ON POSITIVE DEFINITE SEQUENCES AND FUNCTIONS*)

By

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Contents

1. Introduction.

1. Let $F(z) = {}^1/_2 a_0 + \sum_{n=1}^{\infty} a_n z^n$ ($a_0 =$ real) be regular and $\Re F(z) \geqq 0$ for $|z| < 1$, then Herglotz proved that

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta), \tag{1}$$

where $\mu(\theta)$ is an increasing function for $0 \leqq \theta < 2\pi$, so that

$$a_n = \frac{1}{\pi} \int_0^{2\pi} e^{-in\theta} d\mu(\theta) = \int_0^{2\pi} e^{in\theta} d\nu(\theta), \qquad (n = 0, 1, 2, \cdots), \tag{2}$$

where $\nu(\theta) = -\frac{1}{\pi} \mu(2\pi - \theta)$ is an increasing function for $0 \leqq \theta < 2\pi$. Hence the Hermitian forms:

$$H_n(x) = \sum_{0}^{n} a_{\nu-\mu} x_\nu \bar{x}_\mu, \qquad (a_{-\nu} = \bar{a}_\nu)^{1)} \tag{3}$$

are non-negative for $n = 0, 1, 2, \cdots$.

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*) Received August 5, 1949.

1) In this paper $\bar{\alpha}$ means the conjugate complex of α.

2) C. Carathéodory: Über die Variabiltätsbereich der Fourierkonstanten von positiven harmonischen Funktionen. Rendiconti del circolo mat. Palermo. 32 (1911). O. Toeplitz: Über die Fouriersche Entwicklung positiver harmonischen Funktionen. Rendiconti del circolo mat. Palermo. **32** (1911). I. Schur: Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind. Crelle. **147** (1917). O. Szász: Über harmonischen Funktionen und L-Formen. Math. Zeitr. **1** (1918). M. Tsuji: On a regular function, whose real part is positive in a unit circle. Proc. Japan Acad. 21. No. 3-10. (1945).

We call such a sequences $\{a_\nu\}$ a positive definite sequence.

Conversely, if $H_n(x) \geqq 0$ for $n = 0, 1, 2, \cdots$, then $|a_\nu| \leqq a_0$, so that $F(z)$ is regular and $\Re F(z) \geqq 0$ for $|z| < 1$.[2]

If $H_0(x), H_1(x), \cdots, H_{k-1}(x)$ are positive definite and $H_k(x), H_{k+1}(x), \cdots$ are positive semi-definite, then $v(\theta)$ is a step-function, so that

$$F(z) = \sum_{\nu=1}^{k} \frac{r_\nu}{2} \cdot \frac{1 + \varepsilon_\nu z}{1 - \varepsilon_\nu z}, \qquad (|\varepsilon_\nu| = 1,\ \varepsilon_i \neq \varepsilon_j\ (i \neq j),\ r_\nu > 0). \tag{4}$$

We remark that

$$\Re F(z) = \frac{1}{2} \sum_{-\infty}^{\infty} a_\nu\, r^{|\nu|}\, e^{i\nu\theta}, \qquad (z = re^{i\theta},\ a_{-\nu} = \bar{a}_\nu). \tag{5}$$

2. Let $f(t)$ be a bounded measurable function defined for $-\infty < t < \infty$, such that $f(-t) = \overline{f(t)}$ and $q(t)$ be a bounded measurable function, which vanishes for large $|t|$. We call such $q(t)$ a function of class (L^0).

If for all $q(t)$ of class (L^0), the Hermitian forms:

$$H(q) = \iint_{-\infty}^{\infty} f(t - s)\, q(t)\, \overline{q(s)}\, dt\, ds \geqq 0, \tag{6}$$

we call $f(t)$ a positive definite function.

Bochner[3] proved that, if $f(t)$ is a positive definite function, then

$$f(t) \sim \int_{-\infty}^{\infty} e^{it\lambda}\, dv(\lambda),^{4)} \tag{7}$$

where $v(\lambda)$ is an increasing function for $-\infty < \lambda < \infty$, such that

$$\int_{-\infty}^{\infty} dv(\lambda) \leqq M = \underset{-\infty < t < \infty}{\text{upper limit}} |f(t)|. \tag{8}$$

If we normalize $v(\lambda)$, such that $v(-\infty) = 0$, $v(\lambda - 0) = v(\lambda)$, then $v(\lambda)$ is unique.

Let $F(z)$ be the Laplace transform of $f(t)$:

$$F(z) = \int_0^\infty f(t)\, e^{-tz}\, dt, \qquad (z = x + iy), \tag{9}$$

which is regular for $x > 0$.

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3) Bochner: Vorlesungen über Fouriersche Integrale. Kap. IV.

4) $f(t) \sim g(t)$ means that $f(t) = g(t)$ almost everywhere.

If $\Re F(z) \geqq 0$ for $x > 0$, then we can prove (Theorem 5) that $f(t)$ can be expressed in the form (7), so that

$$F(z) = \int_{-\infty}^{\infty} \frac{d\nu(\lambda)}{z - i\lambda}. \tag{10}$$

Conversely, if $f(t)$ is a positive definite function, then $\Re F(z) \geqq 0$ for $x > 0$, so that we have (7) and hence

$$\Re F(z) = \int_{-\infty}^{\infty} \frac{x}{|z - i\lambda|^2} d\nu(\lambda) \geqq 0, \qquad x > 0. \tag{11}$$

Hence a necessary and sufficient condition, that $f(t)$ is a positive definite function is that $\Re F(z) \geqq 0$ for $x > 0$.

We remark that in virtue of $f(-t) = \overline{f(t)}$,

$$\Re F(z) = {}^1/_2 \int_{-\infty}^{\infty} f(t) e^{-(|t|x + ity)} dt, \qquad (z = x + iy). \tag{12}$$

In this paper, we will define a positive definite sequence of n indices and a positive definite function of n variables and prove analogous theorems.

2. Positive definite sequence of n indices.

1. Let $a_{\nu_1, \dots, \nu_n}$ be defined for $\nu_1, \cdots, \nu_n = 0, \pm 1, \pm 2, \cdots$, such that

$$a_{-\nu_1, \dots, -\nu_n} = \bar{a}_{\nu_1, \dots, \nu_n}, \quad |a_{\nu_1, \dots, \nu_n}| \leqq M \text{ for all } \nu_1, \nu_2, \cdots, \nu_n. \tag{1}$$

If for all $N = 0, 1, 2, \cdots$, Hermitian form:

$$H_N(x^{(1)}, \cdots, x^{(n)}) = \sum_0^N \cdots \sum_0^N a_{\nu_1 - \mu_1, \dots, \nu_n - \mu_n} x_{\nu_1}^{(1)} \bar{x}_{\mu_1}^{(1)} \cdots x_{\nu_n}^{(n)} \bar{x}_{\mu_n}^{(n)} \geqq 0, \tag{2}$$

then we call $\{a_{\nu_1, \dots, \nu_n}\}$ a positive definite sequence of n indices. Analogous to (5) of § 1, we put

$$u_I(z_1, \cdots, z_n) = u(z_1, \cdots, z_n) = \sum_{-\infty}^{\infty} \cdots \sum_{-\infty}^{\infty} a_{\nu_1, \dots, \nu_n} r_1^{|\nu_1|} \cdots r_n^{|\nu_n|} e^{i(\nu_1\theta_1 + \dots + \nu_n\theta_n)},$$

$$(z_k = r_k e^{i\theta_k}), \tag{3}$$

then by (1), $u(z_1, \cdots, z_n)$ is convergent for $|z_k| < 1$ and is real. From (3),

$$a_{\nu_1, \dots, \nu_n} r_1^{|\nu_1|} \cdots r_n^{|\nu_n|} = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int u(r_1 e^{-i\theta_1}, \cdots, r_n e^{-i\theta_n}) e^{i(\nu_1\theta_1 + \dots + \nu_n\theta_n)} d\theta_1 \cdots d\theta_n. \tag{4}$$

Writing $\rho_k - \varphi_k$, instead of r_k, θ_k in (4) and inserting $a_{\nu_1,\dots,\nu_n}$ in (3), we see that $u(z_1,\dots,z_n)$ can be expressed by a Poisson integral:

$$u(r_1 e^{i\theta_1},\cdots,r_n e^{i\theta_n})$$
$$= \frac{1}{(2\pi)^n}\int_0^{2\pi}\!\!\cdots\!\int u(\rho_1 e^{i\varphi_1},\cdots,\rho_n e^{i\varphi_n})\prod_{k=1}^{n}\frac{\rho_k^2 - r_k^2}{\rho_k^2 - 2\rho_k r_k\cos(\varphi_k-\theta_k)+r_k^2}\,d\varphi_1\cdots d\varphi_n,$$
$$(0\leqq r_k<\rho_k<1). \qquad (5)$$

