, and G_{i+1} is an elementary reduction or a *weak* Shestakov-Umirbaev reduction of G_i for each i. We have a more precise result as follows.

COROLLARY 8.1. *For each $F \in \mathrm{T}_k\, k[\mathbf{x}]$ with $\deg F > |\mathbf{w}|$, there exists a sequence $(G_i)_{i=0}^r$ of elements of $\mathrm{T}_k\, k[\mathbf{x}]$ for some $r \in \boldsymbol{N}$ such that $G_0 = F$, $\deg G_r = |\mathbf{w}|$, and G_{i+1} is an elementary reduction or a Shestakov-Umirbaev reduction of G_i for each i.*

PROOF. Let $\mathcal{B}$ be the set of $F \in \mathrm{T}_k\, k[\mathbf{x}]$ with $\deg F > |\mathbf{w}|$ for which there does not exist a sequence as claimed. Suppose to the contrary that $\mathcal{B}$ is not empty. Then, we can find $F \in \mathcal{B}$ such that $\deg F = \min\{\deg H;\, H \in \mathcal{B}\} > |\mathbf{w}|$, since Σ is a well-ordered set by Lemma 6.1(ii). Since F is tame, there exists $G \in \mathrm{T}_k\, k[\mathbf{x}]$ which is an elementary reduction or a Shestakov-Umirbaev reduction of F by Theorem 2.1. Then, $\deg G < \deg F$ by (P6). Hence, G does not belong to $\mathcal{B}$ by the minimality of $\deg F$. It follows from the definition of $\mathcal{B}$ that $\deg G = |\mathbf{w}|$ or there exists a sequence as claimed for G. In either case, F cannot be an element of $\mathcal{B}$, a contradiction. Therefore, $\mathcal{B}$ is empty. □

For each $F \in \mathrm{T}_k\, k[\mathbf{x}]$ with $\deg F > |\mathbf{w}|$ and a sequence $\mathcal{G} = (G_i)_{i=0}^r$ as in Corollary 8.1, we define $\mathrm{SU}_{\mathbf{w}}(F; \mathcal{G})$ to be the number of $i \in \{1, \ldots, r\}$ such that G_{i+1} is a Shestakov-Umirbaev reduction of G_i. We define the *Shestakov-Umirbaev number* $\mathrm{SU}_{\mathbf{w}}(F)$ for the weight $\mathbf{w}$ to be the minimum among $\mathrm{SU}_{\mathbf{w}}(F; \mathcal{G})$ for the sequences $\mathcal{G} = (G_i)_{i=0}^r$ as in Corollary 8.1. It may be an interesting question to ask whether $\mathrm{SU}_{\mathbf{w}}(F; \mathcal{G}) = \mathrm{SU}_{\mathbf{w}}(F)$ for any $F \in \mathrm{T}_k\, k[\mathbf{x}]$ and $\mathcal{G} = (G_i)_{i=0}^r$.

When G_i admits a Shestakov-Umirbaev reduction, the possibilities for G_{i+1} are limited as described in the following propositions.

PROPOSITION 8.2. *If* (F, G^1) *and* (F, G^2) *satisfy the Shestakov-Umirbaev condition for* $F, G^1, G^2 \in \mathcal{T}$, *then* $g_i^1 = g_i^2$ *for* $i = 1, 2$, *and* $g_3^1 - g_3^2$ *is contained in* $k[g_2^1]$, *where* $G^j = (g_1^j, g_2^j, g_3^j)$ *for* $j = 1, 2$.

