Tohoku Math. J.
61 (2009), 589–601

INTEGRAL POINTS ON THREEFOLDS AND OTHER VARIETIES

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(Received October 28, 2008, revised July 3, 2009)

Abstract. We prove sufficient conditions for the degeneracy of integral points on certain threefolds and other varieties of higher dimension. In particular, under a normal crossings assumption, we prove the degeneracy of integral points on an affine threefold with seven ample divisors at infinity. Analogous results are given for holomorphic curves. As in our previous works [2], [5], the main tool involved is Schmidt's Subspace Theorem, but here we introduce a technical novelty which leads to stronger results in dimension three or higher.

Introduction. In the present paper we build upon the work appearing in [1], [3] and [5], by providing sufficient conditions for the degeneracy of integral points on certain threefolds and other varieties of higher dimension.

The underlying method of proof, as in the aforementioned papers, is to embed the variety of interest into a suitable projective space of high dimension, and then apply Schmidt's Subspace Theorem with an appropriately chosen set of hyperplanes. In carrying out this method, a technical difficulty appears in dealing with integral points which, for some place v, are v-adically close to several divisors at infinity.

In the case of surfaces, under the assumption that the divisors at infinity are in 'general' position, this technical difficulty can be overcome. In this case, an integral point can be v-adically close to at most two divisors at infinity; then by applying a simple lemma from linear algebra on vector spaces with two filtrations (see [3], Lemma 3.2), one reduces, essentially, to the case of integral points which are v-adically close to only a single divisor at infinity. In dimension $q > 2$, where a point can be v-adically close to q divisors at infinity, this technique breaks down as the analogous linear algebra lemma for $q > 2$ filtrations is no longer true.

However, as in the paper [5], the linear algebra lemma for two filtrations can still be applied in higher dimensions by dividing the set of divisors at infinity into two sets, whose sums represent two divisors to which the lemma is applied; this reduces the problem essentially to the case where $[(q+1)/2]$ divisors are at infinity and again leads to some improvement with respect to a naive approach. This method is also adopted in [1], with some further technical devices, including consideration of the rate of v-adic convergence of a sequence of integral points to the approached divisors.

By means of these methods, for instance, the following result has been proved, which we state after introducing a bit of notation, used throughout this paper.

2000 *Mathematics Subject Classification.* Primary 11G35; Secondary 14G25, 32H30, 11J97.

Key words and phrases. Integral points, holomorphic curves, Schmidt Subspace Theorem, Diophantine approximation.

We let k be a number field, S be a finite set of places of k, including the archimedean ones, and denote by $\mathcal{O}_S$ the ring of S-integers in k. Let $\tilde{\mathcal{X}}$ be a projective nonsingular irreducible variety of dimension q defined over k. Let $D_1, \ldots, D_r$ be reduced effective divisors, such that at most q of them contain a given point. We define $\mathcal{X} := \tilde{\mathcal{X}} \setminus \bigcup_{i=1}^r D_i$.

THEOREM A (Autissier [1]). *Suppose that $D_1, \ldots, D_r$ are ample and that $q > 1$ and $r \geq q^2$. Then no set of S-integral points of $\mathcal{X}$ is Zariski-dense in $\mathcal{X}$. Furthermore, there exists a proper Zariski-closed subset $\mathcal{Y} \subset \mathcal{X}$, independent of k and S, such that $\mathcal{X} \setminus \mathcal{Y}$ has only finitely many S-integral points of $\mathcal{X}$.*

The case $q = 2$ of Theorem A was proved in [3] under certain numerical assumptions on the divisors D_i. The general case of $q = 2$ was proved in [5] (along with the case $r > 2q[(q+1)/2]$ for $q \geq 3$).

The purpose of this paper is to introduce a new approach in higher dimensions, to cope with the absence of the mentioned linear algebra lemma. In all of the related previous approaches, a key element is constructing rational functions which vanish to a high order along the divisors at infinity which contain a given point. In the present paper, we modify this step by considering functions which vanish to a high order along the intersection of such divisors; actually, in this paper we apply this method only when the said intersection consists of isolated points. This will be advantageous because imposing vanishing conditions at a point is significantly less restrictive than imposing vanishing along whole divisors.

Throughout this paper we shall work under the assumption that the divisors at infinity have normal crossings; this kind of assumption is very common in this context and appears, for instance, in the celebrated conjectures of Vojta. Under the normal crossings assumption, we shall improve Theorem A by proving the following:

THEOREM CLZ. *Suppose that $D_1, \ldots, D_r$ are ample, that $\sum_{i=1}^r D_i$ is a reduced normal crossings divisor, and that $q > 2$ and $r > q^2 - q$. Then no set of S-integral points of $\mathcal{X}$ is Zariski-dense in $\mathcal{X}$. Furthermore, there exists a proper Zariski-closed subset $\mathcal{Y} \subset \mathcal{X}$, independent of k and S, such that $\mathcal{X} \setminus \mathcal{Y}$ has only finitely many S-integral points of $\mathcal{X}$.*

We remark that an entirely similar method of proof leads to an analogous result for holomorphic curves (requiring now only that all objects involved be defined over $\boldsymbol{C}$). The only substantial modification consists in using Vojta's version of Cartan's Second Main Theorem [8] in place of the Schmidt Subspace Theorem (see, e.g., [5]). We explicitly state such a result (omitting a proof).

