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BIHARMONIC MAPS AND MORPHISMS FROM CONFORMAL MAPPINGS

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Abstract. Inspired by the all-important conformal invariance of harmonic maps on two-dimensional domains, this article studies the relationship between biharmonicity and conformality. We first give a characterization of biharmonic morphisms, analogues of harmonic morphisms investigated by Fuglede and Ishihara, which, in particular, explicits the conditions required for a conformal map in dimension four to preserve biharmonicity and helps producing the first example of a biharmonic morphism which is not a special type of harmonic morphism. Then, we compute the bitension field of horizontally weakly conformal maps, which include conformal mappings. This leads to several examples of proper (i.e., non-harmonic) biharmonic conformal maps, in which dimension four plays a pivotal role. We also construct a family of Riemannian submersions which are proper biharmonic maps.

1. Introduction. A central feature of harmonic maps is their conformal invariance in dimension two. Not only this allows defining harmonic maps on Riemann surfaces but it also is the starting point to many properties of minimal branched immersions.

In the higher-order theory of biharmonic maps, one could expect similar properties in dimension four. While this dimension certainly enjoys a special role for biharmonicity, as illustrated by the conformal deformation of harmonic maps of [4], the study of the biharmonic stress-energy tensor or the characterization of biharmonic morphisms of Section 3, no conformal invariance of any sort has ever been observed.

Nevertheless, the interaction between conformality and biharmonicity remains a rich subject and provides an interesting source of new examples.

In the theory of harmonic maps, a particularly fruitful approach has been to consider maps which preserve local harmonic functions, called harmonic morphisms, because their characterization as horizontally weakly conformal (a generalization of Riemannian submersions) harmonic maps confers them a more geometrical flavour, which counterweighs the analytical nature of harmonic maps [11, 14].

Their numerous geometrical properties have earned harmonic morphisms a choice place among harmonic maps. The counterparts of these are biharmonic morphisms (see [18]), maps which pull back local biharmonic functions onto biharmonic functions (and also, as it turns out, maps) and we characterize them as horizontally weakly conformal biharmonic maps

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which are 4-harmonic and satisfy an additional equation, whose significance remains largely enigmatic.

While the number of conditions is directly due to the order of the problem, the appearance of 4-harmonicity is yet another clue to the specificity of dimension four.

This characterization also solves one aspect of the original question on the conformal invariance of biharmonic maps on four dimensional domains, since two extra conditions are required.

2. p-Harmonicity and biharmonicity. At the origin of this work, lie the search and study of maps selected as extremals of a measured quantity, and the most natural class of functionals on the space of maps between Riemannian manifolds is the p-energies.

DEFINITION 2.1. Let $\phi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds and assume (M, g) compact then, for $p \in \boldsymbol{R}$ $(p > 1)$, its *p-energy* is

$$E_p(\phi) = \frac{1}{p} \int_M |d\phi|^p \, v_g \, .$$

The critical points of E_p are called *p-harmonic maps* and simply *harmonic maps* for $p = 2$. Standard arguments yield the associated Euler-Lagrange equation, the vanishing of the *p-tension field*:

$$\tau_p(\phi) = |d\phi|^{p-4}[|d\phi|^2 \tau(\phi) + \frac{p-2}{2} d\phi(\text{grad}\, |d\phi|^2)] = 0 \, ,$$

where

$$\tau(\phi) = \text{trace}\, \nabla d\phi$$

is the tension field.

REMARK 2.2. (i) A fundamental feature of p-harmonicity for $p \neq 2$, is the collapse of ellipticity at critical points and its negative consequences for regularity properties (cf. [13]). For $p = 2$, smoothness of continuous harmonic maps is ensured by boot-strap methods and strong uniqueness, for example, follows.

(ii) The main existence result for p-harmonic maps ($p > 2$) is due to Duzaar and Fuchs in [9] and generalizes the Eells-Sampson theorem.

(iii) Working directly with the functional, one sees that, if $p = \dim M$, E_p is conformally invariant, a situation that seems more natural since, when the target has trivial p-th homotopy group, p-harmonic maps exist in each homotopy class (cf. [16]).

While harmonic maps have been extensively studied and shown to exist in numerous circumstances (cf. [10]), in some situations they cannot exist or are very limited. It is therefore interesting to turn to an alternative measuring the default of harmonicity.

DEFINITION 2.3. Let $\phi : (M, g) \to (N, h)$ be a smooth map between compact Riemannian manifolds. Define its *bienergy* as

$$E^2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \, v_g \, .$$

Critical points of the functional E^2 are called *biharmonic maps* and its associated Euler-Lagrange equation is the vanishing of the *bitension field*

$$\tau^2(\phi) = -\Delta^\phi \tau(\phi) - \mathrm{trace}_g\, R^N(d\phi, \tau(\phi))d\phi\,,$$

where $\Delta^\phi = -\,\mathrm{trace}_g(\nabla^\phi\nabla^\phi - \nabla^\phi_\nabla)$ is the Laplacian on sections of $\phi^{-1}TN$ and R^N the Riemann curvature operator on (N, h).

REMARK 2.4. (i) Clearly harmonic maps are automatically biharmonic, actually absolute minimums of E^2. For compact domains and negatively curved targets, the converse holds (cf. [15]).

(ii) An alternative to E^2 is to view (N, h) isometrically immersed in $\boldsymbol{R}^N$ and, considering ϕ as a vector, take the L^2-norm of Δ^ϕ and call the critical points (*extrinsic*) *biharmonic maps*. Except for flat targets, the two definitions are distinct. The regularity of both types of biharmonic maps has been extensively studied in [7, 20, 21, 22]

(iii) If M is non compact, we extend all these definitions by integrating over compact subsets.

A natural generalization of isometric immersions are weakly conformal maps, i.e., whose differential either vanishes or is injective and conformal. Of particular interest is their harmonicity since it characterizes minimality of the image and defines minimal branched immersions. They also explain the conformal invariance of harmonic maps on surfaces, directly at the level of the tension field.

Dual to this notion is horizontal weak conformality where, pointwise, the differential is required to vanish or be surjective and conformal. This reverses the constraint on the dimensions and enables a preservation of harmonicity in higher dimensions.

DEFINITION 2.5. Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. For any point $x \in M$, let $\mathcal{V}_x = \ker d\phi_x$ be the vertical space at x and $\mathcal{H}_x = (\mathcal{V}_x)^\perp$ the horizontal space at x. These spaces define a vertical and a horizontal distribution. The map ϕ is called *horizontally weakly conformal* if for any point $x \in M$ either $d\phi_x = 0$ or $d\phi_x$ is surjective and conformal from $\mathcal{H}_x$ to $T_{\phi(x)}N$, i.e.,

$$h(d\phi_x(X), d\phi_x(Y)) = \lambda^2(x)g(X, Y)\,,$$

for all $X, Y \in \mathcal{H}_x$ and the function λ is called the dilation of ϕ. For a vector $X \in TM$, $X^\mathcal{H}$ and $X^\mathcal{V}$ will denote the horizontal and vertical parts of X. Points where $d\phi_x \neq 0$ are called *regular*.

REMARK 2.6. (i) If $m < n$, then ϕ is horizontally weakly conformal if and only if it is constant.

(ii) By extending the function λ by zero over critical points, we obtain a smooth function λ^2 defined on the whole of M. Besides $|d\phi|^2 = n\lambda^2$.

(iii) Harmonic morphisms, i.e., maps which preserve local harmonic functions by composition on the right-hand side, were characterized by Fuglede and Ishihara, as horizontally weakly conformal harmonic maps (cf. [11, 14]).

Conventions: We will systematically use the Einstein convention on summing repeated indices. Our convention for the Riemann curvature tensor will be

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

and the Laplacian on functions has been chosen with negative eigenvalues, i.e., $\Delta f =$ trace ∇df but on vector fields $\Delta X = -\operatorname{trace} \nabla^2 X$.

3. The characterization of biharmonic morphisms. In this section, we give an improvement of the characterization of biharmonic morphisms and show that the inversion in the unit sphere $\boldsymbol{R}^n \setminus \{0\} \to \boldsymbol{R}^n$ is a biharmonic morphism if and only if $n = 4$, thus providing an example which, unlike any of the previously known ones, is not a harmonic morphism.

In light of the theory of harmonic morphisms, cf. the monograph [5], it only seemed natural to define and study their biharmonic counterparts.

DEFINITION 3.1. A continuous map $\phi : (M, g) \to (N, h)$ is a *biharmonic morphism* if, for any biharmonic function $f : U \subset N \to \boldsymbol{R}$, such that $\phi^{-1}(U) \neq \emptyset$, the pull-back function $f \circ \phi : \phi^{-1}(U) \subset M \to \boldsymbol{R}$ is also biharmonic.

REMARK 3.2. (i) Clearly constant maps and isometries are biharmonic morphisms and the composition of two biharmonic morphisms is again a biharmonic morphism.

(ii) Since harmonic functions are automatically biharmonic, a biharmonic morphism will pull-back harmonic functions onto biharmonic ones, but not necessarily harmonic. Likewise, there is no reason to believe that a harmonic morphism should be a biharmonic morphism. This distinction is clarified by Theorem 3.3 (see also [17]). Nevertheless, all the previously known examples of biharmonic morphisms came from special types of harmonic morphisms (cf. [17, 18, 19]).

(iii) The existence of local harmonic coordinates on Riemannian manifolds [8] implies that, in such a coordinate system, the components of a biharmonic morphism are continuous biharmonic functions, hence smooth by standard properties of elliptic partial differential equations. Therefore a biharmonic morphism is always smooth.