We put for $0\leqq r_k<1$,

$$H_N^{r_1\cdots r_n}(x^{(1)},\cdots,x^{(n)})$$
$$=\sum_0^N\cdots\sum_0^N a_{\nu_1-\mu_1,\dots,\nu_n-\mu_n} r_1^{|\nu_1-\mu_1|}\cdots r_n^{|\nu_n-\mu_n|}x_{\nu_1}^{(1)}\bar{x}_{\mu_1}^{(1)}\cdots x_{\nu_n}^{(n)}\bar{x}_{\mu_n}^{(n)},$$
$$(0\leqq r_k<1), \qquad (6)$$

then by (4),

$$H_N^{r_1\cdots r_n}(x^{(1)},\cdots,x^{(n)})$$
$$=\frac{1}{(2\pi)^n}\int_0^{2\pi}\!\!\cdots\!\int u(r_1e^{-i\theta_1},\cdots,r_ne^{-i\theta_n})\prod_{k=1}^{n}|x_0^{(k)}+x_1^{(k)}e^{i\theta_k}+\cdots+x_N^{(k)}e^{iN\theta_k}|^2\,d\theta_1\cdots d\theta_n. \qquad (7)$$

We will prove:

THEOREM 1. *If* $u(z_1,\cdots,z_n)\geqq 0$ *for* $|z_k|<1$, *then* $a_{\nu_1,\dots,\nu_n}$ *can be expressed by*

$$a_{\nu_1,\dots,\nu_n}=\int_0^{2\pi}\!\!\cdots\!\int e^{i(\nu_1\theta_1+\dots+\nu_n\theta_n)}\,dv(\theta_1,\cdots,\theta_n), \qquad (8$$

where $v(e)$ *is a positive additive set function defined for Borel sets* e *in* $(\theta_1,\cdots,\theta_n)$ *space, such that*

$$\int_0^{2\pi}\!\!\cdots\!\int dv(\theta_1,\cdots,\theta_n)\leqq M=\underset{(\nu_1,\nu_2,\dots,\nu_n)}{\text{upper limit}}|a_{\nu_1,\dots,\nu_n}|$$

and such $v(e)$ *is unique.*

From (8) *and* (2), *we have*

$$H_N(x^{(1)},\cdots,x^{(n)})=\int_0^{2\pi}\!\!\cdots\!\int\prod_{k=1}^{n}\left|x_0^{(k)}+x_1^{(k)}e^{i\theta_k}+\cdots+x_N^{(k)}e^{iN\theta_k}\right|^2dv(\theta_1,\cdots,\theta_n)\geqq 0,$$

so that $\{a_{\nu_1,\cdots,\nu_n}\}$ *is a positive definite sequence and from* (3),

$$u(r_1e^{-i\theta_1},\cdots,r_ne^{-i\theta_n})=\int_0^{2\pi}\!\!\cdots\!\int\prod_{k=1}^{n}(1-r_k^2)(1-2r_k\cos(\varphi_k-\theta_k)+r_k^2)\,dv(\varphi_1,\cdots,\varphi_n),$$

$$(0\leqq r_k<1). \qquad (9)$$

PROOF. From (4) and (1), we have

$$0\leqq a_{0,\cdots,0}=\frac{1}{(2\pi)^n}\int_0^{2\pi}\!\!\cdots\!\int u(re^{-i\theta_1},\cdots,r_ne^{-i\theta_n})\,d\theta_1\cdots d\theta_n\leqq M. \qquad (10)$$

Let e be a Borel set in $(\theta_1,\cdots,\theta_n)$-space and we define a positive additive set function $v_\rho(e)$ $(0<\rho<1)$ by

$$v_\rho(e)=\frac{1}{(2\pi)^n}\int\!\!\cdots\!\int_e u(\rho e^{-i\theta_1},\cdots,\rho e^{-i\theta_n})\,d\theta_1\cdots d\theta_n. \qquad (11)$$

Since $u\geqq 0$, we have $v_\rho(e)\leqq a_{0,\cdots,0}\leqq M$, so that $v_\rho(e)$ is uniformly bounded for $0<\rho<1$, hence we can find $\rho_1<\rho_2<\cdots<\rho_n\to 1$, such that $\lim_{n\to\infty}v_{\rho_n}(e)=v(e)$, where $v(e)$ is a positive additive set function, such that

$$\int_0^{2\pi}\!\!\cdots\!\int dv(\theta_1,\cdots,\theta_n)\leqq M. \qquad (12)$$

Hence from (4),

$$a_{\nu_1,\cdots,\nu_n}=\int_0^{2\pi}\!\!\cdots\!\int e^{i(\nu_1\theta_1+\cdots+\nu_n\theta_n)}\,dv(\theta_1\cdots,\theta_n). \qquad (13)$$

Next we will prove the uniqueness of $v(e)$. It suffices to prove that if for all $\nu_1,\cdots,\nu_n$,

$$\int_0^{2\pi}\!\!\cdots\!\int e^{i(\nu_1\theta_1+\cdots+\nu_n\theta_n)}\,dv(\theta_1,\cdots,\theta_n)=0, \qquad (14)$$

then $v(e)=0$ for any Borel set e, where $dv(e)$ is an additive set function. From (14),

$$\int_0^{2\pi}\!\!\cdots\!\int P(\theta_1,\cdots,\theta_n)\,dv(\theta_1,\cdots,\theta_n)=0 \qquad (15)$$

for any trigonometrical polynomial $P(\theta_1,\cdots,\theta_n)$.

Let Q_0: $0\leqq\theta_1\leqq 2\pi,\cdots,0\leqq\theta_n\leqq 2\pi$ and Q: $\alpha_1\leqq\theta_1\leqq\beta_1,\cdots,\alpha_n\leqq\theta_n\leqq\beta_n$ $(0<\beta_k-\alpha_k\leqq 2\pi)$. If Q has points outside Q_0, we replace them by points in Q_0 (mod. 2π). Let $f(\theta_1,\cdots,\theta_n)=1$ in Q and $=0$ outside Q. Then for any $\varepsilon>0$, we can find, by Fejér's mean, a trigonometrical polynomial $P(\theta_1,\cdots,\theta_n)$, such that $0\leqq P(\theta_1,\cdots,\theta_n)\leqq 1$ in Q_0 and $|f(\theta_1,\cdots,\theta_n)-P(\theta_1,\cdots,\theta_n)|<\varepsilon$, except in a neighbourhood of the boundary of $\underset{\sim}{Q}$. From this and (15), we see that $v(\underset{\sim}{Q}) = 0$, if the boundary of $\underset{\sim}{Q}$ does not contains a mass. Since any $\underset{\sim}{Q}$ can be approximated by such $\underset{\sim}{Q}$, we see that $v(\underset{\sim}{Q}) = 0$ for any $\underset{\sim}{Q}$. From this, we conclude that $v(e) = 0$ for any Borel set e.

2. To prove the converse of Theorem 1, we use the following lemmas.

LEMMA 1.[5] *If* $\tau(\theta) = a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) > 0$ *in* $[0, 2\pi]$, *then* $\tau(\theta)$ *can be expressed in the form:*

$$\tau(\theta) = |\gamma_0 + \gamma_1 e^{i\theta} + \cdots + \gamma_n e^{in\theta}|^2.$$

LEMMA 2.[6] *Let* $H_N(x^{(1)}, \cdots, x^{(n)})$, $H_N^{r_1 \cdots r_n}(x^{(1)}, \cdots, x^{(n)})$ *be defined by* (2) *and* (6). *If for all* $x^{(k)}$, *such that* $|x_0^{(k)}|^2 + \cdots + |x_N^{(k)}|^2 = 1$, $(k = 1, 2, \cdots n)$,

$$\alpha \leqq H_N(x^{(1)}, \cdots, x^{(n)}) \leqq \beta,$$

then

$$\alpha \leqq H_N^{r_1 \cdots r_n}(x^{(1)}, \cdots, x^{(n)}) \leqq \beta.$$

PROOF. Let

$$b_{\nu_1, \cdots, \nu_n} = \rho_1^{|\nu_1|} \cdots \rho_n^{|\nu_n|} \qquad (0 \leqq \rho_k < 1),$$

then, for $|z_k| < 1$,

$$u_b(z_1, \cdots, z_n) = \prod_{k=1}^{n} (1 - (\rho_k r_k)^2)/|1 - \rho_k r_k e^{i\theta_k}|^2 > 0, \qquad (z_k = r_k e^{i\theta_k}),$$

so that by Theorem 1,

$$\rho_1^{|\nu_1|} \cdots \rho_n^{|\nu_n|} = b_{\nu_1, \cdots, \nu_n} = \int_0^{2\pi} \cdots \int e^{i(\nu_1\theta_1 + \cdots \nu_n\theta_n)} dv(\theta_1, \cdots, \theta_n), \tag{16}$$

hence

$$H_N^{\rho_1 \cdots \rho_n}(x^{(1)}, \cdots, x^{(n)}) = \int_0^{2\pi} \cdots \int H_N(\xi^{(1)}, \cdots, \xi^{(n)}) dv(\theta_1, \cdots, \theta_n), \tag{17}$$

where $\xi_\nu^{(k)} = e^{i\nu\theta_k} x_\nu^{(k)}$, so that

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5) L. Fejér: Über trigonometrische Polynome. Crelle 146 (1916).