THEOREM CLZ (holomorphic version). *Suppose that $D_1, \ldots, D_r$ are ample, that $\sum_{i=1}^r D_i$ is a reduced normal crossings divisor, and that $q > 2$ and $r > q^2 - q$. Then there does not exist a holomorphic map $f : \boldsymbol{C} \to \mathcal{X}$ with Zariski-dense image. Furthermore, there exists a proper Zariski-closed subset $\mathcal{Y} \subset \mathcal{X}$ such that the image of any non-constant holomorphic map $f : \boldsymbol{C} \to \mathcal{X}$ is contained in $\mathcal{Y}$.*

1. Proofs. Suppose we have any infinite set of S-integral points on $\mathcal{X}$, and let us consider a positive-dimensional irreducible component $\mathcal{Z}$ of its Zariski closure; it is a well-known easy fact that we may find a sequence (P_i) of points in our set such that every infinite subsequence has Zariski closure $\mathcal{Z}$. Then, by replacing the sequence with a subsequence if necessary, we may assume that for each $v \in S$, P_i converges v-adically to some point $P_v \in \tilde{\mathcal{X}}(k_v)$.

We now explain in more detail our strategy for improvement on previous results, restricting for clarity to the case $q = 3$, which is the first case which can't be treated by the simple linear algebra approach.

As in previous approaches based on the Schmidt Subspace Theorem, one considers an appropriate linear combination D of the ample divisors $D_1, \dots, D_r$ and the space $L(nD)$ of rational functions on $\tilde{\mathcal{X}}$ whose pole-divisor is $\geq -nD$. We want to construct for each place $v \in S$ a basis of $L(nD)$ which consists of functions that 'on average' vanish at P_v. This vanishing implies that certain linear forms corresponding to this basis take small values at P_i, which allows the successful application of Diophantine Approximation.

Now, this construction depends on the set of divisors among $D_1, \dots, D_r$ which contain P_v. For instance, when P_v lies on a single such divisor D_i, since $k(P_v)$ may be a transcendental extension of k, to obtain functions which vanish at P_v we are forced to construct functions which vanish along the whole divisor D_i. The efficiency of this construction depends on the numerical properties of D and D_i.

When P_v lies on two distinct divisors D_i, D_j, the previously quoted papers argue as follows: one considers the two filtrations of $L(nD)$ with respect to the order of vanishing along D_i and D_j, and by using the linear algebra [3, Lemma 3.2] one can combine these two filtrations to define a basis of $L(nD)$ having on average the said vanishing property simultaneously with respect to D_i and D_j; effectively, this reduces the present situation to the case of a single divisor. This linear algebra lemma is of a general nature and does not use any peculiar property of $L(nD)$.

For surfaces, one often assumes that no three of the divisors intersect, so no further case appears. However, already for threefolds, the linear algebra lemma is insufficient to cope with the situation when P_v lies on three distinct divisors D_i; in fact, one can construct counterexamples to the analogous conclusion of the lemma for three filtrations of a vector space, not only in the general linear algebra case, but also in our relevant geometric context. This shows that in higher dimensions some other tools are needed to replace the lemma.

As mentioned above, the method of the paper [5] is to divide the set of divisors into two subsets, and then to apply the lemma to the two filtrations coming from the two subsets. A similar idea occurs in [1], where one considers the filtration corresponding to the pair of divisors which are most rapidly approached v-adically by the sequence (P_i).

In the present paper we adopt a still different approach: for instance, in dimension three and when three divisors intersect at P_v, in many situations (clarified in the detailed proof) we construct a filtration corresponding to a pointwise vanishing rather than a vanishing along a whole divisor. In these cases the filtration lemma will be avoided completely.

These filtrations may be constructed naively, by using linear algebra on the power series locally representing the functions; however, in some cases one may improve the orders of vanishing by taking into account that the functions lie in the space $L(nD)$, which restricts the possibilities for the coefficients of the said power series (see Remark 1.2).

This new pointwise method will be adopted only when the sequence of integral points converges in a certain way to some divisors at infinity; when the convergence is of a different type, we use the method of the papers [1] and [5] (e.g., Corollary 1.4).

In the rest of the paper, we work with the notation and assumptions of the Introduction. For integers $a_1, \dots, a_r > 0$, we define $D = \sum_{i=1}^r a_i D_i$ and we assume that $\sum_{i=1}^r D_i$ is a reduced normal crossings divisor on $\tilde{\mathcal{X}}$. By $D_i.D_j$ or D^q we shall denote intersection products, and we let $l(nD) = \dim L(nD)$.

We let k be a number field and, for a place v of k, we let k_v be the completion and $|\cdot|_v$ be a corresponding absolute value normalized so that the absolute Weil height of $x \in k^*$ is $h(x) = \sum_v \log^+ |x|_v$.