PROOF. By (SU1), there exist $a^j, b^j, c^j \in k$ such that $g_1^j = f_1 + a^j f_3^2 + c^j f_3$ and $g_2^j = f_2 + b^j f_3$ for $j = 1, 2$. By the last statement of (P11), it follows that $a^1 = a^2$, $b^1 = b^2$ and $c^1 = c^2$. Hence, we have $g_i^1 = g_i^2$ for $i = 1, 2$. Put $\phi := g_3^1 - g_3^2 = (g_3^1 - f_3) + (f_3 - g_3^2)$. Then, ϕ belongs to $k[g_1^1, g_2^1] = k[g_1^2, g_2^2]$, since so does $g_3^j - f_3$ for $j = 1, 2$ by (SU1). Suppose to the contrary that ϕ belongs to $k[g_1^1, g_2^1] \setminus k[g_2^1]$. Then, since $\deg \phi \leq \max\{\deg g_3^1, \deg g_3^2\} < \deg f_3 \leq \deg g_1^1$ by (SU5) and (SU4), we get $\deg \phi < \deg^U \phi$, where $U = \{g_1^1, g_2^1\}$. In view of (SU3), it follows from Lemma 3.2(i) that

$$\deg \phi \geq 2 \deg g_1^1 + \deg dg_1^1 \wedge dg_2^1 - \deg g_1^1 - \deg g_2^1 = \deg g_1^1 - \deg g_2^1 + \deg dg_1^1 \wedge dg_2^1 .$$

Since $\deg \phi \leq \max\{\deg g_3^1, \deg g_3^2\}$, this contradicts (SU6). Therefore, $g_3^1 - g_3^2$ belongs to $k[g_2^1]$. □

The following proposition gives a necessary condition on automorphisms to admit both an elementary reduction and a Shestakov-Umirbaev reduction simultaneously.

PROPOSITION 8.3. *Assume that* (F, G) *satisfies the Shestakov-Umirbaev condition for* $F, G \in \mathcal{T}$. *Then,* $f_i^{\mathbf{w}}$ *does not belong to* $k[S_i]^{\mathbf{w}}$ *for* $i = 1$ *if* $f_1^{\mathbf{w}} \not\approx (f_3^{\mathbf{w}})^2$, *for* $i = 2$, *and for* $i = 3$ *if* $(f_1, f_2) \neq (g_1, g_2)$.

PROOF. In each case, we will find $h_0, h_1 \in k[S_i]$ such that $k[h_0, h_1] = k[S_i]$, $\gamma_i' := \deg dh_0 \wedge dh_1 > s\delta$, $h_j^{\mathbf{w}}$ does not belong to $k[h_l^{\mathbf{w}}]$ for $(j, l) = (0, 1), (1, 0)$, and $f_i^{\mathbf{w}}$ does not belong to $k[h_0^{\mathbf{w}}, h_1^{\mathbf{w}}]$. Then, it follows that $f_i^{\mathbf{w}}$ does not belong to $k[S_i]^{\mathbf{w}}$. In fact, supposing that $f_i^{\mathbf{w}} = \phi^{\mathbf{w}}$ for some $\phi \in k[S_i] = k[h_0, h_1]$, we have $\deg \phi < \deg^U \phi$ for $U = \{h_0, h_1\}$, since $\phi^{\mathbf{w}} = f_i^{\mathbf{w}}$ does not belong to $k[h_0^{\mathbf{w}}, h_1^{\mathbf{w}}]$. Since $h_j^{\mathbf{w}}$ does not belong to $k[h_l^{\mathbf{w}}]$ for $(j, l) = (0, 1), (1, 0)$, we get $\deg \phi > \gamma_i'$ by Lemma 3.3(i). Thus, $\deg f_i = \deg \phi > \gamma_i' > s\delta$. This contradicts (P7). Therefore, $f_i^{\mathbf{w}}$ does not belong to $k[S_i]^{\mathbf{w}}$ if such h_0 and h_1 exist.