6) The case $n = 1$ was proved by I. Schur: Bemerkungen zur Theorie der beschränkten Bilinearform mit unendlich vielen Veränderlichen. Crelle 140 (1911). O. Szász: l.c. 2)

$$|\xi_0^{(k)}|^2+\cdots+|\xi_N^{(k)}|^2=|x_0^{(k)}|^2+\cdots+|x_N^{(k)}|^2=1, \qquad (k=1,2,\cdots,n),$$

hence by the hypothesis, $\alpha \leqq H_N(\xi^{(1)},\cdots,\xi^{(n)}) \leqq \beta$.

Since by (16),

$$1=b_{0,\cdots,0}=\int_0^{2\pi}\cdots\int d\nu(\theta_1,\cdots,\theta_n),$$

we have from (17), $\alpha \leqq H_N^{\rho_1\cdots\rho_n}(x^{(1)},\cdots,x^{(n)}) \leqq \beta$.

3. We will prove the converse of Theorem 1.

THEOREM 2. *If $\{a_{\nu_1},\cdots,a_{\nu_n}\}$ is a positive definite sequence, then*

$$u_a(z_1,\cdots,z_n) \geqq 0 \qquad \textit{or } |z_k|<1.$$

Hence by Theorem 1,

$$a_{\nu_1,\cdots,\nu_n}=\int_0^{2\pi}\cdots\int e^{i(\nu_1\theta_1+\cdots+\nu_n\theta_n)}d\nu(\theta_1,\cdots,\theta_n). \tag{18}$$

where

$$\int_0^{2\pi}\cdots\int d\nu(\theta_1,\cdots,\theta_n) \leqq M=\text{upper limit}_{(\nu_1,\cdots,\nu_n)}|a_{\nu_1,\cdots,\nu_n}|.$$

Raikov[7] proved that if $\{a_{\nu_1},\cdots,a_{\nu_n}\}$ is a positive definite sequence, then it can be expressed in the form (18).

PROOF. Since by the hypothesis, $H_N(x^{(1)},\cdots,x^{(n)}) \geqq 0$, we have by Lemma 2 and (7),

$$0 \leqq H_N^{r_1\cdots r_n}(x^{(1)},\cdots,x^{(n)})$$

$$=\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int u(r_1e^{-i\theta_1},\cdots,r_ne^{-i\theta_n})\prod_{k=1}^{n}\left|x_0^{(k)}+x_1^{(k)}e^{i\theta_k}+\cdots+x_N^{(k)}e^{iN\theta_k}\right|^2 d\theta_1\cdots d\theta_n. \tag{19}$$

Let $(\theta_1^0,\cdots,\theta_n^0)$ be any point. For any $\delta>0$, $\delta'>0$, $\eta>0$, we define $v_k(\theta)$, $g_k(\theta)$ as follows:

$$v_k(\theta)=1 \text{ for } |\theta-\theta_k^0| \leqq \delta, \quad v_k(\theta)=0 \text{ for } |\theta-\theta_k^0|>\delta, \tag{20}$$

7). Γ. Raikov: Positive definite functions on discrete commutative groups, C. R. Acad. Sci. URSS., **27** (1940).

$g_k(\theta) = 1$ for $|\theta - \theta_k^0| \leqq \delta$; $g_k(\theta) = \eta\,(>0)$ for $|\theta - \theta_k^0| \geqq \delta + \delta'$; $g_k(\theta)$ is a linear function for $\delta \leqq |\theta - \theta_k^0| \leqq \delta + \delta'$, such that

$$g_k(\theta_k^0 \pm \delta) = 1, \quad g_k(\theta_k^0 \pm (\delta + \delta')) = \eta. \tag{21}$$

Since $g_k(\theta)$ is continuous, we can find a trigonometrical polynomial $\tau_k(\theta)$ of order N, such that in $[0, 2\pi]$

$$|g_k(\theta) - \tau_k(\theta)| < \varepsilon < \eta, \tag{22}$$

then $\tau_k(\theta) > 0$ in $[0, 2\pi]$, so that by Lemma 1, for suitable $x^{(k)}$

$$\tau_k(\theta) = |x_0^{(k)} + x_1^{(k)} e^{i\theta} + \cdots + x_N^{(k)} e^{iN\theta}|^2. \tag{23}$$

For such $x^{(k)}$, we have from (19),

$$0 \leqq H_N^{r_1 \cdots r_n}(x^{(1)}, \cdots, x^{(n)})$$

$$= \frac{1}{(2\pi)^n} \int_0^{2\pi} \!\!\cdots\! \int u(r_1 e^{-i\theta_1}, \cdots, r_n e^{-i\theta_n}) \prod_{k=1}^{n} \tau_k(\theta_k)\, d\theta_1 \cdots d\theta_n. \tag{24}$$

Hence for $\varepsilon \to 0$, $\delta' \to 0$, $\eta \to 0$, we have

$$0 \leqq \int_0^{2\pi} \!\!\cdots\! \int u(r_1 e^{-i\theta_1}, \cdots, r_n e^{-i\theta_n}) \prod_{k=1}^{n} v_k(\theta_k)\, d\theta_1 \cdots d\theta_n$$

$$= \int \!\!\cdots\! \int_{|\theta_k - \theta_k^0| \leqq \delta} u(r_1 e^{-i\theta_1}, \cdots, r_n e^{-i\theta_n})\, d\theta_1 \cdots d\theta_n = (2\delta)^n u(r_1 e^{-i\theta'_1}, \cdots, r_n e^{-i\theta'_n}),$$

where $|\theta'_k - \theta_k^0| \leqq \delta$, so that $u(r_1 e^{-i\theta'_1}, \cdots, r_n e^{-i\theta'_n}) \geqq 0$, hence for $\delta \to 0$ we have $u(r_1 e^{-i\theta_1^0}, \cdots, r_n e^{-i\theta_n^0}) \geqq 0$. Since $(\theta_1^0, \cdots, \theta_n^0)$ is arbitrary, $u(z_1, \cdots, z_n) \geqq 0$ for $|z_k| < 1$, which proves the theorem.

3. Positive definite functions of n variables

1. Let $f(t_1, \cdots, t_n)$ be a bounded measurable function defined for $-\infty < t_k < \infty$ $(k = 1, 2, \cdots, n)$, such that

$$f(-t_1, \cdots, -t_n) = \overline{f(t_1, \cdots, t_n)}, \tag{1}$$

$$\underset{-\infty < t_k < \infty}{\text{upper limit}}\, |f(t_1, \cdots, t_n)| = M < \infty.$$

If for all $q_k(t)$ of class (L°), Hermitian forms:

$$H(q_1, \cdots, q_n) = \int_{-\infty}^{\infty} \!\!\cdots\! \int f(t_1 - s_1, \cdots, t_n - s_n) \prod_{k=1}^{n} \left(q_k(t_k)\, \overline{q_k(s_k)} \right) dt_1 \cdots ds_n \geqq 0, \tag{2}$$

then we call $f(t_1, \cdots, t_n)$ a positive definite function of n variables. Analogous to (12) of § 1, we put

$$u_f(z_1, \cdots, z_n) = u(z_1, \cdots, z_n)$$
$$= \int_{-\infty}^{\infty}\cdots\int f(t_1, \cdots, t_n) \exp(-\sum_k (|t_k|\, x_k + it_k y_k))\, dt_1 \cdots dt_n,$$
$$(z_n = x_k + iy_k), \qquad (3)$$

which is convergent and is real for $x_k > 0$ by (1). From (1), we have

$$|u(z_1, \cdots, z_n)| \leqq M\int_{-\infty}^{\infty}\cdots\int \exp(-\sum_N |t_k|\, x_k)\, dt_1 \cdots dt_n = \frac{2^n M}{x_1 \cdots x_n} \qquad (4)$$

especially,

$$x_1 \cdots x_n\, |u(x_1, \cdots, x_n)| \leqq 2^n M, \qquad (x_k > 0). \qquad (5)$$