We start with some lemmas.

LEMMA 1.1. *Let v be a place of k and (P_m) be a sequence in $\mathcal{X}(k_v)$ such that*:

1. *(P_m) converges v-adically to a point $P_v \in \tilde{\mathcal{X}}(k) \subset \tilde{\mathcal{X}}(k_v)$.*
2. *The limit P_v lies in the intersection of the q distinct divisors $D_1, \dots, D_q$.*
3. *Letting for $i = 1, \dots, q$, $\phi_i \in k(\tilde{\mathcal{X}})$ be a rational function defining D_i locally at P_v, the limit*

$$\lim_{m\to\infty} \frac{\log|\phi_i(P_m)|_v}{\sum_{l=1}^q \log|\phi_l(P_m)|_v} \tag{1.1}$$

exists and is nonzero for $i = 1, \dots, q$. Denote the limit in (1.1) *by t_i.*

4.

$$D^q > \frac{\left(\sum_{i=1}^q a_i t_i\right)^q (1+1/q)^q}{\prod_{i=1}^q t_i}. \tag{1.2}$$

Given these conditions, there exists $\varepsilon > 0$ such that for all large n there exists a basis $f_1, \dots, f_{l(nD)} \in k(\tilde{\mathcal{X}})$ of $L(nD)$ with

$$\lim_{m\to\infty} |\phi_i(P_m)|_v^{-\varepsilon} \left| \prod_{j=1}^{l(nD)} f_j(P_m) \right|_v = 0, \qquad i = 1, \dots, q. \tag{1.3}$$

PROOF. Since $\sum_{i=1}^r D_i$ is a reduced normal crossings divisor, it follows that the maximal ideal $\mathfrak{m}_{P_v}$ of $\mathcal{O}_{\tilde{\mathcal{X}},P_v}$ is generated by $\phi_1, \dots, \phi_q$ and that (over k) the completion of $\mathcal{O}_{\tilde{\mathcal{X}},P_v}$ with respect to $\mathfrak{m}_{P_v}$ is naturally isomorphic to the power series ring $k[[\phi_1, \dots, \phi_q]]$. For $\mathbf{i} = (i_1, \dots, i_q) \in N^q$, let $\phi^{\mathbf{i}} = \prod_{j=1}^q \phi_j^{i_j}$. If f is regular at P_v, let $\sum_{\mathbf{i}\in N^q} c_{\mathbf{i}}(f)\phi^{\mathbf{i}}$ be its canonical image in $k[[\phi_1, \dots, \phi_q]]$. Let $\mathbf{a} = (a_1, \dots, a_q)$ and $\mathbf{t} = (t_1, \dots, t_q)$. Then for any $f \in L(nD)$, the function $\phi^{n\mathbf{a}} f$ is regular at P_v. Let V_j be the subspace of $L(nD)$ given by

$$V_j = \{f \in L(nD)\,;\ c_{\mathbf{i}}(\phi^{n\mathbf{a}} f) = 0 \quad \text{if } \mathbf{i}\cdot\mathbf{t} < j\}.$$

This construction is motivated by the fact that

$$\lim_{m\to\infty} \frac{\log|\phi^{\mathbf{b}}(P_m)|_v}{\sum_{l=1}^{q} \log|\phi_l(P_m)|_v} = \mathbf{b}\cdot\mathbf{t}\,.$$

Let N be the largest positive integer such that $V_N \neq 0$ (this exists since $L(nD)$ is finite-dimensional). Note that $V_0 = L(nD)$ and that we have a vector space filtration $L(nD) = V_0 \supset V_1 \supset \cdots \supset V_N \supset 0$. Let $f_1, \ldots, f_{l(nD)}$ be a basis of $L(nD)$ with respect to this filtration. That is, take a basis of V_N and successively complete it to bases of V_{N-1}, $V_{N-2}, \ldots, V_0 = L(nD)$.

We have

$$\prod_{j=1}^{l(nD)} f_j = \frac{\prod_{j=1}^{l(nD)} \phi^{n\mathbf{a}} f_j}{\phi^{l(nD)n\mathbf{a}}}\,.$$

Then to show (1.3) it suffices to show that

$$\lim_{m\to\infty} \frac{\log\prod_{j=1}^{l(nD)} |\phi^{n\mathbf{a}} f_j(P_m)|_v}{\log|\phi^{l(nD)n\mathbf{a}}(P_m)|_v} > 1\,.$$

If $f \in V_j$, then

$$\lim_{m\to\infty} \frac{\log|\phi^{n\mathbf{a}} f(P_m)|_v}{\sum_{l=1}^{q} \log|\phi_l(P_m)|_v} \geq j\,.$$