The geometric method used by Ishihara in [14] to characterize harmonic morphisms as horizontally weakly conformal harmonic maps, and co-opted for the semi-Riemannian case in [12] by Fuglede (who had his own approach for Riemannian metrics) is extended here to biharmonicity. As one should expect, four conditions are needed to describe biharmonic morphisms, with the last one still very much unfathomable.

THEOREM 3.3. *Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. Then ϕ is a biharmonic morphism if and only if it is a horizontally weakly conformal biharmonic* 4*-harmonic map*, *of dilation λ*, *such that*

$$\begin{aligned} |\tau(\phi)|^4 - 2\Delta\lambda^2 |\tau(\phi)|^2 + 4\Delta\lambda^2 \operatorname{div}\langle d\phi, \tau(\phi)\rangle + n(\Delta\lambda^2)^2 \\ + 2\langle d\phi, \tau(\phi)\rangle(\nabla|\tau(\phi)|^2) + |S|^2 = 0\,, \end{aligned} \tag{1}$$

where $S \in \odot^2\phi^{-1}TN$ is the symmetrization of the g-trace of $d\phi \otimes \nabla^{\phi}\tau(\phi)$ and $\langle d\phi, \tau(\phi)\rangle(X) = \langle d\phi(X), \tau(\phi)\rangle$.

PROOF. Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds, then the statement of Theorem 3.3 is that ϕ is a biharmonic morphism if and only if

$$h(d\phi(X), d\phi(Y)) = \lambda^2 g(X, Y) \quad \text{for all } X, Y \in \mathcal{H}, \tag{2}$$

$$-\Delta^{\phi}\tau(\phi) - \operatorname{trace} \mathrm{R}^N(d\phi, \tau(\phi))d\phi = 0, \tag{3}$$

$$\lambda^2\tau(\phi) + d\phi \operatorname{grad}\lambda^2 = 0, \tag{4}$$

$$\begin{aligned} &|\tau(\phi)|^4 - 2\Delta\lambda^2|\tau(\phi)|^2 + 4\Delta\lambda^2 \operatorname{div}\langle d\phi, \tau(\phi)\rangle + n(\Delta\lambda^2)^2 \\ &+ 2\langle d\phi, \tau(\phi)\rangle(\nabla|\tau(\phi)|^2) + |S|^2 = 0. \end{aligned} \tag{5}$$

Equip the manifold (M, g) with harmonic coordinates $(x^i)_{1\leqslant i\leqslant m}$ centered at the point p and (N, h) with harmonic coordinates $(y^{\alpha})_{1\leqslant\alpha\leqslant n}$ around the point $\phi(p)$. Let $f : U \subset N \to \boldsymbol{R}$ be a local function on N, then

$$\begin{aligned} \Delta^2(f \circ \phi) = {} & \frac{\partial^4 f}{\partial y^{\alpha}\partial y^{\beta}\partial y^{\gamma}\partial y^{\delta}}\left[g^{ij}g^{kl}\frac{\partial\phi^{\alpha}}{\partial x^i}\frac{\partial\phi^{\beta}}{\partial x^j}\frac{\partial\phi^{\gamma}}{\partial x^k}\frac{\partial\phi^{\delta}}{\partial x^l}\right] \\ & + \frac{\partial^3 f}{\partial y^{\alpha}\partial y^{\beta}\partial y^{\gamma}}\left[\left(\frac{\partial^2\phi^{\gamma}}{\partial x^i\partial x^j}\frac{\partial\phi^{\beta}}{\partial x^k}\frac{\partial\phi^{\alpha}}{\partial x^l} + \frac{\partial^2\phi^{\beta}}{\partial x^i\partial x^k}\frac{\partial\phi^{\gamma}}{\partial x^j}\frac{\partial\phi^{\alpha}}{\partial x^l} + \frac{\partial^2\phi^{\alpha}}{\partial x^i\partial x^l}\frac{\partial\phi^{\gamma}}{\partial x^j}\frac{\partial\phi^{\beta}}{\partial x^k}\right.\right. \\ & + \left.\frac{\partial^2\phi^{\beta}}{\partial x^k\partial x^j}\frac{\partial\phi^{\gamma}}{\partial x^i}\frac{\partial\phi^{\alpha}}{\partial x^l} + \frac{\partial^2\phi^{\alpha}}{\partial x^l\partial x^j}\frac{\partial\phi^{\gamma}}{\partial x^i}\frac{\partial\phi^{\beta}}{\partial x^k} + \frac{\partial^2\phi^{\alpha}}{\partial x^l\partial x^k}\frac{\partial\phi^{\gamma}}{\partial x^i}\frac{\partial\phi^{\beta}}{\partial x^j}\right)g^{ij}g^{kl} \\ & \left. + 2g^{ij}\frac{\partial g^{kl}}{\partial x^i}\left(\frac{\partial\phi^{\gamma}}{\partial x^j}\frac{\partial\phi^{\beta}}{\partial x^k}\frac{\partial\phi^{\alpha}}{\partial x^l}\right)\right] \\ & + \frac{\partial^2 f}{\partial y^{\alpha}\partial y^{\beta}}\left[\left(\frac{\partial^3\phi^{\beta}}{\partial x^i\partial x^j\partial x^k}\frac{\partial\phi^{\alpha}}{\partial x^l} + \frac{\partial^2\phi^{\beta}}{\partial x^j\partial x^k}\frac{\partial^2\phi^{\alpha}}{\partial x^l\partial x^i} + \frac{\partial^2\phi^{\beta}}{\partial x^i\partial x^k}\frac{\partial^2\phi^{\alpha}}{\partial x^l\partial x^j}\right.\right. \\ & + \left.\frac{\partial^3\phi^{\alpha}}{\partial x^i\partial x^j\partial x^l}\frac{\partial\phi^{\beta}}{\partial x^k} + \frac{\partial^2\phi^{\beta}}{\partial x^i\partial x^j}\frac{\partial^2\phi^{\alpha}}{\partial x^l\partial x^k} + \frac{\partial^3\phi^{\alpha}}{\partial x^i\partial x^k\partial x^l}\frac{\partial\phi^{\beta}}{\partial x^j} + \frac{\partial^3\phi^{\alpha}}{\partial x^j\partial x^k\partial x^l}\frac{\partial\phi^{\beta}}{\partial x^i}\right)g^{ij}g^{kl} \\ & \left. + \left(\frac{\partial^2\phi^{\alpha}}{\partial x^k\partial x^l}\frac{\partial\phi^{\beta}}{\partial x^j} + \frac{\partial^2\phi^{\alpha}}{\partial x^j\partial x^l}\frac{\partial\phi^{\beta}}{\partial x^k} + \frac{\partial^2\phi^{\beta}}{\partial x^j\partial x^k}\frac{\partial\phi^{\alpha}}{\partial x^l}\right)2g^{ij}\frac{\partial g^{kl}}{\partial x^i} + g^{ij}\frac{\partial^2 g^{kl}}{\partial x^i\partial x^j}\frac{\partial\phi^{\alpha}}{\partial x^l}\frac{\partial\phi^{\beta}}{\partial x^k}\right] \\ & + \frac{\partial f}{\partial y^{\alpha}}\left[\frac{\partial^2\phi^{\alpha}}{\partial x^k\partial x^l}g^{ij}\frac{\partial^2 g^{kl}}{\partial x^i\partial x^j} + \frac{\partial^3\phi^{\alpha}}{\partial x^j\partial x^k\partial x^l}2g^{ij}\frac{\partial g^{kl}}{\partial x^i} + \frac{\partial^4\phi^{\alpha}}{\partial x^i\partial x^j\partial x^k\partial x^l}g^{ij}g^{kl}\right]. \end{aligned} \tag{6}$$

Plugging carefully chosen local biharmonic test functions, as given by [1, Proposition 2.4], into Equation (6) shows that ϕ is a biharmonic morphism if and only if

$$g^{ij}\frac{\partial\phi^{\alpha}}{\partial x^i}\frac{\partial\phi^{\beta}}{\partial x^j} = \lambda^2 h^{\alpha\beta}, \tag{7}$$

$$d\phi^{\gamma}\operatorname{grad}(\lambda^2) + \lambda^2\tau^{\gamma}(\phi) = 0, \tag{8}$$

$$h^{\alpha\beta}\Delta(\lambda^2) + \operatorname{div}(\tau^\beta(\phi)d\phi^\alpha) + \operatorname{div}(\tau^\alpha(\phi)d\phi^\beta) - \tau^\alpha(\phi)\tau^\beta(\phi) = \lambda^2 dh^{\alpha\beta}(\tau(\phi)), \tag{9}$$

$$g^{ij}\frac{\partial^2}{\partial x^i \partial x^j}\tau^\alpha(\phi) = 0, \tag{10}$$

for all $\alpha, \beta = 1, \ldots, n$. Clearly Equations (7) and (8) mean that ϕ is horizontally weakly conformal and 4-harmonic.