For $q_k(t)$ of class (L^0) and $\sigma_k > 0$, we put as (6) of § 2,

$$H^{\sigma_1 \cdots \sigma_n}(q_1, \cdots, q_n)$$
$$= \int_{-\infty}^{\infty}\cdots\int f(t_1 - s_1, \cdots, t_n - s_n) \exp(-\sum_k |t_k - s_k|\, \sigma_k) \prod_{k=1}^{n} \left(q_k(t_k)\overline{q_k(s_k)}\right) dt_1 \cdots ds_n. \qquad (6)$$

2. We will prove some lemmas.

Lemma 3. *$u(z_1, \cdots, z_n)$ can be expressed by a Poisson integral:*

$$u(z_1, \cdots, z_n)$$
$$= \frac{1}{\pi^n}\int_{-\infty}^{\infty}\cdots\int u(\sigma_1 + i\lambda_1, \cdots, \sigma_n + i\lambda_n) \prod_{k=1}^{n} (x_k - \sigma_k)/((x_k - \delta_k)^2 + (y_k - \lambda_k)^2)\, d\lambda_1 \cdots d\lambda_n,$$
$$(z_k = x_k + iy_k, \quad 0 < \sigma_k < x_k).$$

This is an analogue of (5) of § 2.

Proof. From

$$u(\sigma_1 + i\lambda_1, \cdots, \sigma_n + i\lambda_n) = \int_{-\infty}^{\infty}\cdots\int f(t_1, \cdots, t_n) \exp(-\sum_k (|t_k|\, \sigma_k + it_k \lambda_k))\, dt_1 \cdots dt_n$$

and

$$\int_{-\infty}^{\infty} (x_k - \sigma_k)\, e^{-it_k\lambda_k}/((x_k - \sigma_k)^2 + (y_k - \lambda_k)^2)\, d\lambda_k = \pi e^{-(|t_k|(x_k - \sigma_k) + it_k y_k)},$$

$$(x_k > \sigma_k > 0),$$

we have

$$\int_{-\infty}^{\infty}\!\!\cdots\!\int u(\sigma_1+i\lambda_1,\cdots,\sigma_n+i\lambda_n)\prod_{k=1}^{n}(x_k-\sigma_k)\,((x_k-\sigma_k)^2+(y_k-\lambda_k)^2)\,d\lambda_1\cdots$$

$$=\int_{-\infty}^{\infty}\!\!\cdots\!\int f(t_1,\cdots,t_n)\exp\Big(-\sum_k|t_k|\,\sigma_k\Big)\left(\prod_{k=1}^{n}\int_{-\infty}^{\infty}(x_k-\sigma_k)\,e^{-it_k\lambda_k}/((x_k-\sigma_k)^2\right.$$

$$\left.+(y_k-\lambda_k)^2)\,d\lambda_k\right)dt_1\cdots dt_n$$

$$=\pi^n\int_{-\infty}^{\infty}\!\!\cdots\!\int f(t_1,\cdots,t_n)\exp\Big(-\sum_k(|t_k|\,x_k+it_ky_k)\Big)\,dt_1\cdots dt_n=\pi^n\,u(z_1,\cdots,z_n).$$

LEMMA 4. *Almost everywhere,*

$$f(t_1,\cdots,t_n)\exp\Big(-\sum_k|t_k|\,\sigma_k\Big)=\lim_{T\to\infty}\frac{1}{T^n}\int_0^T\!\!\cdots\!\int I(a_1,\cdots,a_n)\,da_1\cdots da_n,\ (\sigma_k>0),$$

where

$$I(a_1,\cdots,a_n)=\frac{1}{(2\pi)^n}\int_{-a_k}^{a_k}\!\!\cdots\!\int\exp\Big(i\sum_k t_k\lambda_k\Big)\,u(\sigma_1+i\lambda_1,\cdots,\sigma_n+i\lambda_n)\,d\lambda_1\cdots d\lambda_n.$$

Hence $f(t_1,\cdots,t_n)$ is uniquely determined by $u(z_1,\cdots,z_n)$. Especially, if $u(z_1,\cdots,z_n)\equiv 0$ for $x_k>0$, then $f(t_1,\cdots,t_n)\sim 0$.

PROOF. From

$$u(\sigma_1+i\lambda_1,\cdots,\sigma_n+i\lambda_n)=\int_{-\infty}^{\infty}\!\!\cdots\!\int f(\tau_1,\cdots,\tau_n)\exp\Big(-\sum_k(|\tau_k|\,\sigma_k+i\tau_k\lambda_k)\Big)\,d\tau_1\cdots d\tau_n$$

$$=\int_{-\infty}^{\infty}\!\!\cdots\!\int\Phi(\tau_1,\cdots,\tau_n)\exp\Big(-i\sum_k\tau_k\lambda_k\Big)\,d\tau_1\cdots d\tau_n,\qquad(7)$$

where

$$\Phi(t_1,\cdots,t_n)=f(t_1,\cdots,t_n)\exp\Big(-\sum_k|t_k|\,\sigma_k\Big),\qquad(8)$$

we have

$$I(a_1,\cdots,a_n)=\frac{1}{\pi^n}\int_{-\infty}^{\infty}\!\!\cdots\!\int\Phi(\tau_1,\cdots,\tau_n)\prod_{k=1}^{n}(\sin a_k(\tau_k-t_k))/(\tau_k-t_k))\,d\tau_1\cdots d\tau_n$$

$$=\frac{1}{\pi^n}\int_{-\infty}^{\infty}\!\!\cdots\!\int\Phi(t_1+v_1,\cdots,t_n+v_n)\prod_{k=1}^{n}((\sin a_k v_k)/v_k)\,dv_1\cdots dv_n,$$

so that

$$J(T, t_1,\dots,t_n) = \frac{1}{T^n}\int_0^T\cdots\int I(a_1, \cdots, a_n)\, da_1, \cdots, da_n$$

$$= \frac{1}{\pi^n}\int_{-\infty}^{\infty}\cdots\int \Phi(t_1 + 2u_1/T, \cdots, t_n + 2u_n/T)\prod_{k=1}^{n}((\sin u_k)/u_k)^2\, du_1 \cdots du_n. \qquad (9)$$

Since $|\Phi(t_1, \cdots, t_n)| \leqq |f(t_1, \cdots, t_n)| \leqq M$ and

$$\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^2 dx = \pi, \qquad (10)$$

we have

$$|J(T, t_1, \dots, t_n)| \leqq M. \qquad (11)$$

From (9), (10),

$$J(T, t_1, \dots, t_n) - \Phi(t_1, \cdots, t_n)$$

$$= \frac{1}{\pi^n}\int_{-\infty}^{\infty}\cdots\int \Big[\Phi(t_1 + 2u_1/T, \cdots, t_n + 2u_n/T) - \Phi(t_1, \cdots, t_n)\Big]\prod_{k=1}^{n}((\sin u_k)/u_k)^2\, du_1 \cdots du_n.$$

We take N so large that

$$\int_{|u|\geqq N}\left(\frac{\sin u}{u}\right)^2 du < \varepsilon,$$

then, since $|\Phi(t_1 + 2u_1/T, \cdots, t_n + 2u_n/T) - \Phi(t_1, \cdots, t_n)| \leqq 2M$, we have

$$|J(T, t_1, \dots, t_n) - \Phi(t_1, \cdots, t_n)|$$

$$\leqq \frac{1}{\pi^n}\int_{-N}^{N}\cdots\int |\Phi(t_1 + 2u_1/T, \cdots, t_n + 2u_n/T_n) - \Phi(t_1, \cdots, t_n)|\, du_1 \cdots du_n + O(\varepsilon)$$

$$= \left(\frac{T}{2\pi}\right)^n \int_{-2N/T}^{2N/T}\cdots\int |\Phi(t_1 + \tau_1, \cdots, t_n + \tau_n) - \Phi(t_1, \cdots, t_n)|\, d\tau_1 \cdots d\tau_n + O(\varepsilon). \qquad \cdots (12)$$

Since by Lebesgue's theorem,

$$\lim_{T\to\infty} T^n \int_{-2N/T}^{2N/T}\cdots\int |\Phi(t_1 + \tau_1, \cdots, t_n + \tau_n) - \Phi(t_1, \cdots, t_n)|\, d\tau_1 \cdots d\tau_n = 0$$

almost everywhere, we conclude from (12) easily that

$$\lim_{T\to\infty} J(T t_1, \dots, t_n) = \Phi(t_1, \cdots, t_n) = f(t_1, \cdots, t_n)\exp\big(-\sum_k |t_k|\ \sigma_k\big)$$

almost everywhere.