Thus

$$\begin{aligned}
\lim_{m\to\infty} \frac{\log\prod_{j=1}^{l(nD)} |\phi^{n\mathbf{a}} f_j(P_m)|_v}{\log|\phi^{l(nD)n\mathbf{a}}(P_m)|_v} &= \lim_{m\to\infty} \frac{\dfrac{\log\prod_{j=1}^{l(nD)} |\phi^{n\mathbf{a}} f_j(P_m)|_v}{\sum_{l=1}^{q}\log|\phi_l(P_m)|_v}}{\dfrac{\log|\phi^{l(nD)n\mathbf{a}}(P_m)|_v}{\sum_{l=1}^{q}\log|\phi_l(P_m)|_v}} \\
&= \frac{1}{l(nD)n\mathbf{a}\cdot\mathbf{t}} \lim_{m\to\infty} \sum_{j=1}^{l(nD)} \frac{\log|\phi^{n\mathbf{a}} f_j(P_m)|_v}{\sum_{l=1}^{q}\log|\phi_l(P_m)|_v} \\
&= \frac{1}{l(nD)n\mathbf{a}\cdot\mathbf{t}} \lim_{m\to\infty} \sum_{j=0}^{N} \sum_{f_h\in V_j\setminus V_{j+1}} \frac{\log|\phi^{n\mathbf{a}} f_h(P_m)|_v}{\sum_{l=1}^{q}\log|\phi_l(P_m)|_v} \\
&\geq \frac{1}{l(nD)n\mathbf{a}\cdot\mathbf{t}} \sum_{j=0}^{N} j(\dim V_j - \dim V_{j+1}) \\
&\geq \frac{1}{l(nD)n\mathbf{a}\cdot\mathbf{t}} \sum_{j=1}^{N} \dim V_j\,.
\end{aligned}$$

So it suffices to show that

$$\sum_{j=1}^{N} \dim V_j > l(nD)n\mathbf{a}\cdot\mathbf{t}\,.$$

From the definition of V_j it is clear that

$$\dim V_j \geq l(nD) - \#\{\mathbf{i} \in \boldsymbol{N}^q\,;\ \mathbf{i}\cdot\mathbf{t} < j\}\,.$$

Let $U = \{\mathbf{x} \in (\boldsymbol{R}_{\geq 0})^q\,;\ \mathbf{x}\cdot\mathbf{t} < 1\}$. By standard lemmas on counting lattice points in homogeneously expanding domains, we have

$$\#\{\mathbf{i} \in \boldsymbol{N}^q\,;\ \mathbf{i}\cdot\mathbf{t} < j\} = \mathrm{Vol}(U)j^q + O(j^{q-1})\,.$$

It is easily calculated that $\mathrm{Vol}(U) = \left(q!\prod_{i=1}^q t_i\right)^{-1}$. By Riemann-Roch, since D is ample,

$$l(nD) = \frac{D^q n^q}{q!} + O(n^{q-1})\,.$$

Thus

$$\dim V_j \geq \frac{D^q n^q}{q!} - \frac{j^q}{q!\prod_{i=1}^q t_i} + O(n^{q-1})\,.$$

So, for $M = \left[n\sqrt[q]{D^q\prod_{i=1}^q t_i}\right]$,

$$\begin{aligned}
\sum_{j=1}^N \dim V_j &\geq \frac{1}{q!}\sum_{j=1}^M\left(D^q n^q - \frac{j^q}{\prod_{i=1}^q t_i} + O(n^{q-1})\right)\\
&\geq \frac{1}{q!}\left(\left(D^q\prod_{i=1}^q t_i\right)^{1/q} D^q n^{q+1} - \frac{\left(D^q\prod_{i=1}^q t_i\right)^{1/q} D^q n^{q+1}}{(q+1)}\right) + O(n^q)\\
&\geq \frac{q\left(D^q\prod_{i=1}^q t_i\right)^{1/q} D^q n^{q+1}}{(q+1)!} + O(n^q)\,.
\end{aligned}$$

Then (1.3) holds if

$$\frac{q\left(D^q\prod_{i=1}^q t_i\right)^{1/q} D^q n^{q+1}}{(q+1)!} > \frac{D^q\mathbf{a}\cdot\mathbf{t}\, n^{q+1}}{q!} + O(n^q)\,.$$

This holds for a sufficiently large integer n precisely when (1.2) is satisfied. □

REMARK 1.2. By consideration of the dimensions of various Riemann-Roch spaces, it is possible in some cases to improve Lemma 1.1. For instance, when $\mathbf{t} = (t_1, \ldots, t_q) = (1, 0, \ldots, 0)$, the estimate

$$\dim V_j \geq l(nD) - \#\{\mathbf{i} \in \boldsymbol{N}^q\,;\ \mathbf{i}\cdot\mathbf{t} < j\}$$

used in the proof of Lemma 1.1 is always trivial. On the other hand, in this case we have the obvious estimate

$$\dim V_j \geq l(nD - jD_1)\,.$$

Using refinements of this observation, one may give an improvement of inequality (1.2) which takes into account the information coming from the relevant Riemann-Roch spaces.

The following result corresponds to [1, Theorem 4.4], stated therein in a different notation. It provides a convenient choice of the weights a_i, so that two certain inequalities are satisfied. The first one will be used in applying Lemma 1.1, whereas the second one will be used when the convergence properties assumed in Lemma 1.1 do not hold.