On the other hand, Equation (9) is equivalent to the vanishing of

$$A^{\alpha\beta} = h^{\alpha\beta}\Delta(\lambda^2) + g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\tau^\beta(\phi)}{\partial x^j} + g^{ij}\frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\tau^\alpha(\phi)}{\partial x^j} + \tau^\alpha(\phi)\tau^\beta(\phi) - \lambda^2\frac{\partial h^{\alpha\beta}}{\partial y^p}\tau^p(\phi),$$

whose norm is easily shown to be

$$\begin{aligned}
|A|^2 &= n(\Delta\lambda^2)^2 + |\tau(\phi)|^4 - 2(\Delta\lambda^2)|\tau(\phi)|^2 + 4(\Delta\lambda^2)\operatorname{div}\langle d\phi, \tau(\phi)\rangle \\
&\quad + 4g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\tau^\beta(\phi)}{\partial x^j}h_{\alpha\delta}h_{\beta\mu}\tau^\delta(\phi)\tau^\mu(\phi) + 2\lambda^2\tau^\alpha(\phi)\tau^\beta(\phi)\frac{\partial h_{\alpha\beta}}{\partial y^p}\tau^p(\phi) \\
&\quad + 2g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\tau^\beta(\phi)}{\partial x^j}h_{\alpha\delta}h_{\beta\mu}g^{kl}\frac{\partial\phi^\delta}{\partial x^k}\frac{\partial\tau^\mu(\phi)}{\partial x^l} \\
&\quad + 2g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\tau^\beta(\phi)}{\partial x^j}h_{\alpha\delta}h_{\beta\mu}g^{kl}\frac{\partial\phi^\mu}{\partial x^k}\frac{\partial\tau^\delta(\phi)}{\partial x^l} \\
&\quad + 4\lambda^2 g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\tau^\beta(\phi)}{\partial x^j}\frac{\partial h_{\alpha\beta}}{\partial y^p}\tau^p(\phi) + \lambda^4\frac{\partial h^{\alpha\beta}}{\partial y^p}h_{\alpha\delta}h_{\beta\mu}\frac{\partial h^{\delta\mu}}{\partial y^q}\tau^p(\phi)\tau^q(\phi),
\end{aligned}$$

since

$$h^{\alpha\beta}h_{\alpha\delta}h_{\beta\mu} = h_{\delta\mu} \text{ and } h_{\alpha\delta}h_{\beta\mu}\frac{\partial h^{\alpha\beta}}{\partial y^p} = -\frac{\partial h_{\alpha\beta}}{\partial y^p}.$$

But

$$\langle d\phi, \tau(\phi)\rangle(\nabla|\tau(\phi)|^2) = 2g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\tau^\beta(\phi)}{\partial x^j}h_{\alpha\delta}h_{\beta\mu}\tau^\delta(\phi)\tau^\mu(\phi) + \lambda^2\tau^\alpha(\phi)\tau^\beta(\phi)\frac{\partial h_{\alpha\beta}}{\partial y^p}\tau^p(\phi)$$

and the components of $S \in \odot^2\phi^{-1}TN$, the symmetrization of $g(d\phi, \nabla^\phi\tau(\phi))$, are

$$S^{\alpha\beta} = g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\tau^\beta(\phi)}{\partial x^j} + \lambda^2 h^{\alpha\delta}\Gamma^\beta_{\gamma\delta}\tau^\gamma(\phi) + g^{ij}\frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\tau^\alpha(\phi)}{\partial x^j} + \lambda^2 h^{\beta\delta}\Gamma^\alpha_{\gamma\delta}\tau^\gamma(\phi).$$

So its norm is

$$\begin{aligned}
|S|^2 &= 2g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\tau^\beta(\phi)}{\partial x^j}h_{\alpha\delta}h_{\beta\mu}g^{kl}\frac{\partial\phi^\delta}{\partial x^k}\frac{\partial\tau^\mu(\phi)}{\partial x^l} + 2g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\tau^\beta(\phi)}{\partial x^j}h_{\alpha\delta}h_{\beta\mu}g^{kl}\frac{\partial\phi^\mu}{\partial x^k}\frac{\partial\tau^\delta(\phi)}{\partial x^l} \\
&\quad + 4\lambda^2 g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\tau^\beta(\phi)}{\partial x^j}\frac{\partial h_{\alpha\beta}}{\partial y^p}\tau^p(\phi) + \lambda^4\frac{\partial h^{\alpha\beta}}{\partial y^p}h_{\alpha\delta}h_{\beta\mu}\frac{\partial h^{\delta\mu}}{\partial y^q}\tau^p(\phi)\tau^q(\phi).
\end{aligned}$$

Hence (9) is equivalent to Equation (1).

To obtain that Equation (10) is biharmonicity, observe that

$$-(\Delta^\phi\tau(\phi))^\alpha = g^{ij}\left[\frac{\partial^2\tau^\alpha(\phi)}{\partial x^i\partial x^j} + \frac{\partial\tau^\gamma(\phi)}{\partial x^j}\frac{\partial\phi^\beta}{\partial x^i}\Gamma^\alpha_{\beta\gamma} + \frac{\partial\tau^\gamma(\phi)}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\Gamma^\alpha_{\beta\gamma} + \tau^\gamma(\phi)\frac{\partial^2\phi^\beta}{\partial x^i\partial x^j}\Gamma^\alpha_{\beta\gamma}\right.$$

$$+\tau^{\gamma}(\phi)\frac{\partial\phi^{\beta}}{\partial x^{j}}\frac{\partial\Gamma^{\alpha}_{\beta\gamma}}{\partial y^{\mu}}\frac{\partial\phi^{\mu}}{\partial x^{i}}+\tau^{\mu}(\phi)\frac{\partial\phi^{\beta}}{\partial x^{j}}\Gamma^{\gamma}_{\mu\beta}\frac{\partial\phi^{\delta}}{\partial x^{i}}\Gamma^{\alpha}_{\gamma\delta}-\Gamma^{k}_{ij}\frac{\partial\tau^{\alpha}(\phi)}{\partial x^{k}}-\Gamma^{k}_{ij}\tau^{\gamma}(\phi)\frac{\partial\phi^{\beta}}{\partial x^{k}}\Gamma^{\alpha}_{\beta\gamma}\Big]$$

$$=2g^{ij}\frac{\partial\tau^{\gamma}(\phi)}{\partial x^{i}}\frac{\partial\phi^{\beta}}{\partial x^{j}}\Gamma^{\alpha}_{\beta\gamma}+\tau^{\gamma}(\phi)\tau^{\beta}(\phi)\Gamma^{\alpha}_{\beta\gamma}+\lambda^{2}\tau^{\gamma}(\phi)h^{\beta\mu}\frac{\partial\Gamma^{\alpha}_{\beta\gamma}}{\partial y^{\mu}}+\lambda^{2}h^{\beta\delta}\tau^{\mu}(\phi)\Gamma^{\gamma}_{\mu\beta}\Gamma^{\alpha}_{\gamma\delta},$$

since ϕ is horizontally weakly conformal,

$$g^{ij}\frac{\partial^{2}}{\partial x^{i}\partial x^{j}}\tau^{\alpha_0}(\phi)=0$$

and we are working with harmonic coordinates.

On the other hand, $\operatorname{trace}_g R^N(d\phi,\tau(\phi))d\phi=\lambda^2\mathrm{Ric}^N\tau(\phi)$ and

$$-\lambda^{2}\mathrm{Ric}^{N}\tau(\phi)=\lambda^{2}\Big[-\frac{\partial h^{\alpha\beta}}{\partial y^{\gamma}}\Gamma^{\delta}_{\alpha\beta}-h^{\alpha\beta}\frac{\partial\Gamma^{\delta}_{\alpha\gamma}}{\partial y^{\beta}}-\sum_{\mu=1}^{n}h^{\alpha\beta}\Gamma^{\mu}_{\alpha\gamma}\Gamma^{\delta}_{\mu\beta}\Big]\tau^{\gamma}(\phi).$$

Thus

$$\tau^{2}(\phi)=\Big[2g^{ij}\frac{\partial\tau^{\gamma}(\phi)}{\partial x^{i}}\frac{\partial\phi^{\beta}}{\partial x^{j}}\Gamma^{\alpha}_{\beta\gamma}+\tau^{\gamma}(\phi)\tau^{\beta}(\phi)\Gamma^{\alpha}_{\beta\gamma}-\lambda^{2}\frac{\partial h^{\delta\beta}}{\partial y^{\gamma}}\Gamma^{\alpha}_{\delta\beta}\tau^{\gamma}(\phi)\Big]\frac{\partial}{\partial y^{\alpha}}=0.$$

Of course, Conditions (7)–(10) are also sufficient. □

It is well-known that harmonic morphisms into 2-dimensional manifolds have an interesting link to the geometry of the fibres, which can be stated as: a horizontally conformal submersion $(M^m,g)\to(N^2,h)$ is harmonic (hence a harmonic morphism) if and only if it has minimal fibres. The corresponding result for biharmonic morphisms is partially true.

COROLLARY 3.4. *A biharmonic morphism* $\phi:(M^m,g)\to(N^4,h)$ $(m\geqslant 4)$, *with a* 4*-dimensional target*, *always has minimal fibres. However there exist horizontally conformal submersions which are* 4*-harmonic*, *have minimal fibres but are not biharmonic morphisms.*

PROOF. By Theorem 3.3, a biharmonic morphism is a 4-harmonic morphism. The first statement then follows from a result in [3] that a horizontally weakly conformal map $(M^m,g)\to(N^n,h)$ is a p-harmonic map (hence a p-harmonic morphism) with $p=n=\dim N$ if and only if it has minimal fibres. It is well-known that the radial projection $\varphi:\boldsymbol{R}^5\setminus\{0\}\to\boldsymbol{S}^4$, $\varphi(x)=x/|x|$ is a horizontally homothetic submersion with totally geodesic fibres and hence a harmonic morphism (see [5]). It is also a p-harmonic morphism for any p and, in particular, a 4-harmonic morphism. However, the radial projection $\varphi:\boldsymbol{R}^m\setminus\{0\}\to\boldsymbol{S}^{m-1}$, $\varphi(x)=x/|x|$ is a biharmonic morphism if and only if $m=4$ [19]. □

REMARK 3.5. (1) A straightforward computation shows that if ϕ is a biharmonic morphism then there exists a continuous function λ on M such that

$$\tau^{2}(\psi\circ\phi)=\lambda^{4}\tau^{2}(\psi)\circ\phi,$$

for any map ψ (including functions).