We remark that $\gamma_i := \deg f_j \wedge f_l > s\delta$ in each case, where $j, l \in \{1, 2, 3\} \setminus \{i\}$ with $j < l$. Actually, $\gamma_1 > s\delta$ and $\gamma_2 \geq \delta + \gamma_1 > (s + 1)\delta$ by the last two conditions of (P12). If $i = 3$, then $(f_1, f_2) \neq (g_1, g_2)$ by assumption. Hence, the first condition of (P12) implies that γ_3 is equal to one of $\deg f_3 + \gamma_1$, γ_2 and γ_1, which are greater than $s\delta$.

We set $(h_0, h_1) = (f_2, f_3)$ if $i = 1$, and $(h_0, h_1) = (f_1, f_2)$ if $i = 3$. Then, $k[h_0, h_1] = k[S_i]$ and $\gamma_i' = \gamma_i > s\delta$ in either case. Moreover, $h_j^{\mathbf{w}}$ does not belong to $k[h_l^{\mathbf{w}}]$ for $(j, l) = (0, 1), (1, 0)$ by (P8). We check that $f_i^{\mathbf{w}}$ does not belong to $k[h_0^{\mathbf{w}}, h_1^{\mathbf{w}}]$. This holds for $i = 3$ because $f_3^{\mathbf{w}}$ does not belong to $k[f_l^{\mathbf{w}}]$ for $l = 1, 2$ by (P8), and we have $\deg f_3 \leq \deg f_1 < \deg f_1 f_2$ by (P7). Suppose to the contrary that $f_1^{\mathbf{w}}$ belongs to $k[f_2^{\mathbf{w}}, f_3^{\mathbf{w}}]$. Then, $f_1^{\mathbf{w}}$ must belong to $k[f_2^{\mathbf{w}}]$ or $k[f_3^{\mathbf{w}}]$, since

$$\deg f_1 \leq \deg g_1 = s\delta = 2\delta + (s - 2)\delta < \deg f_2 + \deg f_3 = \deg f_2 f_3$$

by (SU2) and (P2). It follows from (P8) that $f_1^{\mathbf{w}}$ does not belong to $k[f_2^{\mathbf{w}}]$, and so $f_1^{\mathbf{w}}$ belongs to $k[f_3^{\mathbf{w}}]$ and $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$. This contradicts the assumption that $f_1^{\mathbf{w}} \not\approx (f_3^{\mathbf{w}})^2$. Thus, $f_1^{\mathbf{w}}$ does not belong to $k[h_0^{\mathbf{w}}, h_1^{\mathbf{w}}]$ if $i = 1$. Therefore, h_0 and h_1 satisfy the required conditions, and thereby $f_i^{\mathbf{w}}$ does not belong to $k[S_i]^{\mathbf{w}}$ for $i = 1, 3$ as mentioned above.

In the case $i = 2$, set $h_0 = f_3$, and $h_1 = f_1$ if $f_1^{\mathbf{w}} \not\approx (f_3^{\mathbf{w}})^2$, while $h_1 = f_1 - cf_3^2$ otherwise, where $c \in k$ is such that $f_1^{\mathbf{w}} = c(f_3^{\mathbf{w}})^2$. Then, $k[h_0, h_1] = k[S_2]$ and $\gamma_2' = \gamma_2 > (s+1)\delta$. If $f_1^{\mathbf{w}} \not\approx (f_3^{\mathbf{w}})^2$, then $h_1 = f_1$, and so $h_j^{\mathbf{w}}$ does not belong to $k[h_l^{\mathbf{w}}]$ for $(j, l) = (0, 1), (1, 0)$ by (P8). If $f_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$, then $f_1^{\mathbf{w}}$ belongs to $k[f_3^{\mathbf{w}}]$. By (P8), we get $s = 3$ and $\deg h_0 = \deg f_3 = (3/2)\delta$. Since $\deg h_0 + \deg h_1 \geq \gamma_2' > (s+1)\delta = 4\delta$ by (2.3), we have $\deg h_1 > 4\delta - (3/2)\delta = (5/2)\delta > \deg h_0$. Hence, $h_0^{\mathbf{w}}$ does not belong to $k[h_1^{\mathbf{w}}]$. It follows that $(5/2)\delta < \deg h_1 = \deg(f_1 - cf_3^2) < \deg f_3^2 = 3\delta$. Since $5/2 < (3/2)l < 3$ does not hold for any $l \in \boldsymbol{N}$, we conclude that $h_1^{\mathbf{w}}$ does not belong to $k[h_0^{\mathbf{w}}]$. For both $h_1 = f_1$ and $h_1 = f_1 - cf_3^2$, it holds that $\deg f_2 = 2\delta < \deg h_1$. Hence, $f_2^{\mathbf{w}}$ does not belong to $k[h_0^{\mathbf{w}}, h_1^{\mathbf{w}}] \setminus k[h_0^{\mathbf{w}}]$. By (P8), $f_2^{\mathbf{w}}$ does not belong to $k[h_0^{\mathbf{w}}] = k[f_3^{\mathbf{w}}]$. Thus, $f_2^{\mathbf{w}}$ does not belong to $k[h_0^{\mathbf{w}}, h_1^{\mathbf{w}}]$. Therefore, h_0 and h_1 satisfy the required conditions, thereby $f_2^{\mathbf{w}}$ does not belong to $k[S_2]^{\mathbf{w}}$. □