LEMMA 1.3 (Autissier). *Suppose that $D_1, \dots, D_r$ are ample divisors. Let $\varepsilon > 0$. Then there exist positive integers $a_1, \dots, a_r$ such that if $D = \sum_{i=1}^r a_i D_i$, then*

$$\left| \frac{D^q}{a_i D_i . D^{q-1}} - r \right| < \varepsilon\,, \quad i = 1, \dots, r\,, \tag{1.4}$$

and for every index i in $\{1, \dots, r\}$ and sufficiently large integer n,

$$\frac{\sum_{k \geq 1} l(nD - k a_i D_i)}{l(nD)} > \frac{rn}{2q}\,. \tag{1.5}$$

PROOF. This is essentially [1, Theorem 4.4], up to the fact that inequality (1.4) is not given in the statement of the theorem, but it is contained in its proof. For the reader's convenience, we repeat in our notation Autissier's proof (for another proof of (1.4), see [5, Lemma 9.7]).

Let $\Delta \subset \boldsymbol{R}^r$ be defined as $\Delta = \{\mathbf{t} = (t_1, \dots, t_r)\,;\, t_1, \dots, t_r \geq 0, t_1 + \dots + t_r = 1\}$. For every $\mathbf{t} \in \Delta$, let $D_{\mathbf{t}}$ be the $\boldsymbol{R}$-divisor $D_{\mathbf{t}} = \sum_{i=1}^r t_i D_i$ and put $\phi(\mathbf{t}) = (\sum_{i=1}^r (D_{\mathbf{t}}^{q-1}.D_i)^{-1})^{-1}$. Finally, let us define the continuous map $f : \Delta \to \Delta$ by

$$f(\mathbf{t}) = \left(\frac{\phi(\mathbf{t})}{(D_t^{q-1}.D_1)}, \dots, \frac{\phi(\mathbf{t})}{(D_t^{q-1}.D_r)} \right).$$

By Brouwer's fixed point theorem, there exists a point $\mathbf{t} = (t_1, \dots, t_r)$ which is fixed for f. This means that for each index i in $\{1, \dots, r\}$, $\phi(\mathbf{t})/(D_{\mathbf{t}}^{q-1}.D_i) = t_i$, so $(D_{\mathbf{t}}^{q-1}.t_i D_i)$ is independent of i. Then $D_{\mathbf{t}}^q/(D_{\mathbf{t}}^{q-1}.t_i D_i)$ must be equal to r. By approximating $\mathbf{t}$ by a rational point of the form $(a_1/m, \dots, a_r/m)$, we can achieve inequality (1.4). Inequality (1.5) follows by using [1, Corollary 4.3], applied with $E = a_i D_i$, exactly as in the last part of the proof of [1, Theorem 4.4]. □

For a rational function $f \in k(\tilde{\mathcal{X}})^*$, we let $\mathrm{div}(f)_0$ denote the divisor of zeros of f. From Autissier's Lemma above we derive the following.

COROLLARY 1.4. *Suppose that $D_1, \dots, D_r$ are ample divisors. Let $\varepsilon > 0$. Then there exist positive integers $a_1, \dots, a_r$ such that if $D = \sum_{i=1}^r a_i D_i$, then*

$$\left| \frac{D^q}{a_i D_i . D^{q-1}} - r \right| < \varepsilon\,, \quad i = 1\,, \dots, r\,,$$

and for a sufficiently large integer n and any i_1, i_2 in $\{1, \dots, r\}$ with $i_1 \neq i_2$, there exists a basis $f_1, \dots, f_{l(nD)}$ of $L(nD)$ such that

$$\mathrm{div}\Bigl(\prod_{i=1}^{l(nD)} f_i \Bigr)_0 > \left(\left(\frac{r}{2q} - 1 \right) \frac{D^q}{q!} n^{q+1} + O(n^q) \right) (a_{i_1} D_{i_1} + a_{i_2} D_{i_2})\,. \tag{1.6}$$

PROOF. Let us choose the integers $a_1, \dots, a_r$ as provided by Lemma 1.3, so that (1.4) holds and the first inequality of Corollary 1.4 is proved. Let $n > 0$ be an integer sufficiently large so that inequality (1.5) of Lemma 1.3 holds. Let us fix two indices i_1, i_2 with $i_1 \neq i_2$, and consider the two filtrations of the space $L(nD)$ defined for $j = 1, 2$ by the chain

$$L(nD) \supset L(nD - D_{i_j}) \supset L(nD - 2D_{i_j}) \supset \cdots \supset \{0\}.$$

By the linear algebra [3, Lemma 3.2], there exists a basis $f_1, \dots, f_{l(nD)}$ of $L(nD)$ which contains a basis of all the nonzero subspaces of the form $L(nD - kD_{i_j})$ for $j = 1, 2$ and $k \geq 0$. Then the order of vanishing of the product $\prod_{i=1}^{l(nD)} f_i$ at each of the divisors $a_{i_j} D_{i_j}$ is at least

$$\begin{aligned}
&\sum_{k\geq 0} (k - n) \dim(L(nD - ka_{i_j} D_{i_j})/L(nD - (k+1)a_{i_j} D_{i_j})) \\
&\quad = -nl(nD) + \sum_{k\geq 1} k \dim(L(nD - ka_{i_j} D_{i_j})/L(nD - (k+1)a_{i_j} D_{i_j})) \\
&\quad = -nl(nD) + \sum_{k\geq 1} l(nD - ka_{i_j} D_{i_j}).
\end{aligned}$$

Since, as we already observed, we have $l(nD) = D^q n^q/q! + O(n^{q-1})$, the sought inequality (1.6) follows from the lower bound (1.5) for the term $\sum l(nD - ka_{i_j} D_{i_j})$, provided by Lemma 1.3. □

We now prove Theorem CLZ.

