A NOTE ON GENERAL TOPOLOGICAL SPACES.*)

By

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1. If for any subset A of the fundamental set S we can assign a "closure" $\bar{A}$ satisfying some proper conditions, then the set S is said to be a space. In general there are two methods defining the closure, that is;

(I) When there corresponds a family "neighbourhoods" V_x to every point x in S, $x \in \bar{A}$ is, by definition, that no $V_x \cap A$ is vacuous.

(II) When there is a family of "sequences" $\{x_\alpha\}$ in S[1] for which it is always decided that $\{x_\alpha\}$ converge to x or not; $x \in \bar{A}$ is by definition, that there is a sequence in A convergent to x.

S is said to be a neighbourhood space or convergent space according as it is topologized by a system of neighbourhoods or a family of convergent sequences. When convergence of sequences are suitably defined by means of system of neibourhoods, the neighbourhood space becomes a convergence space. For example, if in a neighbourhood space S convergency of the sequence $\{x_\alpha\}$ is defined by

(III) $\{x_\alpha\}$ converges to x if and only if for each neighbourhood V_x of x, there exists an $\alpha_0 = \alpha_0(V_x)$ such that $\alpha > \alpha_0$ implies $x_\alpha \in V_x$, then S becomes a convergence space.

In this paper we intoduce the notion of "φ-closure" (in Definition 2), by which neighbourhood space turns to the space with "φ-topology". Main results concerning φ-topology are cotained in Theorem 4.

But if we consider some set A such as $\{x_\alpha\} \subset A \subset S$, we obtain many interesting results, for instance, all convergence topologies defined in S is a Boolean algebra[2] by some order relation.

2. Let φ be a set-function on 2^S (=family of all subsets in S) such that

(2, 1) for any subset A in S, $A \subset \varphi(A)$,

(2, 2) $A \subset B$ implies $\varphi(A) \subset \varphi(B)$.

And let ϕ be the class of all such φ.

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(1) For any finite or infinite directed set.

(2) G. Birkhoff, Fund Math., XXVI(1936).

For any φ_1 and φ_2 in ϕ, we write $\varphi_1 < \varphi_2$ if and only if

$$\varphi_1(A) \subset \varphi_2(A)$$

for all subsets A of S. φ is a partially ordered system, that is,

(2, 3) $\varphi_1 < \varphi_2 < \varphi_3$ implies $\varphi_1 < \varphi_3$,

(2, 4) $\varphi_1 < \varphi_2 < \varphi_1$ implies $\varphi_1 = \varphi_2$.

Further ϕ is a lattice, and

(2, 5) $(\varphi_1 \wedge \varphi_2)\ A = \varphi_1(A) \cap \varphi_2(A)$,

(2, 6) $(\varphi_1 \vee \varphi_2)\ A = \varphi_1(A) \cup \varphi_2(A)$.

If we define O and I by

(2, 7) $O(A) \equiv A$ for all subsets A of S,

(2, 8) $I(A) \equiv S$ for all subsets A of S,

then ϕ becomes a Boolean algebra.

3. Closure with respect to φ. Let S be a neighbourhood space and denote its points by $x, y, \ldots\ldots$ Suppose that for each x in S there corresponds at least one "neighbourhood" V_x of x such that

(N. 1) for each $x \in S$, V_x contains x;

(N. 2) if U_x and V_x are neighbourhoods of x, $W_x = U_x \cap V_x$ is also a neighbourhood of x.

We shall now introduce an equivalent sequencial topology and weaker ones into . The convergence of the sequence in S is defined by

Definition 1. If a sequence $\{x_\alpha\}$ is contained in a certain fixed set A, and for each neighbourhood V_x of x containing A^c there exists an $\alpha_0 = \alpha_0(V_x)$ such that $\alpha > \alpha_0$ implies $x_\alpha \in V_x$, then $\{x_\alpha\}$ is said to be *convergent to* x with respect to A and denote it by

$$x_\alpha \longrightarrow x\ (A)^{(3)}.$$

In this definition if we take $\equiv S$, then $x \longrightarrow x(\ )$ coincides with ordinary convergence in (III). As easily may be seen by example, $x_\alpha \longrightarrow x\ (A)$ does not imply $x \longrightarrow x\ (B)$ in general if $A \neq B$.

Lemma 1. *If $\{x\}$ converges to x with respect to A and $\{x\}$ is a cofinal subsequence of $\{x\}$, then $\{x\}$ converges to x with respect to A.*

Proof is easy.

Lemma 2. *If $\{x_\alpha\} \subset B \subset A$ and $x \longrightarrow x\ (A)$, then $x_\alpha \longrightarrow x\ (B)$.*

Proof. Each neighbourhood V_x of x containing B^c is that of x containing A^c. Since $x_\alpha \longrightarrow x\ (A)$ for any neighbourhood V_x containing B^c, there exists an $\alpha_0 = \alpha_0(V_x)$ such that $\alpha > \alpha_0$ implies $x_\alpha \in V_x$, that is, $x_\alpha \longrightarrow x\ (B)$.

Definition 2. Let $\varphi \in \phi$. If there exists at least one sequence $\{x_\alpha\}$ of

(3) If such neighbourhood does not exist, $\{x_\alpha\}$ converges to x with respect to A.

points in A such as $x_\alpha \longrightarrow x$ $(\varphi(A))$, then we say that x is a *limiting point of A with respect to* φ-topology and denote it by $x \in A^\varphi$. And A^φ is said to be φ-closure of A.(4)

Specially if $\varphi \equiv I$, it coincides with (III).