3. We will prove:

THEOREM 3. *If* $u_f(z_1, \cdots, z_n) \geqq 0$ *for* $x_k > 0$, *then*

$$f(t_1, \cdots, t_n) \sim \int_{-\infty}^{\infty} \cdots \int \exp(i(t_1 \lambda_1 + \cdots + t_n \lambda_n))\, dv(\lambda_1, \cdots, \lambda_n),$$

where $v(e)$ *is a positive additive set function defined for Borel sets* e *in* $(\lambda_1, \cdots, \lambda_n)$-*space, such that*

$$\int_{-\infty}^{\infty} \cdots \int dv(\lambda_1, \cdots, \lambda_n) \leqq M = \text{upper limit}_{-\infty < t_k < \infty} |f(t_1, \cdots, t_n)|$$

and such $v(e)$ *is unique.*

Hence by (2),

$$H(q_1, \cdots, q_n) = \int_{-\infty}^{\infty} \cdots \int \prod_{k=1}^{n} \left| \int_{-\infty}^{\infty} \exp(it_k \lambda_k)\, q_k(t_k)\, dt_k \right|^2 dv(\lambda_1, \cdots, \lambda_n) \geqq 0, \tag{13}$$

so that $f(t_1, \cdots, t_n)$ *is a positive definite function and by* (3) and

$$u(x_1 + iy_1, \cdots, x_n + iy_n)$$
$$= 2^n \int_{-\infty}^{\infty} \cdots \int \prod_{k=1}^{n} x_k / (x_k^2 + (y_k - \lambda_k)^2)\, dv(\lambda_1, \cdots, \lambda_n). \tag{14}$$

PROOF. By Lemma 3,

$$u_f(z_1, \cdots, z_n)$$
$$= \frac{1}{\pi^n} \int_{-\infty}^{\infty} \cdots \int \prod_{k=1}^{n} (x_k - \sigma_k)/((x_k - \sigma_k)^2 + (y_k - \lambda_k)^2)\, u(\sigma_1 + i\lambda_1, \cdots, \sigma_n + i\lambda_n)\, d\lambda_1 \cdots d\lambda_n,$$
$$(z_k = x_k + iy_k,\ 0 < \sigma_k < x_k), \tag{15}$$

and from $u \geqq 0$ and (5), we have for $T > 0$.

$$2^n M \geqq x_1 \cdots x_n\, u(x_1, \cdots, x_n)$$
$$\geqq \frac{1}{\pi^n} \int_{-T}^{T} \cdots \int \left(\prod_{k=1}^{n} x_k (x_k - \sigma_k)/((x_k - \sigma_k)^2 + \lambda_k^2) \right) u(\sigma_1 + i\lambda_1, \cdots, \sigma_n + i\lambda_n)\, d\lambda_1 \cdots d\lambda_n,$$

so that for $x_k \to \infty$,

$$2^n M \geqq \frac{1}{\pi_n} \int_{-T}^{T} \cdots \int u(\sigma_1 + i\lambda_1, \cdots, \sigma_n + i\lambda_n)\, d\lambda_1 \cdots d\lambda_n,$$

hence

$$\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int u(\sigma_1 + i\lambda_1, \cdots, \sigma_n + i\lambda_n)\, d\lambda_1 \cdots d\lambda_n \leqq M. \tag{16}$$

For any Borel set e in $(\lambda_1, \cdots, \lambda_n)$-space, we define a positive additive set function $v_\sigma(e)$ $(\sigma > 0)$ by

$$v_\sigma(e) = \frac{1}{(2\pi)^n}\int_e\cdots\int u(\sigma + i\lambda_1, \cdots, \sigma + i\lambda_n)\, d\lambda_1 \cdots d\lambda_n \ (\sigma > 0). \tag{17}$$

Since by (16), $v_\sigma(e)$ is uniformly bounded for $\sigma > 0$, we can find $\sigma_1 > \sigma_2 > \cdots > \sigma_n \to 0$, such that $\lim_{n\to\infty} v_{\sigma_n}(e) = v(e)$, where $v(e)$ is a positive additive set function, such that by (16)

$$\int_{-\infty}^{\infty}\cdots\int dv(\lambda_1, \cdots, \lambda_n) \leqq M. \tag{18}$$

Hence from (15,)

$$u_f(z_1, \cdots, z_n) = 2^n \int_{-\infty}^{\infty}\cdots\int \Big(\prod_{k=1}^{n} x_k/(x_k^2 + (y_k - \lambda_k)^2)\Big)\, dv(\lambda_1, \cdots, \lambda_n). \tag{19}$$

Let

$$g(t_1, \cdots, t_n) = \int_{-\infty}^{\infty}\cdots\int \exp(i(t_1\lambda_1 + \cdots t_n\lambda_n))\, dv(\lambda_1, \cdots, \lambda_n), \tag{20}$$

then by (3),

$$u_g(z_1, \cdots, z_n) = 2^n \int_{-\infty}^{\infty}\cdots\int \Big(\prod_{k=1}^{n} x_k/(x_k^2 + (y_k - \lambda_k)^2)\Big)\, dv(\lambda_1, \cdots, \lambda_n),$$

so that

$$u_g(z_1, \cdots, z_n) \equiv u_f(z_1, \cdots, z_n), \text{ or } u_{f-g}(z_1, \cdots, z_n) \equiv 0,$$

hence by Lemma 4, $g(t_1, \cdots, t_n) \sim f(t_1, \cdots, t_n)$, or

$$f(t_1, \cdots, t_n) \sim \int_{-\infty}^{\infty}\cdots\int \exp(i(t_1\lambda_1 + \cdots + t_n\lambda_n))\, dv(\lambda_1, , \lambda_n). \tag{21}$$

Next we will prove the uniqueness of $v(e)$. It suffices to prove that if

$$\int_{-\infty}^{\infty}\cdots\int \exp(i(t_1\lambda_1 + \cdots + t_n\lambda_n))\, dv(\lambda_1, \cdots, \lambda_n) \sim 0, \tag{22}$$

then $\nu(e) = 0$ for any Borel set e, where $\nu(e)$ is an additive set function.

Let $(\lambda_1^0, \cdots, \lambda_n^0)$ be any point. For any $\delta_k > 0$, we define $v_k(\lambda)$ as follows:

$$\left.\begin{aligned} v_k(\lambda) &= 1 \text{ for } |\lambda - \lambda_k^0| < \delta_k, \\ v_k(\lambda_k^0 \pm \delta_k) &= 1/2, \\ v_k(\lambda) &= 0 \text{ for } |\lambda - \lambda_k^0| > \delta_k, \end{aligned}\right\} \tag{23}$$

then

$$v_k(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(it\lambda)\, q_k(t)\, dt, \tag{24}$$

where

$$q_k(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-it\lambda)\, v_k(\lambda)\, d\lambda = \sqrt{\frac{2}{\pi}} \exp(-it\lambda_k^0)(\sin t\delta_k)/t, \tag{25}$$

$$\int_{-\infty}^{\infty} |q_k(t)|^2 dt = \int_{-\infty}^{\infty} |v_k(\lambda)|^2 d\lambda = 2\delta_k.$$

For any $T > 0$,

$$\begin{aligned} \int_{-T}^{T} e^{it\lambda} q_k(t)\, dt &= \sqrt{\frac{2}{\pi}} \int_0^T \left[\sin t(\lambda - \lambda_k^0 + \delta_k) - \sin t(\lambda - \lambda_k^0 - \delta_k)\right]/t\, dt \\ &= \sqrt{\frac{2}{\pi}} \int_{T(\lambda - \lambda_k^0 - \delta_k)}^{T(\lambda - \lambda_k^0 + \delta_k)} \frac{\sin u}{u}\, du, \end{aligned} \tag{26}$$

hence

$$\left| \int_{-T}^{T} e^{it\lambda} q_k(t)\, dt \right| \leqq K, \tag{27}$$

where K is a constant independent of T and λ.