Appendix: Reductions of types I, II, III and IV. In this appendix, we explain that the following results are implicit in the theory of Shestakov-Umirbaev [10]:

(A) If $F \in \operatorname{Aut}_k k[\mathbf{x}]$ admits a reduction of one of the types I, II, III and IV, then F admits none of the reductions of the other three types.

(B) If $F \in \operatorname{Aut}_k k[\mathbf{x}]$ admits a reduction of type IV, then there exists an elementary automorphism E such that $F \circ E$ admits a reduction of type IV, but does not admit an elementary reduction.

From (A) and (B), it follows that, if there exists a tame automorphism admitting a reduction of type IV, then there exists a tame automorphism which is not affine and does not admit an elementary reduction nor any one of the reductions of types I, II and III. Actually, an automorphism admitting a reduction of type IV is not affine, and admits none of the reductions of types I, II and III by (A). Theorem 2.1, together with Proposition 7.2, implies that each tame automorphism but an affine automorphism admits an elementary reduction or a reduction of one of the types I, II and III. Thus, we obtain another proof of Theorem 7.1 that no tame automorphism admits a reduction of type IV.

First, we show (A). Recall the definitions of reductions of types I through IV (see the conditions (1) through (4) listed in Section 7). If F satisfies (1), then $\deg f_1 < \deg f_3 \leq \deg f_2$. Moreover, (1) implies that $\deg df_1 \wedge df_2 = \deg df_1 \wedge df_3$ (cf. [10, Proposition 1 (1)]). If F satisfies one of (2), (3) and (4), then $\deg f_3 \leq \deg f_1 < \deg f_2$, where $\deg f_3 = \deg f_1$ holds only in the case (2). Moreover, it follows that

$$(8.1) \qquad \deg df_1 \wedge df_3 = \deg dg_1 \wedge dg_2 + 3l\,, \quad \deg df_2 \wedge df_3 = \deg df_1 \wedge df_3 + l$$

in these cases (cf. [10, Equations (6) and (7)]).