PROOF OF THEOREM CLZ. By enlarging k and S, we can assume without loss of generality that $D_1, \dots, D_r$ are defined over k and that every point in the intersection of any q distinct divisors $D_{i_1}, \dots, D_{i_q}$ is k-rational. By going to an infinite subsequence of the points, we easily reduce to the case of S-integral sets of points $R = \bigcup_{i=1}^{\infty} \{P_i\}$ with the following properties: for each $v \in S$, (P_i) converges v-adically to a point $P_v \in \tilde{\mathcal{X}}(k_v)$, and if $J_v = \{j\,;\, P_v \in D_j\}$ and $\phi_{j,v}$ denotes a function defining D_j in a neighborhood of P_v, then the limit

$$t_{j,v} = \lim_{i\to\infty} \frac{\log|\phi_{j,v}(P_i)|_v}{\sum_{l\in J_v} \log|\phi_{l,v}(P_i)|_v} \tag{1.7}$$

exists for each $j \in J_v$.

For a sufficiently small $\varepsilon > 0$, let a_i, $i = 1, \dots, r$, be positive integers as in Corollary 1.4 and let $D = \sum_{i=1}^r a_i D_i$. Let n be a sufficiently large integer. Let $g_1, \dots, g_{l(nD)}$ be a basis of $L(nD)$. Let $g : \tilde{\mathcal{X}} \setminus D \to \boldsymbol{P}^{l(nD)-1}$ be given by the map $x \mapsto (g_1(x) : \dots : g_{l(nD)}(x))$.

Let $v \in S$. We consider various cases depending on where P_v lies. In each of these cases, we shall prove an inequality of the following type (1.8) with the purpose of applying the Subspace Theorem after summation. For a hyperplane H in a projective space $\boldsymbol{P}^N$, we shall denote by $\lambda_{v,H}$ a Weil function for H. Namely, if H is defined by a linear form $L = 0$

and $P = (\xi_0 : \dots : \xi_N)$ is not in H, we set

$$\lambda_{v,H}(P) = -\log \frac{|L(P)|_v}{\sup |\xi_i|_v} .$$

The alluded to inequality is

$$\sum_{j=1}^{l(nD)} \lambda_{v,H_{j,v}}(g(P_i)) > (l(nD)+\varepsilon) \log \max_j |g_j(P_i)|_v + O(1) , \quad i \in \boldsymbol{N} , \tag{1.8}$$

where the hyperplanes $H_{j,v}$ are in general position.

The easiest case is when P_v does not belong to the support of D. In this case, let $H_{1,v}, \dots, H_{l(nD),v}$ be any $l(nD)$ hyperplanes of $\boldsymbol{P}^{l(nD)-1}$ in general position. Then inequality (1.8) holds trivially. In fact, $P_v \notin D$, so $|g_j(P_i)|_v$ is bounded for all i and j.

Suppose now that P_v lies in the intersection of m (and not more) distinct divisors $D_{i_1}, \dots, D_{i_m}$ with $1 \leq m \leq q-1$. After reindexing, we can assume that $a_{i_1}t_{i_1,v} \geq a_{i_2}t_{i_2,v} \geq \cdots \geq a_{i_m}t_{i_m,v}$. Let $f_1, \dots, f_{l(nD)}$ be the basis from Corollary 1.4 for i_1 and i_2 (choosing $i_2 \neq i_1$ arbitrarily if $m=1$). Then each f_j is a linear form in $g_1, \dots, g_{l(nD)}$. Let $\{L_1, \dots, L_{l(nD)}\}$ be the set of such linear forms and let $\{H_{1,v}, \dots, H_{l(nD),v}\}$ be the corresponding set of hyperplanes in $\boldsymbol{P}^{l(nD)-1}$. It is easily seen that with these choices, (1.8) is equivalent to

$$\max_{j'} |g_{j'}(P_i)|_v^{\varepsilon} \prod_{j=1}^{l(nD)} |f_j(P_i)|_v \ll 1 \tag{1.9}$$

as i goes to ∞. It follows from inequality (1.6) of Corollary 1.4 that, for some rational function ϕ that is regular at P_v,

$$\prod_{j=1}^{l(nD)} f_j = \phi(\phi_{i_1,v}^{a_{i_1}}\phi_{i_2,v}^{a_{i_2}})^{(r/2q-1)D^q n^{q+1}/q!+O(n^q)} \prod_{j=3}^{m} \phi_{i_j,v}^{-a_{i_j}nl(nD)} . \tag{1.10}$$