(2) Taking the trace of Equation (9) yields

$$n\Delta(\lambda^{2})+2\operatorname{div}\langle d\phi,\tau(\phi)\rangle=|\tau(\phi)|^{2}, \tag{11}$$

hence by Stokes' Theorem, biharmonic morphisms on a compact manifold without boundary are exactly homothetic submersions with minimal fibres. See [17] for similar results.

Apart from harmonic morphisms with harmonic dilation, examples of biharmonic morphisms are hard to unearth. Conformal maps in dimension four have the double advantage of satisfying automatically two of the four conditions, namely horizontal weak conformality and 4-harmonicity. Once biharmonicity is secured, remains only Equation (1) to fulfil, though no geometric insight is yet at our disposal.

THEOREM 3.6. *The inversion in the unit sphere*

$$\phi : \boldsymbol{R}^n \setminus \{0\} \to \boldsymbol{R}^n$$
$$x \mapsto \frac{x}{|x|^2}$$

is a biharmonic morphism if and only if $n = 4$.

PROOF. By [4], the inversion in the unit sphere is a biharmonic map if and only if $n = 4$. Moreover the map

$$\phi : \boldsymbol{R}^4 \setminus \{0\} \to \boldsymbol{R}^4$$
$$x \mapsto \frac{x}{|x|^2}$$

is clearly a conformal map of dilation $\lambda^2 = 1/|x|^4$ between spaces of dimension four, hence a 4-harmonic map (cf. [19]) and $\tau(\phi) = -4x/|x|^4$. Using the standard coordinates $\{x^\alpha\}_{\alpha=1,2,3,4}$ on $\boldsymbol{R}^4$, simple computations show that

$$|\tau(\phi)|^2 = \frac{16}{|x|^6}\,, \quad \Delta\lambda^2 = \frac{8}{|x|^6}\,, \quad \langle d\phi, \tau(\phi)\rangle = 4\frac{x}{|x|^6}\,, \quad \mathrm{div}\langle d\phi, \tau(\phi)\rangle = -\frac{8}{|x|^6}\,.$$

As to the symmetric tensor S, we have

$$S^{\alpha\alpha} = -\frac{8}{|x|^8}\left(|x|^2 + 2(x^\alpha)^2\right), \quad S^{\alpha\beta} = -\frac{16}{|x|^8}x^\alpha x^\beta\,,$$

for all $\alpha \neq \beta = 1, \ldots, 4$, so $|S|^2 = 3(16)^2/|x|^{12}$. Therefore ϕ satisfies Equation (1) and is a biharmonic morphism. □

4. Biharmonicity and conformality. Since biharmonic morphisms appear so rigid, we drop two of their characteristic conditions and only keep horizontal weak conformality and biharmonicity. In the light of the Fuglede-Ishihara theorem, such maps are the exact counterparts of harmonic morphisms. Moreover, the expression of the tension field of a horizontally weakly conformal map enlightens the relationship between minimality of the fibres and harmonicity of the map. Though the formula for the bitension field is far more intricate, we can apply it to some special cases to obtain new examples of biharmonic maps.

We will denote by

$$\mu = \frac{1}{m-n}\sum_{s=1}^{m-n}(\nabla_{e_s}e_s)^{\mathcal{H}} \text{ and } \nu = \frac{1}{n}\sum_{i=1}^{n}(\nabla_{e_i}e_i)^{\mathcal{V}}\,,$$

for an orthonormal frame $\{e_i, e_s\}_{i=1,\dots,n,s=1,\dots,m-n}$, with e_i horizontal and e_s vertical, the mean curvatures of the vertical and horizontal distributions, and A and B the second fundamental forms of the horizontal and vertical distributions:

$$A_E F = (\nabla_{E^{\mathcal{H}}} F^{\mathcal{H}})^{\mathcal{V}} \quad B_E F = (\nabla_{E^{\mathcal{V}}} F^{\mathcal{V}})^{\mathcal{H}}, \quad E, F \in \Gamma(TM).$$

THEOREM 4.1. *Let $\phi : (M^m, g) \to (N^n, h)$ $(m \geqslant n \geqslant 2)$ be a horizontally weakly conformal map of dilation λ between Riemannian manifolds, then ϕ is biharmonic if and only if, at every regular point,*

$$\begin{aligned} &d\phi(-\Delta X) + \Delta^{\mathcal{H}}(\ln\lambda)d\phi(X) + (n-2)\nu(\ln\lambda)d\phi(X) + 2d\phi(\nabla_{\mathrm{grad}^{\mathcal{H}}\ln\lambda}X) \\ &\quad + 2d\phi(\mathrm{grad}\, X(\ln\lambda)) - d\phi(\nabla_X(\mathrm{grad}^{\mathcal{H}}\ln\lambda)) - 2(\mathrm{div}^{\mathcal{H}} X)d\phi(\mathrm{grad}\ln\lambda) \\ &\quad + X(\ln\lambda)d\phi(X) - d\phi(\nabla_X \mathrm{grad}\ln\lambda) + \lambda^2 \,\mathrm{Ricci}^N(d\phi(X)) + (n-1)d\phi(\nabla_X\nu) \\ &\quad - (m-n)[\mu(\ln\lambda)d\phi(X) - \langle X, \mu\rangle d\phi(\mathrm{grad}\ln\lambda)] - (m-n)d\phi(\nabla_X\mu) + nd\phi(A^*_X\nu) \\ &\quad + \mathrm{trace}\, d\phi((\nabla A)^*_X - (\nabla A)X + 3A^*_{\nabla X} + (\nabla_X B^*)^* - B_{\nabla X} + 2B_{\nabla_X} - 2A^*_{\nabla_X}) = 0, \end{aligned}$$

where $X = (2-n)\,\mathrm{grad}^{\mathcal{H}}\ln\lambda - (m-n)\mu$, $\mathrm{grad}^{\mathcal{H}}$ being the horizontal gradient.

PROOF. At a regular point, the tension field of a horizontally weakly conformal map $\phi : (M^m, g) \to (N^n, h)$, of dilation λ, is $\tau(\phi) = d\phi(X)$, where $X = (2-n)\,\mathrm{grad}^{\mathcal{H}}\ln\lambda - (m-n)\mu$. Recall that the mean curvature of the horizontal distribution is $\nu = \mathrm{grad}^{\mathcal{V}}\ln\lambda$. Then, for an adapted orthonormal frame $\{e_i, e_s\}_{i=1,\dots,n,s=1,\dots,m-n}$, $e_i \in \mathcal{H}$ and $e_s \in \mathcal{V}$ on (M^m, g),

$$\nabla^{\phi}_{e_i} d\phi(X) = e_i(\ln\lambda)d\phi(X) + X(\ln\lambda)d\phi(e_i) - \langle X, e_i\rangle d\phi(\mathrm{grad}\ln\lambda) + d\phi(\nabla_{e_i}X),$$

so

$$\begin{aligned} \nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} d\phi(X) &= e_i(e_i(\ln\lambda))d\phi(X) + (e_i(\ln\lambda))^2 d\phi(X) + e_i(\ln\lambda)X(\ln\lambda)d\phi(e_i) \\ &\quad - e_i(\ln\lambda)\langle X, e_i\rangle d\phi(\mathrm{grad}\ln\lambda) + e_i(\ln\lambda)d\phi(\nabla_{e_i}X) + e_i(X(\ln\lambda))d\phi(e_i) \\ &\quad + X(\ln\lambda)\nabla^{\phi}_{e_i}d\phi(e_i) - e_i\langle X, e_i\rangle d\phi(\mathrm{grad}\ln\lambda) - \langle X, e_i\rangle e_i(\ln\lambda)d\phi(\mathrm{grad}\ln\lambda) \\ &\quad - \langle X, e_i\rangle(\mathrm{grad}^{\mathcal{H}}\ln\lambda)(\ln\lambda)d\phi(e_i) + \langle X, e_i\rangle\langle\mathrm{grad}\ln\lambda, e_i\rangle d\phi(\mathrm{grad}\ln\lambda) \\ &\quad - \langle X, e_i\rangle d\phi(\nabla_{e_i}\mathrm{grad}^{\mathcal{H}}\ln\lambda) + e_i(\ln\lambda)d\phi(\nabla_{e_i}X) \\ &\quad + (\nabla_{e_i}X)^{\mathcal{H}}(\ln\lambda)d\phi(e_i) - \langle e_i, \nabla_{e_i}X\rangle d\phi(\mathrm{grad}\ln\lambda) + d\phi(\nabla_{e_i}(\nabla_{e_i}X)^{\mathcal{H}}). \end{aligned}$$

Summing on the index i

$$\begin{aligned} (\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla_{e_i}e_i})d\phi(X) &= e_i(e_i(\ln\lambda))d\phi(X) + 2d\phi(\nabla_{\mathrm{grad}^{\mathcal{H}}\ln\lambda}X) \\ &\quad + d\phi(\mathrm{grad}(X(\ln\lambda))) + X(\ln\lambda)(\nabla d\phi)(e_i, e_i) - d\phi(\nabla_X(\mathrm{grad}^{\mathcal{H}}\ln\lambda)) \\ &\quad + (\nabla_{e_i}X)^{\mathcal{H}}(\ln\lambda)d\phi(e_i) - 2\langle e_i, \nabla_{e_i}X\rangle d\phi(\mathrm{grad}\ln\lambda) + d\phi(\nabla_{e_i}(\nabla_{e_i}X)^{\mathcal{H}}) \\ &\quad - (\nabla_{e_i}e_i)^{\mathcal{H}}(\ln\lambda)d\phi(X) + d\phi(\nabla_X(\nabla_{e_i}e_i)^{\mathcal{V}}) - d\phi(\nabla_{\nabla_{e_i}e_i}X). \end{aligned}$$