Now, suppose that F satisfies one of (2), (3) and (4), but admits a reduction of type I, i.e., F_τ satisfies (1) for some $\tau \in \mathfrak{S}_3$. Then, $\deg df_{\tau(1)} \wedge df_{\tau(2)} = \deg df_{\tau(1)} \wedge df_{\tau(3)}$ as mentioned. It follows from the condition on the degrees of f_1, f_2 and f_3 that $\tau = (1, 3)$. Hence, $\deg df_3 \wedge df_2 = \deg df_3 \wedge df_1$, which contradicts the second equation of (8.1). If F satisfies (3) or (4), and admits a reduction of type II, then F satisfies (2) by the conditions on the degrees of f_1, f_2 and f_3. This is impossible, because $(3/2)l < \deg f_3$ if (2), while $\deg f_3 \leq (3/2)l$ if (3) or (4). Finally, we show that F does not admit reductions of types III and IV simultaneously. Suppose that F satisfies (4), and admits a reduction of type III. Then, F satisfies (3), since $\deg f_3 < \deg f_1 < \deg f_2$ in both cases. We remark that $\alpha, \beta, \gamma \in k$ appearing in (3) and (4) are uniquely determined by F (cf. [10, Proposition 3 (1), (2) and (3)]), and hence so are g_1 and g_2. There exist $\sigma^1, \sigma^2 \in k \setminus \{0\}$ and $g^1, g^2 \in k[g_1, g_2] \setminus k$ such that $\deg dg_1 \wedge dg_{3,i} < 3l + \deg dg_1 \wedge dg_2$ for $i = 1, 2$, $\deg g_{3,1} < l + \deg dg_1 \wedge dg_2$, and $\deg(g_2 - \mu g_{3,2}^2) \leq 2l$ for some $\mu \in k \setminus \{0\}$, where $g_{3,i} = \sigma^i f_3 + g^i$ for $i = 1, 2$. We claim that $\deg g_{3,1} < \deg f_3$. In fact, we have $\deg g_{3,1} < l + \deg dg_1 \wedge dg_2$, while the first equation of (8.1) implies $\deg f_3 \geq l + \deg dg_1 \wedge dg_2$, since $\deg f_1 + \deg f_3 \geq \deg df_1 \wedge df_3$ and $\deg f_1 = 2l$. Hence, $\deg g_{3,1} < \deg f_3 \leq (3/2)l$. From $\deg(g_2 - \mu g_{3,2}^2) \leq 2l$, we get $\deg g_{3,2} = (3/2)l$. It follows that $\phi := \sigma^2 g^1 - \sigma^1 g^2 = \sigma^2 g_{3,1} - \sigma^1 g_{3,2}$ is an element of $k[g_1, g_2]$ such that $\deg dg_1 \wedge d\phi < 3l + \deg dg_1 \wedge dg_2$ and $\deg \phi = (3/2)l$. Since $\deg \phi < \deg g_i$ for $i = 1, 2$, and since ϕ is not an element of k, we have $\deg \phi < \deg^U \phi$, where $U = \{g_1, g_2\}$. As $\deg g_1 = 2l$ and $\deg g_2 = 3l$, it follows from Lemma 3.3(ii) that $\deg d\phi \wedge dg_1 \geq 3l + \deg dg_1 \wedge dg_2$, a contradiction. Therefore, F does not admit reductions of types III and IV simultaneously. This completes the proof of (A).

Next, assume that F satisfies (4). From the proof of [10, Lemma 12], we know that each $a \in k[S_i]$ with $\deg a \leq \deg f_i$ can be written as follows: If $i = 1$, then $a = \delta_1 f_3$ (up to a constant term) for some $\delta_1 \in k$. If $i = 2$, then $a = \delta_1 f_3^2 + \sigma_1 f_3 + \mu_1 f_1$ (up to a constant term) for some $\delta_1, \sigma_1, \mu_1 \in k$. If $i = 3$ and $(\alpha, \beta, \gamma) \neq (0, 0, 0)$, then a is an element of k. It is also mentioned in the proof of [10, Lemma 12] that $(f_1, f_2 + a, f_3)$ satisfies (4) for each $a \in k[S_2]$ with $\deg a \leq \deg f_2$. In fact, it is claimed that $(g_1, g_2 + \mu_1 g_1, g_3)$ is a "predreduction" of type IV of $(f_1, f_2 + a, f_3)$.