Corollary. *If $\varphi > \varphi'$ and $x \in A^\varphi$, then $x \in A^{\varphi'}$.*

Proof. By the hypothesis there exists a sequence $\{x_\alpha\}$ in A, snch as $x_\alpha \longrightarrow x(\varphi(A))$. Since $\{x_\alpha\} \subset A \subset \varphi'(A) \subseteq \varphi(A)$, we have $x_\alpha \longrightarrow x$ $(\varphi'(A))$ by Lemma 2.

From this Cor. we see that if $\varphi > \varphi'$, then φ'-topology is not weaker than φ-topology.

4. Fundamental theorems.

Theorem 1. *For any subsets A and B in S,*

$$(A \cup B)^\varphi \subset A^\varphi \cup B^\varphi. \qquad (1)$$

Proof. Let $\{x_\alpha\}$ be a sequence of points in $A \cup B$ such that $x_\alpha \longrightarrow x(\varphi(A \cup B))$. Then at least one of $\{x_\alpha\} \cap A$ and $\{x_\alpha\} \cap B$ must be a cofinal subsequence of $\{x_\alpha\}$.

If $$\{x_\beta\} \equiv \{x_\alpha\} \cap A$$

is so, Lemma 1 and 2 show that $x_\beta \longrightarrow x$ $(\varphi(A \cup B))$ and $x_\beta \longrightarrow x$ $(\varphi(A))$ for $\{x_\beta\} \subset A \subset \varphi(A) \subset \varphi(A \cup B)$. Thus the theorem is proved.

As easily may be seen by example, equality (1) does not hold in general.

Theorem 2. *A set V is a neighbourhood of x if and only if $x \in (V^c)^\varphi$, for any φ in ϕ.*

Proof. It is sufficient to show, by Cor. of Definition 2, $x \in (V^c)^2$. If we suppose that $x \in (V^c)^0$, then for some $\{x_\alpha\}$ in V^c, $\{x_\alpha\}$ converges to x with respect to $(V^c)^c = V^c$. Since $V^{cc} = V$ is a neighbourhood of x containing $V^{cc} = V$, there exists an $\alpha_0 = \alpha_0(V)$ such that $\alpha > \alpha_0$ implies $x_\alpha \in V$. On the other hand all points of $\{x_\alpha\}$ are contained in V^c. Thus we have a contradiction. Conversely let $x \in (V^c)$, then there exists at least one neighbourhood U_x containing V. For, if such neighbourhood U_x does not exist, every sequence $\{x_\alpha\}$ in V^c must converges to x with respect to V, which contradicts to $x \in (V^c)^0$. From all such U_x we can select a set $\{x_U\}$ consisting of points such that $x_U \in U_x \cap V^c$. Since all neighbourhoods of x containing V form a directed system concerning set-implication, $\{x_U\}$ is a sequence. By the construction of $\{x_U\}$ it converges to x with respect to V^c, which contradicts to $x \in (V^c)^0$.

(4) This concept may be considered as a generalization of closure notion. Therefore we may say $A\varphi$ a φ-derived set, instead of φ-closure.

Consequently $U_x \cap U^c = 0$ and $U_x \supset V$, that is, $U_x = V$ is a neighbourhood of x. Q. E. D.

Theorem 3. *$A \subset B$ implies $A^\varphi \subset B^\varphi$ if and only if*[5]

(N. 3) *for any neighbourhood U_x of x, a set containing U_x is also a neighbourhood of x.*

Proof. Let $A \subset B$ and $x \in A^\varphi$, there exists a sequence $\{x_\alpha\}$ in A such that $x_\alpha\}$ converges to x with respect to $\varphi(A)$. By $0 < \varphi$, $x_\alpha \to x$ (A), and then A^c is not a neighbourhood of x.

Since $(\varphi(B))^c \subset B^c \subset A^c$, and by (N. 3), $(\varphi(B))^c$ is not a neighbourhood of x. Moreover intersection of B with each neighbourhood V_x of x containing $(\varphi(B))^c$ is not empty. As in the proof of Theorem 2 there exists a sequence $\{Y_V\}$ such that $Y_V \longrightarrow x$ $(\varphi(B))$. That is, $x \in B^\varphi$. Thus $A \subset B$ implies $A^\varphi \subset B^\varphi$. Conversely, if $U_x \subset V$ then $U_x{}^c \supset V^c$. By the hypothesis of φ-topology, $(U_x{}^c) \supset (V^c)^\varphi$. Since $x \in (U_x{}^c)^\varphi$, $x \in (V^c)^\varphi$. Consequently V is a neighbourhood of x by Theorem 2. Q. E. D.

Summing up the above results we get:

Theorem 4. *If S is a neighbourhood space satisfying* (N. 1)–(N. 3), *then each topology defined by Definition 2 satisfies the following two conditions concerning closure,*

$$(A \cup B)^\varphi \supset A^\varphi \cup B^\varphi,$$

$$(A \cup B)^\varphi \supset A^\varphi \cup B^\varphi,$$

and moreover the φ-topologies form a Boolean algebra.

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