We exploit this formula by showing that on evaluating at P_i we obtain a small value. We have $l(nD) = D^q n^q/q! + O(n^{q-1})$. Since, by assumption,

$$a_{i_1}t_{i_1,v} + a_{i_2}t_{i_2,v} \geq (2/(q-1)) \sum_{j=1}^{m} a_{i_j}t_{i_j,v} ,$$

we have

$$\left(\frac{r}{2q}-1\right)\frac{D^q}{q!}(a_{i_1}t_{i_1,v} + a_{i_2}t_{i_2,v})n^{q+1} - \sum_{j=3}^{m} a_{i_j}t_{i_j,v}\frac{D^q}{q!}n^{q+1}$$
$$\geq \sum_{j=1}^{m} a_{i_j}t_{i_j,v}\frac{D^q}{q!}n^{q+1}\left(\frac{r}{q(q-1)}-1\right).$$

As $r > q(q-1)$, it follows easily from (1.10) and the definition of $t_{j,v}$ (1.7) that the left-hand side of (1.9) is bounded if n has been chosen large enough. Thus (1.8) holds in this case.

Suppose now that P_v belongs to exactly q distinct divisors $D_{i_1}, \dots, D_{i_q}$. We consider two subcases.

SUBCASE A. Suppose that we have $a_{i_1} t_{i_1,v}, \dots, a_{i_q} t_{i_q,v} \geq (q^2-q+1)^{-1} \sum_{j=1}^q a_{i_j} t_{i_j,v}$. It is easily seen that, when $\sum_{j=1}^q a_{i_j} t_{i_j,v}$ is fixed, the minimum of $\prod_{j=1}^q t_{i_j,v}$ subject to these constraints is attained when $q-1$ among the $a_{i_j} t_{i_j,v}$ are equal to $(q^2-q+1)^{-1} \sum_{j=1}^q a_{i_j} t_{i_j,v}$; in such a way we find that

$$\prod_{j=1}^q t_{i_j,v} \geq \frac{q^2-2q+2}{(q^2-q+1)^q \prod_{j=1}^q a_{i_j}} \left(\sum_{j=1}^q a_{i_j} t_{i_j,v} \right)^q .$$

So the right-hand side of (1.2) is less than or equal to

$$\frac{(1+1/q)^q (q^2-q+1)^q}{q^2-2q+2} \prod_{j=1}^q a_{i_j} = \frac{(q^2+1/q)^q}{q^2-2q+2} \prod_{j=1}^q a_{i_j} .$$

We now estimate D^q from below. Using Corollary 1.4, we have

$$(D^q)^q > (r-\varepsilon)^q \prod_{j=1}^q a_{i_j} D_{i_j}.D^{q-1} .$$

By the generalized Hodge index theorem [4, Ch. 1, Ex. 6] for $j = 1, \dots, q$,

$$(D_{i_j}.D^{q-1})^q \geq D_{i_j}^q (D^q)^{q-1} .$$

It follows that

$$(D^q)^q > (r-\varepsilon)^q (D^q)^{q-1} \prod_{j=1}^q a_{i_j} \sqrt[q]{D_{i_j}^q} .$$

By assumption, $r \geq q^2-q+1$ and $D_{i_j}^q \geq 1$ for all j. So

$$D^q > (q^2-q+1-\varepsilon)^q \prod_{j=1}^q a_{i_j} .$$

For small enough ε, it is easily checked that the inequality

$$(q^2-q+1-\varepsilon)^q > (q^2+1/q)^q/(q^2-2q+2)$$

holds for $q \geq 3$. So the inequality (1.2) holds. Then by Lemma 1.1, for all sufficiently large n, there exists a basis $f_1, \dots, f_{l(nD)}$ of $L(nD)$ such that the left-hand side of (1.9) goes to 0 as i goes to ∞. So (1.9) is verified and if $H_{1,v}, \dots, H_{l(nD),v}$ are the hyperplanes corresponding to $f_1, \dots, f_{l(nD)}$ in the basis $g_1, \dots, g_{l(nD)}$, then (1.8) holds.

SUBCASE B. Now suppose that, say, $a_{i_q} t_{i_q} < (q^2-q+1)^{-1} \sum_{j=1}^q a_{i_j} t_{i_j,v}$. This means that the contribution of the corresponding divisor D_{i_q} is very small, so we shall essentially ignore it and proceed as in the case when fewer than q of the divisors meet at our point.