On the other hand, for the vertical bundle,

$$-\nabla^{\phi}_{\nabla_{e_s}e_s} d\phi(X) = -(\nabla_{e_s}e_s)^{\mathcal{H}}(\ln\lambda)d\phi(X) - X(\ln\lambda)d\phi(\nabla_{e_s}e_s) + \langle X, \nabla_{e_s}e_s\rangle d\phi(\operatorname{grad}\ln\lambda) - d\phi(\nabla_{(\nabla_{e_s}e_s)^{\mathcal{H}}}X) - d\phi([(\nabla_{e_s}e_s)^{\mathcal{V}}, X]),$$

so

$$(\nabla^{\phi}_{e_s}\nabla^{\phi}_{e_s} - \nabla^{\phi}_{\nabla_{e_s}e_s})d\phi(X) = d\phi(\nabla_{e_s}\nabla_{e_s}X) - d\phi(\nabla_{e_s}\nabla_X e_s) - d\phi(\nabla_{\nabla_{e_s}X}e_s) + d\phi(\nabla_{\nabla_X e_s}e_s) - (\nabla_{e_s}e_s)^{\mathcal{H}}(\ln\lambda)d\phi(X) - X(\ln\lambda)d\phi(\nabla_{e_s}e_s) + \langle X, \nabla_{e_s}e_s\rangle d\phi(\operatorname{grad}\ln\lambda) - d\phi(\nabla_{\nabla_{e_s}e_s}X) + d\phi(\nabla_X(\nabla_{e_s}e_s)^{\mathcal{V}}).$$

Therefore

$$-\Delta(\tau(\phi)) = d\phi(-\Delta X) - d\phi(\nabla_{e_i}(\nabla_{e_i}X)^{\mathcal{V}}) + \Delta^{\mathcal{H}}(\ln\lambda)d\phi(X) + n\nu(\ln\lambda)d\phi(X) + 2d\phi(\nabla_{\operatorname{grad}^{\mathcal{H}}\ln\lambda}X) + d\phi(\operatorname{grad}X(\ln\lambda)) - d\phi(\nabla_X(\operatorname{grad}^{\mathcal{H}}\ln\lambda)) - 2(\operatorname{div}^{\mathcal{H}}X)d\phi(\operatorname{grad}\ln\lambda) + X(\ln\lambda)(\tau(\phi) + (m-n)d\phi(\mu)) + (\nabla_{e_i}X)^{\mathcal{H}}(\ln\lambda)d\phi(e_i) + nd\phi(\nabla_X\nu) - (m-n)[\mu(\ln\lambda)d\phi(X) + X(\ln\lambda)d\phi(\mu) - \langle X, \mu\rangle d\phi(\operatorname{grad}\ln\lambda)] - d\phi(\nabla_{e_s}\nabla_X e_s) - d\phi(\nabla_{\nabla_{e_s}X}e_s) + d\phi(\nabla_{\nabla_X e_s}e_s) + d\phi(\nabla_X(\nabla_{e_s}e_s)^{\mathcal{V}}).$$

But

$$(\nabla_{e_i}X)^{\mathcal{H}}(\ln\lambda)d\phi(e_i) = d\phi(\operatorname{grad}X(\ln\lambda)) - d\phi(\nabla_X\operatorname{grad}\ln\lambda) - \langle(\nabla_{e_i}X)^{\mathcal{V}}, \operatorname{grad}^{\mathcal{V}}\ln\lambda\rangle d\phi(e_i).$$

However (cf. [5, Proposition 2.5.17])

$$-\langle(\nabla_{e_i}X)^{\mathcal{V}}, \operatorname{grad}^{\mathcal{V}}\ln\lambda\rangle d\phi(e_i) = -\langle(1/2)(\nabla_{e_i}X)^{\mathcal{V}}, \operatorname{grad}^{\mathcal{V}}\ln\lambda\rangle d\phi(e_i) + \langle(1/2)(\nabla_X e_i)^{\mathcal{V}}, \operatorname{grad}^{\mathcal{V}}\ln\lambda\rangle d\phi(e_i) - |\operatorname{grad}^{\mathcal{V}}\ln\lambda|^2 d\phi(X)$$

since $(\mathcal{L}_V g)(X, Y) = -g(\nabla_X Y + \nabla_Y X, V) = -d(\ln\lambda^2)(V)g(X, Y)$. Hence

$$-\langle(\nabla_{e_i}X)^{\mathcal{V}}, \operatorname{grad}^{\mathcal{V}}\ln\lambda\rangle d\phi(e_i) = -d\phi(\nabla_X\operatorname{grad}^{\mathcal{V}}\ln\lambda) - 2|\operatorname{grad}^{\mathcal{V}}\ln\lambda|^2 d\phi(X).$$

Therefore

$$(\nabla_{e_i}X)^{\mathcal{H}}(\ln\lambda)d\phi(e_i) = d\phi(\operatorname{grad}X(\ln\lambda)) - d\phi(\nabla_X\operatorname{grad}\ln\lambda) - d\phi(\nabla_X\nu) - 2|\nu|^2 d\phi(X)$$

and

$$-\Delta(\tau(\phi)) = d\phi(-\Delta X) - d\phi(\nabla_{e_i}(\nabla_{e_i}X)^{\mathcal{V}}) + \Delta^{\mathcal{H}}(\ln\lambda)d\phi(X) + (n-2)\nu(\ln\lambda)d\phi(X) + 2d\phi(\nabla_{\operatorname{grad}^{\mathcal{H}}\ln\lambda}X) + 2d\phi(\operatorname{grad}X(\ln\lambda)) - d\phi(\nabla_X(\operatorname{grad}^{\mathcal{H}}\ln\lambda)) - 2(\operatorname{div}^{\mathcal{H}}X)d\phi(\operatorname{grad}\ln\lambda) + X(\ln\lambda)\tau(\phi) - d\phi(\nabla_X\operatorname{grad}\ln\lambda) + (n-1)d\phi(\nabla_X\nu) - (m-n)[\mu(\ln\lambda)d\phi(X) - \langle X, \mu\rangle d\phi(\operatorname{grad}\ln\lambda)]$$

$$- d\phi(\nabla_{e_s}\nabla_X e_s) - d\phi(\nabla_{\nabla_{e_s}X} e_s) + d\phi(\nabla_{\nabla_X e_s} e_s) + d\phi(\nabla_X(\nabla_{e_s} e_s)^{\mathcal{V}}) .$$

LEMMA 4.2. *Let A and B be the second fundamental forms of the horizontal and vertical distributions. Then*

$$d\phi(\mathrm{trace}_h\, \nabla(\nabla X)^{\mathcal{V}}) = d\phi(\mathrm{trace}(\nabla A)X) ,$$

$$\begin{aligned} &d\phi(-\nabla_{e_s}\nabla_X e_s - \nabla_{\nabla_{e_s}X} e_s + \nabla_{\nabla_X e_s} e_s + \nabla_X(\nabla_{e_s} e_s)^{\mathcal{V}}) = -(m-n)d\phi(\nabla_X \mu) \\ &\quad + \mathrm{trace}\, d\phi((\nabla A)^*_X + 3A^*_{\nabla X} + (\nabla_X B^*)^* - B_{\nabla X} - 2A^*_{\nabla_X} + 2B_{\nabla_X}) + n d\phi(A^*_X \nu) . \end{aligned}$$

PROOF. Consider an adapted orthonormal frame $\{e_i, e_s\}_{i=1,\ldots,n,s=1,\ldots,m-n}$, $e_i \in \mathcal{H}$ and $e_s \in \mathcal{V}$. First observe that

$$d\phi((\nabla_{e_i} A)_{e_i} X) = d\phi(\nabla_{e_i}(\nabla_{e_i} X)^{\mathcal{V}}) ,$$

and $(\nabla_{e_s} A)_{e_s} X$ is vertical. For the second equality, we have

$$\begin{aligned} &\langle -\nabla_{e_s}\nabla_X e_s - \nabla_{\nabla_{e_s}X} e_s + \nabla_{\nabla_X e_s} e_s + \nabla_X \nabla_{e_s} e_s, e_j \rangle \\ &= \langle (\nabla_{e_s} A)_X e_j, e_s \rangle + \langle A_{\nabla_{e_s}X} e_j, e_s \rangle + \langle (\nabla_X B^*)_{e_s} e_j, e_s \rangle + \langle B^*_{\nabla_{e_s}X} e_j, e_s \rangle \\ &\quad + 2\langle \nabla_{[e_s,X]} e_j, e_s \rangle \end{aligned}$$

(confer [5, Th. 11.2.1 iii)].) Moreover

$$\langle \nabla_{[e_s,X]} e_j, e_s \rangle = \langle e_j, A^*_{\nabla_{e_s}X} e_s - A^*_{\nabla_X e_s} e_s - B_{\nabla_{e_s}X} e_s + B_{\nabla_X e_s} e_s \rangle .$$

To conclude, observe that

$$\begin{aligned} \langle (\nabla_{e_i} A)^*_X e_i, e_j \rangle &= -n\langle e_j, A^*_X \nu \rangle , \\ A^*_{\nabla_{e_i}X} e_i &= A^*_{\nabla_X e_i} e_i = 0 , \\ \langle (\nabla_X B^*)^*_{e_i} e_i, e_j \rangle &= B_{\nabla_{e_i}X} e_i = B_{\nabla_X e_i} e_i = 0 . \end{aligned}$$ □

Note that for a horizontally weakly conformal map, the curvature term of the bitension field becomes

$$-\lambda^2\, \mathrm{Ricci}^N(d\phi(X)) .$$

This completes the proof of Theorem 4.1. □

Three special cases are interesting enough to be stated separately.