Now by (22),

$$\begin{aligned} &\int_{-\infty}^{\infty} \cdots \int \prod_{k=1}^{n} \left(\int_{-T}^{T} \exp(it_k \lambda_k)\, q_k(t_k)\, dt_k \right) d\nu(\lambda_1, \cdots, \lambda_n) \\ &= \int_{-T}^{T} \cdots \int \left(\prod_{k=1}^{n} q_k(t_k) \right) dt_1 \cdots dt_n \int_{-\infty}^{\infty} \cdots \int \exp(i \sum_{k=1}^{n} t_k \lambda_k)\, d\nu(\lambda_1, \cdots, \lambda_n) = 0. \end{aligned}$$

In virtue of (27) and Lebesgue's theorem, for $T \to \infty$ we have from (24)

$$\int_{-\infty}^{\infty}\!\!\cdots\!\int\Big(\prod_{k=1}^{n} v_k(\lambda_k)\Big)\,dv(\lambda_1,\cdots,\lambda_n)=\int\!\!\cdots\!\int_{|\lambda_k-\lambda_k^0|\leqq\delta_k} dv(\lambda_1,\cdots,\lambda_n)=v(Q)=0,$$

where Q: $|\lambda_k-\lambda_k^0|\leqq\delta_k\ (k=1,2,\cdots,n)$.

From this we conclude easily that $v(e)=0$ for any Borel set e.

Remark. By the second mean value theorem, we have from (26)

$$\Big|\int_{-T}^{T} e^{it\lambda} q_k(t)\,dt\Big|\leqq\frac{K}{T(|\lambda|+1)},\qquad (|\lambda-\lambda_k^0|\geqq 2\delta_k), \tag{28}$$

where $K=K(\delta_k)$ is a constant, which depends on δ_k only, so that by (27), (28),

$$\int_{-\infty}^{\infty}\Big|\int_{-T}^{T} e^{it\lambda} q_k(t)\,dt\Big|^2 d\lambda\leqq K, \tag{29}$$

where K is a constant independent of T and

$$\int_{|\lambda-\lambda_k|\geqq 2\delta_k}\Big|\int_{-T}^{T} e^{it\lambda} q_k(t)\,dt\Big|^2 d\lambda=O\Big(\frac{1}{T^2}\Big). \tag{30}$$

4. To prove the converse of Theorem 3, we use the following lemma.

Lemma 5. *Let $H(q_1,\cdots,q_n)$, $H^{\sigma_1\cdots\sigma_n}(q_1,\cdots,q_n)$ be defined by* (2) *and* (6). *If for all $q_k(t)$ of class* (L^0), *such that* $\int_{-\infty}^{\infty}|q_k(t)|^2\,dt=1$,

$$\alpha\leqq H(q_1,\cdots,q_n)\leqq\beta,$$

then

$$\alpha\leqq H^{\sigma_1\cdots\sigma_n}(q_1,\cdots,q_n)\leqq\beta.$$

Proof. Let

$$g(t_1,\cdots,t_n)=\exp\Big(-\sum_k |t_k|\,\sigma_k\Big),\qquad (\sigma_k>0), \tag{31}$$

then by (3),

$$u_g(z_1,\cdots,z_n)=2^n\prod_{k=1}^{n}(x_k+\sigma_k)/((x_k+\sigma_k)^2+y_k^2)>0\qquad\text{for } x_k>0,$$
$$(z_k=x_k+iy_k),$$

so that by Theorem 3, since $g(t_1,\cdots,t_n)$ is continuous,

$$\exp\Big(-\sum_k|t_k|\,\sigma_k\Big)=g(t_1,\cdots,t_n)=\int_{-\infty}^{\infty}\!\!\cdots\!\int\exp\Big(i\sum_{k=1}^{n}t_k\lambda_k\Big)\,dv(\lambda_1,\cdots,\lambda_n). \tag{32}$$

Hence

$$H^{\sigma_1 \cdots \sigma_n}(q_1, \cdots, q_n) = \int_{-\infty}^{\infty} \cdots \int H(p_1, \cdots, p_n)\, dv(\lambda_1, \cdots, \lambda_n), \tag{33}$$

where $p_k(t) = e^{it\lambda_k} q_k(t)$, so that

$$\int_{-\infty}^{\infty} \left| p_k(t) \right|^2 dt = \int_{-\infty}^{\infty} \left| q_k(t) \right|^2 dt = 1,$$

hence by the hypothesis, $\alpha \leqq H(p_1, \cdots, p_n) \leqq \beta$.

Since

$$1 = g(0, \cdots, 0) = \int_{-\infty}^{\infty} \cdots \int dv(\lambda_1, \cdots, \lambda_n),$$

we have from (33), $\alpha \leqq H^{\sigma_1 \cdots \sigma_n}(q_1, \cdots, q_n) \leqq \beta$.

5. We will prove the converse of Theorem 3.

THEOREM 4. *If $f(t_1, \cdots, t_n)$ is a positive definite function, then $u_f(z_1, \cdots, z_n) \geqq 0$ for $x_k > 0$.*

Hence by Theorem 3,

$$f(t_1, \cdots, t_n) \sim \int_{-\infty}^{\infty} \cdots \int \exp(i \sum_{k=1}^{n} t_k \lambda_k)\, dv(\lambda_1, \cdots, \lambda_n), \tag{34}$$

where

$$\int_{-\infty}^{\infty} \cdots \int dv(\lambda_1, \cdots, \lambda_n) \leqq M = \text{upper limit}_{-\infty < t_k < \infty} \left| f(t_1, \cdots, t_n) \right|.$$

Powzner and Raikov[8] proved that if $f(t_1, \cdots, t_n)$ is a positive definite function, then it can be expressed in the form (34).

PROOF. Let $(z_1^0, \cdots, z_n^0)$ $(z_k^0 = \sigma_k^0 + i\lambda_k^0, \sigma_k^0 > 0)$ be any point.

By Lemma 4, almost everywhere,

$$f(t_1, \cdots, t_n) \exp(-\sum_k |t_k| \sigma_k^0) = \lim_{T \to \infty} J(T, t_1, \cdots, t_n), \tag{35}$$

where

8) A. Powzner: Über positive Funktionen auf einer Abelschen Gruppe C. R. Acad. Sci. URSS **28** (1940). D. Raikov: Positive definite functions on commutative groups with an invariant measure, C. R. Acad. Sci. URSS **28** (1940).

$$J(T, t_1, \cdots, t_n) = \frac{1}{T^n}\int_0^T \cdots \int I(a, t_1, \cdots, t_n)\, da_1 \cdots da_n,$$

$$I(a, t_1, \cdots, t_n) = \frac{1}{(\ \pi)^n}\int_{-a_k}^{a_k} \cdots \int \exp(i \sum_k t_k \lambda_k)\, u(\sigma_1^0 + i\lambda_1, \cdots, \sigma_n^0 + i\lambda_n)\, d\lambda_1 \cdots d\lambda_n. \tag{39}$$

By (11),

$$|J(T, t_1, \cdots, t_n)| \leqq M. \tag{37}$$

We define $v_k(\lambda)$, $q_k(t)$ by (23) and (25), then for $T' > 0$,

$$\int_{-T'}^{T'} \cdots \int J(T, t_1 - s_1, \cdots, t_n - s_n)\Big(\prod_{k=1}^{n} q_k(t_k)\, \overline{q_k(s_k)}\Big)\, dt_1 \cdots ds_n$$

$$= \frac{1}{T^n}\int_0^T \cdots \int K(a_1, \cdots, a_n)\, da_1 \cdots da_n, \tag{38}$$

where

$$K(a_1, \cdots, a_n)$$
$$= \frac{1}{(2\pi)^n}\int_{-ak}^{ak} \cdots \int u(\sigma_1^0 + i\lambda_1, \cdots, \sigma_n^0 + i\lambda_n)\Big(\prod_{k=1}^{n}\Big|\int_{-T'}^{T'} \exp(it_k \lambda_k)\, q_k(t_k)\, dt_k\Big|^2\Big)\, d\lambda_1 \cdots d\lambda_n. \tag{39}$$