We deduce (B) from the facts above. The assertion is clear if F does not admit an elementary reduction. So, assume that $\deg F \circ E < \deg F$ for some $E \in \mathcal{E}_i$, where $i \in \{1, 2, 3\}$. Then, $(F \circ E)(x_i) = f_i + a$ and $\deg(f_i + a) < \deg f_i$ for some $a \in k[S_i]$. Since $\deg a = \deg f_i$, we can write a as stated in the preceding paragraph. Hence, if $i = 1$, then $\deg a = \deg \delta_1 f_3 \leq (3/2)l$. Since $\deg a = \deg f_1 = 2l$, this is impossible. Thus, $i \neq 1$. If $i = 2$, then $\deg a = \deg f_2 > (5/2)l$. Since $\deg f_3 \leq (3/2)l$, we have $\delta_1 \neq 0$ and $\deg f_2 = \deg a = 2 \deg f_3$. This implies that $\deg f_2 = 3l$ and $\deg f_3 = (3/2)l$, since $\deg f_2 = 3l$ if $\deg f_3 < (3/2)l$, and $\deg f_3 = (3/2)l$ if $\deg f_2 < 3l$. If $i = 3$, then $\alpha = \beta = \gamma = 0$. and so $g_1 = f_1$ and $g_2 = f_2$. We show that $F \circ E$ admits a reduction of type IV, but does not admit an elementary reduction in the cases $i = 2$ and $i = 3$.

Assume that $i = 2$. Then, $\deg(f_2 + a) < \deg f_2 = 3l$. Moreover, $F \circ E = (f_1, f_2 + a, f_3)$ satisfies (4) as mentioned, in which $\alpha \in k$ involved in the condition cannot be zero, since

$\deg(f_2 + a) < 3l$. By applying to $F \circ E$ the argument in the preceding paragraph, we know that there does not exist $E' \in \mathcal{E}_j$ with $\deg F \circ E \circ E' < \deg F \circ E$ for $j = 1$, for $j = 2$, since $\deg(f_2 + a) \neq 3l$, and for $j = 3$, since the constant α is not zero. Thus, $F \circ E$ does not admit an elementary reduction.

Assume that $i = 3$. Without loss of generality, we may assume that $(F \circ E)(x_3)^{\mathbf{w}}$ does not belong to $k[f_1, f_2]^{\mathbf{w}}$ by replacing E if necessary. We show that $F \circ E = (f_1, f_2, f_3 + a)$ satisfies (4) by using the assumption that F satisfies (4) for $\alpha = \beta = \gamma = 0$. We claim that $\deg(f_3 + a) \geq l + \deg dg_1 \wedge dg_2$. In fact, if not, we can check that $(f_1, f_2 + f_3, f_3)$ satisfies (3) and (4) by the assumption that F satisfies (4) for $\alpha = \beta = \gamma = 0$. This contradicts (A). Hence, $l < \deg(f_3 + a) \leq (3/2)l$, as required in the assumption of (4). Let $g' := g_3 - \sigma(f_3 + a) = \sigma f_3 - g - \sigma(f_3 + a)$. Then, g' belongs to $k[g_1, g_2] = k[f_1, f_2]$, since so do g and a. It follows that $\deg g' = (3/2)l$, since $\deg(f_3 + a) < \deg f_3 \leq (3/2)l$ and $\deg g_3 = (3/2)l$. Hence, g is not an element of k. Moreover, we can express $g_3 = \sigma(f_3 + a) + g'$. This shows that $F \circ E$ satisfies (4). Consequently, there does not exist $E' \in \mathcal{E}_j$ such that $\deg F \circ E \circ E' < \deg F \circ E$ for $j = 1$, and for $j = 2$, since $\deg(f_3 + a) \neq (3/2)l$. This also holds for $j = 3$ as we choose E so that $(F \circ E)(x_3)^{\mathbf{w}}$ does not belong to $k[f_1, f_2]^{\mathbf{w}}$. Therefore, $F \circ E$ does not admit an elementary reduction. This completes the proof of (B).

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