COROLLARY 4.3. (i) *A conformal map $\phi : (M^m, g) \to (N^n, h)$ of conformal factor λ, between manifolds of equal dimensions ($m = n > 2$), is biharmonic if and only if*

$$\begin{aligned} &- d\phi(\Delta(\mathrm{grad}\ln\lambda)) - \Delta(\ln\lambda)d\phi(\mathrm{grad}\ln\lambda) + 2d\phi(\mathrm{grad}\,|\,\mathrm{grad}\ln\lambda|^2) \\ &\quad + (2-n)|\,\mathrm{grad}\ln\lambda|^2 d\phi(\mathrm{grad}\ln\lambda) + \lambda^2\,\mathrm{Ricci}^N(d\phi(\mathrm{grad}\ln\lambda)) = 0 . \end{aligned} \tag{12}$$

(ii) *A horizontally conformal map $\phi : (M^m, g) \to (N^2, h)$ of dilation λ into a surface ($m > n = 2$) is biharmonic if and only if*

$$\begin{aligned} &d\phi(-\Delta\mu) + \Delta^{\mathcal{H}}(\ln\lambda)d\phi(\mu) + 2d\phi(\nabla_{\mathrm{grad}^{\mathcal{H}}\ln\lambda}\mu) + 2d\phi(\mathrm{grad}\,\mu(\ln\lambda)) \\ &- d\phi(\nabla_\mu(\mathrm{grad}^{\mathcal{H}}\ln\lambda)) - 2(\mathrm{div}^{\mathcal{H}}\,\mu)d\phi(\mathrm{grad}\ln\lambda) \end{aligned}$$

$$+ (m-2)|\mu|^2 d\phi(\operatorname{grad}\ln\lambda) - d\phi(\nabla_\mu \operatorname{grad}\ln\lambda) + \lambda^2 \operatorname{Ricci}^N(d\phi(\mu))$$
$$+ d\phi(\nabla_\mu \nu) + (2-m)d\phi(\nabla_\mu \mu) - 2(m-2)\mu(\ln\lambda)d\phi(\mu)$$
$$+ \operatorname{trace} d\phi((\nabla A)^*_\mu - (\nabla A)\mu + 3A^*_{\nabla\mu} + (\nabla_\mu B^*)^* + 2B_{\nabla_\mu} - B_{\nabla\mu} - 2A^*_{\nabla_\mu})$$
$$+ 2d\phi(A^*_\mu \nu) = 0.$$

(iii) *A homothetic submersion* $\phi : (M^m, g) \to (N^n, h)$ $(m > n)$, *i.e.*, λ *is constant*, *is biharmonic if and only if*

$$d\phi(\Delta\mu) - \lambda^2 \operatorname{Ricci}^N(d\phi(\mu)) + (m-n)d\phi(\nabla_\mu \mu)$$
$$- (\operatorname{trace} d\phi((\nabla A)^*_\mu - (\nabla A)\mu + 3A^*_{\nabla\mu} + (\nabla_\mu B^*)^* + 2B_{\nabla_\mu} - B_{\nabla\mu} - 2A^*_{\nabla_\mu})) = 0\,.$$

REMARK 4.4. (i) The case $m = n = 2$ is trivial since any conformal map between surfaces is harmonic, hence biharmonic.

(ii) When $\phi : (M^m, g) \to (N^m, h)$ is the identity and $h = e^{2\rho}g$, Equation (12) becomes [6, Equation (3.1)]:

$$0 = \operatorname{trace}_g \nabla^2 \operatorname{grad}\rho + (-2\Delta\rho + (2-m)|\operatorname{grad}\rho|^2)\operatorname{grad}\rho$$
$$+ \frac{6-m}{2}\operatorname{grad}(|\operatorname{grad}\rho|^2) + \operatorname{Ricci}^g(\operatorname{grad}\rho).$$

(iii) A formula similar to the one of Theorem 4.1 was obtained in [2].

The first case of Corollary 4.3 is the most prolific and the following examples indicate not only that horizontally weakly conformal biharmonic maps are easier to come by than biharmonic morphisms, but also that, even for this situation, dimension four emerges as remarkable.

PROPOSITION 4.5. *The inverse stereographic projection* σ_{N}^{-1} *from* $(\boldsymbol{R}^n, ds^2)$ *into* $(\boldsymbol{S}^n \setminus \{\mathrm{N}\}, ds^2)$, *where* N *is the north pole and* ds^2 *the Euclidean metric*, *is a biharmonic map if and only if* $n = 4$. *Similarly*, *the identity map from* $(\boldsymbol{B}^n, ds^2)$ *into* $(\boldsymbol{B}^n, (4/(1-|x|^2)^2)ds^2)$, *where* $\boldsymbol{B}^n$ *is the open unit ball of* $\boldsymbol{R}^n$, *is a biharmonic map if and only if* $n = 4$.

Furthermore, *in either case*, *the biharmonic map is not a biharmonic morphism.*

PROOF. For the first case, the map is given by $\sigma_{\mathrm{N}}^{-1}(x) = (1/(1+|x|^2))(1-|x|^2, 2x)$ and is isometric to the identity map from the Euclidean space $(\boldsymbol{R}^n, ds^2)$ into $(\boldsymbol{R}^n, (4/(1+|x|^2)^2)ds^2)$. We can treat both examples at once, by considering the identity map ϕ on $\boldsymbol{R}^n$ or $\boldsymbol{B}^n$, from the Euclidean metric ds^2 into the conformal metric $(4/(1+\varepsilon|x|^2)^2)ds^2$ ($\varepsilon = 1$ for the sphere and $\varepsilon = -1$ for the ball).
The map ϕ is clearly conformal of dilation $\lambda = 2/(1+\varepsilon|x|^2)$, so

$$\operatorname{grad}\ln\lambda = -\frac{2\varepsilon}{1+\varepsilon|x|^2}x\,, \quad |\operatorname{grad}\ln\lambda|^2 = \frac{4|x|^2}{(1+\varepsilon|x|^2)^2}\,,$$
$$\operatorname{grad}|\operatorname{grad}\ln\lambda|^2 = \frac{8(1-\varepsilon|x|^2)}{(1+\varepsilon|x|^2)^3}x\,, \quad \Delta\ln\lambda = -\frac{2\varepsilon}{(1+\varepsilon|x|^2)^2}(n+(n-2)\varepsilon|x|^2)\,,$$
$$(\operatorname{trace}\nabla^2)(\operatorname{grad}\ln\lambda) = \frac{4}{(1+\varepsilon|x|^2)^3}(n+2+(n-2)\varepsilon|x|^2)x\,,$$

and Equation (12) becomes

$$-8(n-4)\frac{(1-\varepsilon|x|^2)}{(1+\varepsilon|x|^2)^3}x = 0\,,$$

so both maps, into the sphere or the Poincaré model, are biharmonic if and only if $n = 4$.

To know whether ϕ is also a biharmonic morphism we only need to check Equation (5), whose constituents are

$$\tau(\phi) = \frac{4\varepsilon}{1+\varepsilon|x|^2}x\,, \quad |\tau(\phi)|^2 = \frac{4.16}{(1+\varepsilon|x|^2)^4}|x|^2\,,$$
$$\Delta\lambda^2 = \frac{-16\varepsilon}{(1+\varepsilon|x|^2)^4}(4-2\varepsilon|x|^2)\,, \quad \mathrm{div}\langle d\phi, \tau(\phi)\rangle = \frac{16\varepsilon}{(1+\varepsilon|x|^2)^4}(4-2\varepsilon|x|^2)\,,$$
$$\nabla|\tau(\phi)|^2 = \frac{8.16(1-3\varepsilon|x|^2)}{(1+\varepsilon|x|^2)^5}x\,, \quad \langle d\phi, \tau(\phi)\rangle(\nabla|\tau(\phi)|^2) = \frac{8.16^2(1-3\varepsilon|x|^2)}{(1+\varepsilon|x|^2)^8}|x|^2\,,$$

therefore

$$|\tau(\phi)|^4 - 2\Delta\lambda^2|\tau(\phi)|^2 + 4\Delta\lambda^2\,\mathrm{div}\,\langle d\phi, \tau(\phi)\rangle + 4(\Delta\lambda^2)^2 + 2\langle d\phi, \tau(\phi)\rangle(\nabla|\tau(\phi)|^2)$$
$$= \frac{16^3}{(1+\varepsilon|x|^2)^8}|x|^2(1+2\varepsilon-3\varepsilon|x|^2)\,,$$

so ϕ cannot be a biharmonic morphism. □

PROPOSITION 4.6. *The identity map from $(\boldsymbol{B}^n, ds^2)$ to $(\boldsymbol{B}^n, h)$, where $\boldsymbol{B}^n$ is the unit ball in $\boldsymbol{R}^n$, ds^2 its Euclidean metric and $h_x = 4(1-|x|^2)^{-2}ds^2$ gives the hyperbolic space, is a biharmonic map if and only if $n = 4$. Furthermore, in either case, the biharmonic map is not a biharmonic morphism.*