By (29), (30) and (4), we have uniformly for a_k,

$$K(a_1, \cdots, a_n)$$
$$= \frac{1}{(2\pi)^n}\int_{|\lambda_k - \lambda_{k_k}| \leqq 2\delta_k} \cdots \int u(\sigma_1^0 + i\lambda_1, \cdots, \sigma_n^0 + i\lambda_n)\Big(\prod_{k=1}^{n}\Big|\int_{-T'}^{T'} \exp(it_k \lambda_k)\, q_k(t_k)\, dt_k\Big|^2\Big)\, d\lambda_1 \cdots d\lambda_n$$
$$+ O(T'^{-2}),$$

so that by (38),

$$\int_{-T'}^{T'} \cdots \int J(T, t_1 - s_1, \cdots, t_n - s_n)\Big(\prod_{k=1}^{n} q_k(t_k)\, \overline{q_k(s_k)}\Big)\, dt_1 \cdots ds_n$$

$$= \frac{1}{(2\pi)^n}\int_{|\lambda_k - \lambda_k| \leqq 2\delta_k} \cdots \int u(\sigma_1^0 + \lambda i_1, \cdots, \sigma_n^0 + i\lambda_n)\Big(\prod_{k=1}^{n}\Big|\int_{-T'}^{T'} \exp(it_k \lambda_k)\, q_k(t_k)\, dt_k\Big|^2\Big)\, d\lambda_1 \cdots d\lambda_n$$
$$+ O(T'^{-2}). \tag{40}$$

If we make $T \to \infty$, then by (35), (37) and Lebesgue's theorem,

$$\int_{-T'}^{T'}\cdots\int f(t_1 - s_1, \cdots, t_n - s_n) \exp(-\sum_k |t_k - s_k| \sigma_k^0)\Big(\prod_{k=1}^{n} q_k(t_k) \overline{q_k(s_k)}\Big) dt_1 \cdots ds_n$$

$$= \frac{1}{(\pi)} \int_{|\lambda_k - \lambda_k^0| \leqq 2\delta_k}\cdots\int u(\sigma_1^0 + i\lambda_1, \cdots, \sigma_n^0 + i\lambda_n)\Big(\prod_{k=1}^{n}\Big|\int_{-T'}^{T'} \exp(it_k\lambda_k) q_k(t_k) dt_k\Big|^2 d\lambda_1 \cdots d\lambda_n$$

$$+ O(T'^{-2}). \tag{41}$$

Next we make $T' \to \infty$, then by (24), (27) and Lebesgue's theorem,

$$H^{\sigma_1 \cdots \sigma_n}(q_1, \cdots, q_n)$$

$$= \int_{-\infty}^{\infty}\cdots\int f(t_1 - s_1, \cdots, t_n - s_n) \exp(-\sum_k |t_k - s_k| \sigma_k^0)\Big(\prod_{k=1}^{n} q_k(t_k)\overline{q_k(s_k)}\Big) dt_1 \cdots ds_n$$

$$= \int_{|\lambda_k - \lambda_k^0| \leqq 2\delta_k}\cdots\int u(\sigma_1^0 + i\lambda_1, \cdots, \sigma_n^0 + i\lambda_n)\Big(\prod_{k=1}^{n} |v_k(\lambda_k)|^2\Big) d\lambda_1 \cdots d\lambda_n$$

$$= \int_{|\lambda_k - \lambda_k^0| \leqq \delta_k}\cdots\int u(\sigma_1^0 + i\lambda_1, \cdots, \sigma_n^0 + i\lambda_n) d\lambda_1 \cdots d\lambda_n$$

$$= 2^n \delta_1 \cdots \delta_n u(\sigma_1^0 + i\lambda'_1, \cdots, \sigma_n^0 + i\lambda'_n), \ (|\lambda'_k - \lambda_k^0| \leqq \delta_k). \tag{42}$$

$q_k(t)$ is not a function of class (L^0), but if we put $q_k^N(t) = q_k(t)$ for $|t| \leqq N$ and $= 0$ for $|t| > N$, then $q_k^N(t)$ belongs to class (L^0), so that by the hypothesis, $H(q_1^N, \cdots, q_n^N) \geqq 0$, hence by Lemma 5, $H^{\sigma_1 \cdots \sigma_n}(q_1^N, \cdots, q_n^N) \geqq 0$, so that for $N \to \infty$, $H^{\sigma_1 \cdots \sigma_n}(q_1, \cdots, q_n) \geqq 0$.

Hence from (42), $u(\sigma_1^0 + i\lambda'_1, \cdots, \sigma_n^0 + i\lambda'_n) \geqq 0$, so that for $\delta_k \to 0$, we have $u(\sigma_1^0 + i\lambda_1^0, \cdots, \sigma_n^0 + i\lambda_n^0) = u(z_1^0, \cdots, z_n^0) \geqq 0$. Since $(z_1^0, \cdots, z_n^0)$ is arbitrary, we have $u(z_1, \cdots, z_n) \geqq 0$ for $x_k > 0$, which proves the theorem.

4. Some remarks.

1. Let $f(t)$ be a bounded measurable function defined for $0 \leqq t < \infty$ and $F(z)$ be its Leplace transform :

$$F(z) = \int_0^\infty f(t) e^{-tz} dt, \qquad (z = x + iy). \tag{1}$$

If we define $f(t)$ for $t < 0$ by $f(t) = \overline{f(-t)}$, then

$$\Re F(z) = {}^1/_2 \int_{-\infty}^{\infty} f(t) \exp(-(|t|\, x + ity))\, dt = {}^1/_2\, u_f(z). \tag{2}$$

Hence by Theorem 3 and 4, we have

THEOREM 5. *If* $\Re F(z) \geqq 0$ *for* $x > 0$, *then*

$$f(t) \sim \int_{-\infty}^{\infty} e^{it\lambda}\, dv(\lambda), \tag{3}$$

so that

$$F(z) = \int_{-\infty}^{\infty} \frac{dv(\lambda)}{z - i\lambda}, \tag{4}$$

where $v(\lambda)$ *is an increasing function for* $-\infty < \lambda < \infty$, *such that*

$$\int_{-\infty}^{\infty} dv(\lambda) \leqq M = \underset{-\infty < t < \infty}{\text{upper limit}} |f(t)|.$$

If we normalize $v(\lambda)$, *such that* $v(-\infty) = 0$, $v(\lambda - 0) = v(\lambda)$, *then* $v(\lambda)$ *is unique.*

From this as remarked in the introduction, the necessary and sufficient condition, that $f(t)$ is a positive definite function, is that $\Re F(z) \geqq 0$ for $x > 0$.

Let $f(t)$ be a positive definite function of the form (3), then for any $q(t)$ of class (L^0), the Hermitian form:

$$H(q) = \int_{-\infty}^{\infty}\!\!\cdots\!\!\int f(t-s)\, q(t)\, \overline{q(s)}\, dt ds \geqq 0.$$

If $H(q) > 0$ for all $q(t)$, such that $\int_{-\infty}^{\infty} |q(t)|^2 dt > 0$, then we call $H(q)$ a positive definite form. If $H(q_0) = 0$ for a certain $q_0(t)$, such that $\int_{-\infty}^{\infty} |q_0(t)|\, dt > 0$, then we call $H(q)$ a positive semi-definite form.

THEOREM 6. *If* $H(q)$ *is a positive semi-definite form, then* $v(\lambda)$ *is a step-function, such that*

$$f(t) \sim \sum_{\nu} e^{it\lambda_\nu} \left(v(\lambda_\nu + 0) - v(\lambda_\nu - 0)\right),$$

where $\{\lambda_\nu\}$ *have no cluster point in a finite distance.*

PROOF. Let $H(q_0) = 0$ for a certain $q_0(t)$ of class (L^0), such that

$$\int_{-\infty}^{\infty} \left| q_0(t) \right|^2 dt > 0, \tag{5}$$

then

$$H(q_0) = \iint_{-\infty}^{\infty} f(t-s)\, q_0(t)\, \overline{q_0(s)}\, dt\, ds = \int_{-\infty}^{\infty} dv(\lambda) \left| \int_{-\infty}^{\infty} e^{it\lambda} q_0(t)\, dt \right|^2 = 0. \tag{6}$$

If at a point λ, $v(\lambda+\delta) - v(\lambda-\delta) > 0$ for any $\delta > 0$, we denote $dv(\lambda) > 0$. Let E be the set of λ, such that $dv(\lambda) > 0$, then from (6) at any point λ of E,

$$\int_{-\infty}^{\infty} e^{it\lambda} q_0(t)\, dt = 0.$$

Suppose that E has a cluster point λ_0 in a finite distance, then there exists $\lambda_\nu \to \lambda_0$ $(\lambda_\nu \in E)$, so that

$$\int_{-\infty}^{\infty} e^{it\lambda_\nu} q_0(t)\, dt = 0, \qquad (\nu = 1, 2, \cdots). \tag{7}$$

Let $q_0(t) = 0$ for $|t| \geqq T$, then from (7) we have

$$e^{iT\lambda_\nu} \int_{-T}^{T} e^{it\lambda_\nu} q_0(t)\, dt = \int_0^{2T} e^{i\tau\lambda_\nu} q_0(\tau + T)\, d\tau = 0.$$

Let

$$F(z) = \int_0^{2T} e^{-tz} q_0(t+T)\, dt,$$

then $F(z)$ is an integral function and $F(-i\lambda_\nu) = 0$, so that $F(z) \equiv 0$, a fortiori, $\Re F(z) \equiv 0$, hence by Lemma 4, $q_0(t+T) \sim 0$, or $q_0(t) \sim 0$, which contradicts (5). Hence E has no cluster point in a finite distance, so that $\nu(\lambda)$ is a step-function.