To carry this out, after reindexing, we can assume that $a_{i_1}t_{i_1,v} \geq a_{i_2}t_{i_2,v} \geq \cdots \geq a_{i_q}t_{i_q,v}$. Let $f_1, \ldots, f_{l(nD)}$ be the basis from Corollary 1.4 for i_1 and i_2. Since

$$a_{i_1}t_{i_1,v} + a_{i_2}t_{i_2,v} > \frac{2}{q-1}\left(1 - \frac{1}{q^2-q+1}\right)\sum_{j=1}^{q} a_{i_j}t_{i_j,v} = \frac{2q}{q^2-q+1}\sum_{j=1}^{q} a_{i_j}t_{i_j,v}$$

and $r \geq q^2 - q + 1$, we have

$$\left(\frac{r}{2q} - 1\right)(a_{i_1}t_{i_1,v} + a_{i_2}t_{i_2,v}) - \sum_{j=3}^{q} a_{i_j}t_{i_j,v} > \left(\frac{r}{2q}\frac{2q}{q^2-q+1} - 1\right)\sum_{j=1}^{q} a_{i_j}t_{i_j,v} \geq 0\,.$$

Then by calculations similar to the second case above, it follows that (1.9) holds. Thus, if $H_{1,v}, \ldots, H_{l(nD),v}$ are the hyperplanes corresponding to $f_1, \ldots, f_{l(nD)}$ in the basis $g_1, \ldots, g_{l(nD)}$, then (1.8) holds.

So we have indeed checked that (1.8) holds in every case.

Since R is a set of S-integral points on $\mathcal{X}$, it follows that

$$\sum_{v \in S} \log \max_j |g_j(P_i)|_v = h(g(P_i)) + O(1)$$

for all i. So (1.8) implies, if $n \gg 0$ and ε is sufficiently small,

$$\sum_{v \in S}\sum_{j=1}^{l(nD)} \lambda_{v,H_{j,v}}(g(P_i)) > (l(nD) + \varepsilon)h(g(P_i)) + O(1) \quad \text{for } i \in \boldsymbol{N}\,.$$

By Schmidt's Subspace Theorem (see e.g. [6, Theorem 1.1, Inequality (1.4)]), it follows that $g(R)$ is contained in the union of finitely many hyperplanes. Note that for $n \gg 0$, the map g is an embedding since $D_1, \ldots, D_r$ are ample and that, furthermore, since $g_1, \ldots, g_{l(nD)}$ are linearly independent, $g(\mathcal{X})$ is not contained in any hyperplane of $\boldsymbol{P}^{l(nD)-1}$. It follows that R is not Zariski-dense in $\tilde{\mathcal{X}}$. This proves the first part of the theorem.

To prove the existence of the set $\mathcal{Y}$ in the theorem, a little more care must be taken. First, we have to use Vojta's version [6] of Schmidt's Subspace Theorem; in this version, up to excluding finitely many approximating points (which may depend on the field of definition of the approximating points and the set S), the exceptional hyperplanes can be chosen in a finite set which depends only on the 'target hyperplanes', but neither on the field of definition of the approximating points nor on the set S.

Now, our target hyperplanes we are choosing depend on P_v and the limits $t_{i,v}$ (in the case that P_v belongs to q divisors).

Concerning the dependence on P_v, the construction of the target hyperplanes depends not quite on P_v, but only on a maximal set of divisors D_i containing P_v. Therefore the variation of P_v gives rise only to finitely many possibilities.

Concerning the limits $t_{i_j,v}$ (when P_v belongs to $D_{i_1}, \ldots, D_{i_q}$), it is easily seen that, if $t^*_{i_j,v}$ are quantities which are sufficiently close to $t_{i_j,v}$ (satisfying $\sum_{j=1}^{q} t^*_{i_j,v} = 1$), the target hyperplanes at v constructed in the proof of Theorem CLZ will be identical. Thus, by compactness of $\{(t_1, \ldots, t_q) \in [0,1]^q\,;\ \sum_{i=1}^{q} t_i = 1\}$, it follows that we make use of only finitely many sets of target hyperplanes in the proof of Theorem CLZ. Now Vojta's result on the exceptional hyperplanes in the Subspace Theorem gives the existence of the set $\mathcal{Y}$. □

REMARK 1.5.

1. The results may be in principle applied to study the integral points on a given variety, but defined over a *variable* field of bounded degree, for instance the *quadratic integral points over a surface*. Following a procedure already adopted for curves, one is led to study the usual integral points over a symmetric power of the original variety. A difficulty which appears, compared to the case of curves, is that the symmetric powers are singular if the dimension of the original variety is greater than one.

2. Our result shows that removing seven ample divisors on a threefold suffices to obtain degeneracy of integral points. However, this number may be decreased down to six or even five, provided all of the divisors are assumed to be nearly parallel in the Néron-Severi group. This kind of result may be proved by an entirely similar procedure. (If the divisors are assumed to be actually parallel, then this follows already from Vojta's results on semi-abelian varieties [7].)

3. The results obtained with these methods appear to be strongly ineffective; namely, even in the cases when one can prove finiteness, the methods do not provide a way to effectively bound the number of integral points (in the case of curves, however, see [2]). In general, one can bound the degree of a proper subvariety containing the integral points, but not the height of such a subvariety. As the bound on the degree depends on the height of the original variety, this yields a problem when iterating these methods to effectively bound the number of solutions in a finiteness result. We note, however, that the set $\mathcal{Y}$ in Theorem CLZ can be effectively computed (this follows from the complexity bound on the exceptional hyperplanes given in [6]).

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