PROOF. Call ϕ the identity from $(\boldsymbol{B}^n, ds^2)$ to $(\boldsymbol{B}^n, h)$. Clearly $h = e^{2\rho}ds^2$ for $\rho = \ln(2(1-|x|^2)^{-1})$ and

$$\nabla\rho = \frac{2}{1-|x|^2}x\,, \quad |\nabla\rho|^2 = \frac{4|x|^2}{(1-|x|^2)^2}\,,$$
$$\nabla|\nabla\rho|^2 = 8\frac{1+|x|^2}{(1-|x|^2)^3}x\,, \quad \Delta\rho = \frac{2}{(1-|x|^2)^2}(n(1-|x|^2)+2|x|^2)\,,$$
$$\mathrm{trace}_g\,\nabla^2\nabla\rho = \frac{4x}{(1-|x|^2)^3}(n+2-(n-2)|x|^2)\,.$$

So Equation (12) becomes

$$\frac{8x}{(1-|x|^2)^3}(4-n)(1+|x|^2) = 0$$

and ϕ is biharmonic only in dimension four. However, in dimension four, the Laplacian of its conformal factor $\lambda^2 = 4(1-|x|^2)^{-2}$ is

$$\Delta\lambda^2 = \frac{16}{(1-|x|^2)^4}(4+2|x|^2)$$

and

$$\tau(\phi) = -2\,\mathrm{grad}\,\rho = \frac{-4}{1-|x|^2}x \quad \text{and} \quad |\tau(\phi)|^2 = \frac{64|x|^2}{(1-|x|^2)^4}\,.$$

On the other hand:

$$\langle d\phi, \tau(\phi)\rangle = \frac{-16}{(1-|x|^2)^3}x \quad \text{and} \quad \mathrm{div}\langle d\phi, \tau(\phi)\rangle = \frac{-16}{(1-|x|^2)^4}(4+2|x|^2)\,.$$

Therefore

$$4\Delta\lambda^2 + 2\,\mathrm{div}\langle d\phi, \tau(\phi)\rangle = \frac{32}{(1-|x|^2)^4}(4+2|x|^2) \neq |\tau(\phi)|^2\,,$$

so ϕ is not a biharmonic morphism. □

PROPOSITION 4.7. *The identity map from* $(\boldsymbol{R}^n_+, ds^2)$ *to* $\boldsymbol{H}^n = (\boldsymbol{R}^n_+, \tilde{h} = (1/(x^n)^2)ds^2)$ *from the upper-half Euclidean space to the hyperbolic space is biharmonic if and only if* $n = 4$. *Furthermore, in either case, the biharmonic map is not a biharmonic morphism.*

PROOF. Call ϕ the identity map from $(\boldsymbol{R}^n_+, ds^2)$ to $\boldsymbol{H}^n$. Since $\tilde{h} = e^{2\rho}$ for $\rho = -\ln x^n$ and

$$\mathrm{grad}\,\rho = -\frac{1}{x^n}\frac{\partial}{\partial x^n}\,, \quad |\,\mathrm{grad}\,\rho|^2 = \frac{1}{(x^n)^2}\,,$$
$$\mathrm{grad}\,|\,\mathrm{grad}\,\rho|^2 = -\frac{2}{(x^n)^3}\frac{\partial}{\partial x^n}\,, \quad \Delta\rho = \frac{1}{(x^n)^2},$$

and Equation (12) becomes

$$\frac{2}{(x^n)^3}(n-4)\frac{\partial}{\partial x^n} = 0\,.$$

On the other hand, in dimension four, its dilation is $\lambda^2 = 1/(x^4)^2$ and $\tau(\phi) = -2\,\mathrm{grad}\,\rho = (2/x^4)\partial/\partial x^4$, so

$$|\tau(\phi)|^2 = \frac{4}{(x^4)^4}\,, \quad \Delta\lambda^2 = \frac{6}{(x^4)^4}\,,$$
$$\langle d\phi, \tau(\phi)\rangle = \frac{2}{(x^4)^3}\frac{\partial}{\partial x^4}\,, \quad \mathrm{div}\langle d\phi, \tau(\phi)\rangle = -\frac{6}{(x^4)^4}\,,$$

so

$$4\Delta\lambda^2 + 2\,\mathrm{div}\langle d\phi, \tau(\phi)\rangle = \frac{12}{(x^4)^4} \neq |\tau(\phi)|^2$$

and ϕ is not a biharmonic morphism. □

Not all conformal maps in dimension four are biharmonic.

EXAMPLE 4.8. (1) Consider the identity map ϕ from $(\boldsymbol{H}^4, g) = (\boldsymbol{R}^4_+, (x^4)^{-2}ds^2)$ to $(\boldsymbol{R}^4_+, ds^2)$. Then $ds^2 = e^{2\rho}g$ for $\rho = \ln x^4$ and, using the orthonormal basis $\{e_i = x^4\partial/\partial x^i\}_{i=1,\dots,4}$, we have

$$\mathrm{grad}^g\,\rho = e_4\,, \quad |\,\mathrm{grad}^g\,\rho|^2_g = 1\,, \quad \mathrm{grad}^g\,|\,\mathrm{grad}^g\,\rho|^2 = 0\,,$$

$$\Delta^g \rho = -3\,, \quad \mathrm{trace}_g \nabla^2 \operatorname{grad} \rho = -3e_4\,, \quad \mathrm{Ricci}(\operatorname{grad}^g \rho) = -3e_4\,,$$

since $g = e^{2\alpha} ds^2$ for $\alpha = -\ln x^4$ and

$$\nabla^g_{e_1} e_4 = -e_1\,, \quad \nabla^g_{e_2} e_4 = -e_2\,, \quad \nabla^g_{e_3} e_4 = -e_3\,, \quad \nabla^g_{e_4} e_4 = 0\,,$$
$$\nabla^g_{e_1} e_1 = \nabla^g_{e_2} e_2 = \nabla^g_{e_3} e_3 = e_4\,.$$

Testing Equation (12), we have $-2e_4 \neq 0$, so ϕ is not biharmonic.
Moreover, from simple considerations

$$\tau(\phi) = -2e_4\,, \quad \lambda^2 = (x^4)^2\,, \quad \Delta\lambda^2 = -2(x^4)^2\,,$$
$$\langle d\phi, \tau(\phi)\rangle = -2e_4\,, \quad \operatorname{div}\langle d\phi, \tau(\phi)\rangle = 6,$$

so $4\Delta\lambda^2 + 2\operatorname{div}\langle d\phi, \tau(\phi)\rangle \neq |\tau(\phi)|^2$ and ϕ does not satisfy (5).

(2) Let ϕ be the identity map from $(\boldsymbol{R}^n, g = 4/(1+\varepsilon|x|^2)^2 ds^2)$ to $(\boldsymbol{R}^n, ds^2)$ $(\varepsilon = \pm 1)$. It is clearly conformal of dilation $\lambda^2 = (1+\varepsilon|x|^2)^2/4$ and $g = e^{2\rho} ds^2$ for $\rho = \ln 2 - \ln(1 + \varepsilon|x|^2)$. Then $e_i = ((1+\varepsilon|x|^2)/2)\,\partial/\partial x^i$ is an orthonormal basis for g and

$$\nabla^g_{e_i} e_j = \nabla^{ds^2}_{e_i} e_j + e_i(\rho)e_j + e_j(\rho)e_i - \langle e_i, e_j\rangle \operatorname{grad}\rho = -\varepsilon x^j e_i + \delta_{ij}\varepsilon X$$

where $X = \sum_{k=1}^n x^k e_k$. Therefore

$$\operatorname{grad}^g \ln\lambda = \frac{(1+\varepsilon|x|^2)^2}{4}\frac{\partial}{\partial x^i}(\ln\lambda)\frac{\partial}{\partial x^i} = \varepsilon X\,,$$
$$|\operatorname{grad}^g \ln\lambda|^2_g = \frac{4}{(1+\varepsilon|x|^2)^2}\frac{(1+\varepsilon|x|^2)^2}{4}|x|^2 = |x|^2\,,$$
$$\operatorname{grad}^g |\operatorname{grad}^g \ln\lambda|^2_g = \frac{(1+\varepsilon|x|^2)^2}{4}\frac{\partial}{\partial x^i}(|x|^2)\frac{\partial}{\partial x^i} = (1+\varepsilon|x|^2)X\,.$$

Since $\Delta^g \ln\lambda = \sum_{i=1}^4 e_i(e_i(\ln\lambda)) - (\nabla^g_{e_i} e_i)(\ln\lambda)$ and

$$\sum_{i=1}^n \nabla^g_{e_i} e_i = (n-1)\varepsilon X\,, \quad X(\ln\lambda) = \varepsilon|x|^2\,, \quad \sum_{i=1}^n (\nabla^g_{e_i} e_i)(\ln\lambda) = (n-1)|x|^2\,,$$
$$e_i(\ln\lambda) = \varepsilon x^i\,, \quad e_i(e_i(\ln\lambda)) = \varepsilon(1+\varepsilon|x|^2)/2\,,$$

we have $\Delta^g \ln\lambda = \varepsilon(n + (2-n)\varepsilon|x|^2)/2$.
On the other hand

$$\Delta \operatorname{grad}^g \ln\lambda = -\sum_{i=1}^n \nabla^g_{e_i}\nabla^g_{e_i}(\operatorname{grad}^g \ln\lambda) - \nabla^g_{\nabla^g_{e_i} e_i}(\operatorname{grad}^g \ln\lambda)\,,$$

and

$$-\sum_{i=1}^n \nabla^g_{\nabla^g_{e_i} e_i}(\operatorname{grad}^g \ln\lambda) = -\nabla^g_{(n-1)\varepsilon X}(\varepsilon X) = -(n-1)\frac{1+\varepsilon|x|^2}{2}X\,,$$
$$\nabla^g_{e_i}(\operatorname{grad}^g \ln\lambda) = \varepsilon\left(\frac{1-\varepsilon|x|^2}{2}e_i + \varepsilon x^i X\right),$$

$$\nabla^g_{e_i}\nabla^g_{e_i}(\mathrm{grad}^g \ln\lambda) = -\frac{1+\varepsilon|x|^2}{2}x^i e_i + (1+\varepsilon(x^i)^2)X\,,$$

$$\sum_{i=1}^{n}\nabla^g_{e_i}\nabla^g_{e_i}(\mathrm{grad}^g \ln\lambda) = \frac{2n-1+\varepsilon|x|^2}{2}X\,,$$

so

$$-\Delta\,\mathrm{grad}^g \ln\lambda = \frac{n+(2-n)\varepsilon|x|^2}{2}X.$$

Equation (12) becomes

$$(2+(4-n)\varepsilon|x|^2)X = 0\,,$$

which is impossible, whatever the value of n.

EXAMPLE 4.9. Let (M^2, h) be a Riemannian surface of Gaussian curvature G_h, $\beta : M^2\times \boldsymbol{R} \to \boldsymbol{R}^*$ and $\lambda : \boldsymbol{R}\to\boldsymbol{R}^*$ two positive functions. Consider the doubly twisted product $(\boldsymbol{R}\times M^2, g = \beta^2 dt^2 + \lambda^{-2}h)$. Then, the projection

$$\phi : (\boldsymbol{R}\times M^2, g = \beta^2 dt^2 + \lambda^{-2}h) \to (M^2, h)$$
$$(t, x) \mapsto x$$

is a biharmonic map if and only if

$$\begin{aligned} 0 = {} & -\lambda^4\,\mathrm{grad}_h(\Delta_h \ln\beta) - 2\lambda^4 G_h\,\mathrm{grad}_h \ln\beta + (3/2)V(\lambda^2|\,\mathrm{grad}_h \ln\beta|^2)V \\ & - \lambda^4|\,\mathrm{grad}_h \ln\beta|^2\,\mathrm{grad}_h \ln\beta - \nabla_V\nabla_V[\lambda^2\,\mathrm{grad}_h \ln\beta] - (\lambda^4/2)\,\mathrm{grad}_h\,|\,\mathrm{grad}_h \ln\beta|^2_h \\ & - \lambda^2 V(V(\ln\lambda)\,\mathrm{grad}_h \ln\beta - 2V(\ln\lambda)[\nabla_V(\lambda^2\,\mathrm{grad}_h \ln\beta) - \lambda^2|\,\mathrm{grad}_h \ln\beta|^2_h V]\,. \end{aligned} \tag{13}$$

In particular, for $\beta = c_2\, e^{\int f(x)\,dx}$ with

$$f(x) = \frac{-c_1(1+e^{c_1 x})}{1-e^{c_1 x}}$$

and $c_1, c_2 \in \boldsymbol{R}^*$, we have a family of Riemannian submersions

$$\phi : (\boldsymbol{R}^2\times\boldsymbol{R}, dx^2+dy^2+\beta^2(x)dz^2) \to (\boldsymbol{R}^2, dx^2+dy^2)$$
$$\phi(x,y,z) = (x,y)$$

which are proper biharmonic maps.

In fact, it is easily checked that ϕ is a horizontally conformal submersion of dilation λ^2. Let V be the unit vertical vector $\beta^{-1}d/dt$, using the Koszul formula one can show that for vector fields E and F on $\boldsymbol{R}\times M^2$

$$A_E F = g(E^{\mathcal{H}}, F^{\mathcal{H}})V(\ln\lambda)V\,.$$

On the other hand, if W is a vertical vector field, then

$$B_V W = -\lambda^2\,\mathrm{grad}_h(\ln\beta)\langle W, V\rangle\,.$$

Note also that the mean curvature of the fibres is $\mu = \nabla^g_V V = -\lambda^2\,\mathrm{grad}_h(\ln\beta)$ and the mean curvature of the horizontal distribution is $\nu = V(\ln\lambda)V$.

Choosing a geodesic frame $\{e_1, e_2\}$ around a point $p \in M^2$ and evaluating all subsequent formulas at a point $(p, t) \in M^2 \times \boldsymbol{R}$, straightforward computations show that (summing on repeated indices)

$$\begin{aligned}
&[\operatorname{trace}((\nabla A)^*_\mu - (\nabla A)\mu + 3A^*_{\nabla\mu} + (\nabla_\mu B^*)^* + 2B_{\nabla_\mu} - B_{\nabla\mu} - 2A^*_{\nabla_\mu}) + 2A^*_\mu \nu]^{\mathcal{H}} \\
&= V(V(\ln\lambda))\mu + 3(V(\ln\lambda)^2)\mu - 3V(\ln\lambda)V(\lambda e_i(\ln\beta))\lambda e_i \\
&\quad + \mu(g(\lambda e_j, \mu))\lambda e_j + |\mu|^2\mu.
\end{aligned}$$

Since $\operatorname{grad}^{\mathcal{H}} \ln\lambda = 0$, the only remaining terms are

$$\begin{aligned}
[\nabla_\mu \mu]^{\mathcal{H}} &= (\lambda^4/2)\operatorname{grad}_h(|\operatorname{grad}_h \ln\beta|_h^2)\,, \\
[\nabla_\mu \nu]^{\mathcal{H}} &= -(V(\ln\lambda))^2\mu\,, \\
\Delta^{\mathcal{H}}(\ln\lambda) &= -2(V(\ln\lambda))^2\,, \\
[\nabla_\mu \operatorname{grad}\ln\lambda]^{\mathcal{H}} &= -(V(\ln\lambda))^2\mu\,, \\
[\Delta\mu]^{\mathcal{H}} &= \lambda^4 e_i(e_i(e_j \ln\beta)))e_j - (V(\ln\lambda))^2 \operatorname{grad}^{\mathcal{H}}\ln\beta \\
&\quad - V(\ln\lambda)V(\lambda e_j(\ln\beta))\lambda e_j + V(V(\lambda e_j(\ln\beta)))\lambda e_j \\
&\quad - \lambda^4|\operatorname{grad}_h \ln\beta|_h^2 \operatorname{grad}_h \ln\beta + (\lambda^4/2)\operatorname{grad}_h(|\operatorname{grad}_h \ln\beta|_h^2) \\
&\quad + \lambda e_k(\ln\beta)\lambda^2[\nabla_{\lambda e_i}(\nabla^{M^2}_{e_i} e_k)]^{\mathcal{H}}\,.
\end{aligned}$$

So the projection ϕ is biharmonic if and only if (still summing on repeated indices)

$$\begin{aligned}
0 &= -\lambda^4 e_i(e_i(e_j \ln\beta))e_j - V(V(\lambda e_j(\ln\beta)))\lambda e_j - (\lambda^4/2)\operatorname{grad}_h|\operatorname{grad}_h \ln\beta|_h^2 \\
&\quad - \lambda^4 G_h \operatorname{grad}_h \ln\beta - \lambda^2 V(V(\ln\lambda)\operatorname{grad}_h \ln\beta - 2V(\ln\lambda)V(\lambda e_i(\ln\beta))\lambda e_i \\
&\quad - \lambda e_k(\ln\beta)\lambda^2[\nabla_{\lambda e_i}(\nabla^{M^2}_{e_i} e_k)]^{\mathcal{H}}\,.
\end{aligned}$$

Using

$$\begin{aligned}
&V(V(\lambda e_j(\ln\beta)))\lambda e_j = -(3/2)V(\lambda^2|\operatorname{grad}_h \ln\beta|^2)V + \lambda^4|\operatorname{grad}_h \ln\beta|^2 \operatorname{grad}_h \ln\beta \\
&+ \nabla_V\nabla_V[\lambda^2 \operatorname{grad}_h \ln\beta]\,, \\
&V(\lambda e_j(\ln\beta))\lambda e_j = \nabla_V(\lambda^2 \operatorname{grad}_h \ln\beta) - \lambda^2|\operatorname{grad}_h \ln\beta|_h^2 V\,, \\
&\operatorname{grad}_h(\Delta_h \ln\beta) = e_i e_i e_j(\ln\beta)e_j - G_h \operatorname{grad}_h \ln\beta \\
&- \langle\nabla_{e_1}\nabla_{e_1}e_1 + \nabla_{e_2}\nabla_{e_2}e_1, e_2\rangle e_2(\ln\beta)e_1 - \langle\nabla_{e_1}\nabla_{e_1}e_2 + \nabla_{e_2}\nabla_{e_2}e_2, e_1\rangle e_1(\ln\beta)e_2\,, \\
&\lambda e_k(\ln\beta)\lambda^2[\nabla_{\lambda e_i}(\nabla^{M^2}_{e_i} e_k)]^{\mathcal{H}} = -\lambda^4[e_1(\ln\beta)h(\nabla^{M^2}_{e_1}\nabla^{M^2}_{e_1} e_2 + \nabla^{M^2}_{e_2}\nabla^{M^2}_{e_2} e_2, e_1)e_2 \\
&+ e_2(\ln\beta)h(\nabla^{M^2}_{e_1}\nabla^{M^2}_{e_1} e_1 + \nabla^{M^2}_{e_2}\nabla^{M^2}_{e_2} e_1, e_2)e_1],
\end{aligned}$$

we obtain the biharmonic equation (13) for ϕ.

For the Riemannian submersion

$$\phi : (\boldsymbol{R}^2 \times \boldsymbol{R}, dx^2 + dy^2 + \beta^2(x)dz^2) \to (\boldsymbol{R}^2, dx^2 + dy^2)$$

$$\phi(x, y, z) = (x, y),$$

Equation (13) reduces to

$$ff' + f'' = 0,$$

where $f = \ln \beta$. Solving this equation, we obtain the last statement in the example.

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