?. Let $u(z_1, \cdots, z_n)$, $H_N^{r_1 \cdots r_n}(x^{(1)}, \cdots, x^{(n)})$ be defined by (3) and (6) of §2 and put for $0 \leqq r_k < 1$,

$$\left. \begin{aligned} &\underset{0 \leqq \theta_k \leqq 2\pi}{\text{Min.}}\ u(r_1 e^{i\theta_1}, \cdots, r_n e^{i\theta_n}) = m(r_1, \cdots, r_n), \\ &\underset{0 \leqq \theta_k \leqq 2\pi}{\text{Max.}}\ u(r_1 e^{i\theta_1}, \cdots, r_n e^{i\theta_n}) = M(r_1, \cdots, r_n), \end{aligned} \right\} \tag{8}$$

and for all $x^{(k)}$, such that $|x_0^{(k)}|^2 + \cdots + |x_N^{(k)}|^2 = 1$ $(k = 1, 2, \cdots, n)$, let

$$\left.\begin{aligned}\text{Min.}\, H_N^{r_1 \cdots r_n}(x^{(1)}, \cdots, x^{(n)}) &= g_N(r_1, \cdots, r_n),\\ \text{Max.}\, H_N^{r_1 \cdots r_n}(x^{(1)}, \cdots, x^{(n)}) &= G_N(r_1, \cdots, r_n).\end{aligned}\right\} \tag{9}$$

Since $g_N(r_1, \cdots, r_n)$ decreases and $G_N(r_1, \cdots, r_n)$ increases, when N increases, let

$$\lim_{N\to\infty} g_N(r_1, \cdots, r_n) = g(r_1, \cdots, r_n),\ \lim_{N\to\infty} G_N(r_1, \cdots, r_n) = G(r_1, \cdots, r_n),\quad (0 \leqq r_k < 1). \tag{10}$$

Then we will prove:

THEOREM 7.[9] $g(r_1, \cdots, r_n) = m(r_1, \cdots, r_n)$, $G(r_1, \cdots, r_n) = M(r_1, \cdots, r_n)$.

PROOF. From (21), (22), (23) of § 2, we have

$$\int_0^{2\pi} g_k(\theta)\, d\theta = 2\delta + O(\delta') + O(\eta),$$

$$2\pi(|x_0^{(k)}|^2 + \cdots + |x_N^{(k)}|^2) = \int_0^{2\pi} \tau_k(\theta)\, d\theta = \int_0^{2\pi} g_k(\theta)\, d\theta + O(\varepsilon)$$
$$= 2\delta + O(\delta') + O(\eta) + O(\varepsilon),$$

so that

$$|x_0^{(k)}|^2 + \cdots + |x_N^{(k)}|^2 = \delta/\pi + O(\delta') + O(\eta) + O(\varepsilon). \tag{11}$$

From (24) of § 2,

$$g(r_1, \cdots, r_n) \prod_{k=1}^{n} \left(\left| x_0^{(k)} \right|^2 + \cdots + \left| x_N^{(k)} \right|^2 \right) \leqq$$

$$g_N(r_1, \cdots, r_n) \prod_{k=1}^{n} \left(\left| x_0^{(k)} \right|^2 + \cdots + \left| x_N^{(k)} \right|^2 \right) \leqq H_N^{r_1 \cdots r_n}(x^{(1)}, \cdots, x^{(n)})$$

$$= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int u(r_1 e^{-i\theta_1}, \cdots, r_n e^{-i\theta_n}) \prod_{k=1}^{n} \tau_k(\theta_k)\, d\theta_1 \cdots d\theta_n.$$

If we make $\varepsilon \to 0$, $\delta' \to 0$, $\eta \to 0$, we have from (11),

$$g(r_1, \cdots, r_n)(\delta/\pi)^n \leqq \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int u(r_1 e^{-i\theta_1}, \cdots, r_n e^{-i\theta_n}) \prod_{k=1}^{n} g_k(\theta_k)\, d\theta_1 \cdots d\theta_n$$

$$= \frac{1}{(2\pi)^n} \int \cdots \int_{|\theta_k - \theta'_k| \leqq \delta} u(r_1 e^{-i\theta_1}, \cdots, r_n e^{-i\theta_n})\, d\theta_1 \cdots d\theta_n = (2\delta/2\pi)^n\, u(r_1 e^{-i\theta'_1}, \cdots, r_n e^{-i\theta'_n}),$$

9) The case $n=1$ was proved by O. Szász: l.c. (2).

where $|\theta'_k - \theta^0_k| \leqq \delta$.

Hence $g(r_1, \cdots, r_n) \leqq u(r_1 e^{-i\theta'_1}, \cdots, r_n e^{-i\theta'_n})$, so that for $\delta \to 0$, $g(r_1, \cdots, r_n) \leqq u(r_1 e^{-i\theta^0_1}, \cdots, r_n e^{-i\theta^0_n})$. Since $(\theta^0_1, \cdots, \theta^0_n)$ is arbitrary,

$$g(r_1, \cdots, r_n) \leqq m(r_1, \cdots, r_n). \tag{12}$$

On the other hand, for any $x^{(k)}$, such that $|x^{(k)}_0|^2 + \cdots + |x^{(k)}_N|^2 = 1$ $(k = 1, 2, \cdots, n)$, we have from (7) of § 2,

$$\begin{aligned}
&H_N^{r_1 \cdots r_n}(x^{(1)}, \cdots, x^{(n)}) \\
&= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int u(r_1 e^{-i\theta_1}, \cdots, r_n e^{-i\theta_n}) \prod_{k=1}^{n} \left| x^{(k)}_0 + x^{(k)}_1 e^{i\theta_k} + \cdots + x^{(k)}_N e^{iN\theta_k} \right|^2 d\theta_1 \cdots d\theta_n \\
&\geqq m(r_1, \cdots, r_n)\, (2\pi)^{-n} \int_0^{2\pi} \cdots \int \prod_{k=1}^{n} | x^{(k)}_0 + \cdots + x^{(k)}_N e^{iN\theta_k} |^2 \, d\theta_1 \cdots d\theta_n \\
&= m(r_1, \cdots, r_n) \prod_{k=1}^{n} \left(|x^{(k)}_0|^2 + \cdots + |x^{(k)}_N|^2 \right) = m(r_1, \cdots, r_n),
\end{aligned}$$

so that $g_N(r_1, \cdots, r_n) \geqq m(r_1, \cdots, r_n)$, hence for $N \to \infty$,

$$g(r_1, \cdots, r_n) \geqq m(r_1, \cdots, r_n). \tag{13}$$

From (12), (13), we have $g(r_1, \cdots, r_n) = m(r_1, \cdots, r_n)$. Similarly wa can prove $G(r_1, \cdots, r_n) = M(r_1, \cdots, r_n)$.

3. Let $u(z_1, \cdots, z_n)$, $H^{\sigma_1 \cdots \sigma_n}(q_1, \cdots, q_n)$ be defined by (3) and (6) of § 3 and we put for $\sigma_k > 0$,

$$\left.\begin{aligned}
&\operatorname*{lower\ limit}_{-\infty < \lambda_k < \infty} u(\sigma_1 + i\lambda_1, \cdots, \sigma_n + i\lambda_n) = m(\sigma_1, \cdots, \sigma_n), \\
&\operatorname*{upper\ limit}_{-\infty < \lambda_k < \infty} u(\sigma_1 + i\lambda_1, \cdots, \sigma_n + i\lambda_n) = M(\sigma_1, \cdots, \sigma_n),
\end{aligned}\right\} \tag{14}$$

and for all $q_k(t)$ of class (L^0), such that $\int_{-\infty}^{\infty} |q_k(t)|^2 dt = 1$, let

$$\left.\begin{aligned}
&\text{lower limit } H^{\sigma_1 \cdots \sigma_n}(q_1, \cdots, q_n) = g(\sigma_1, \cdots, \sigma_n), \\
&\text{upper limit } H^{\sigma_1 \cdots \sigma_n}(q_1, \cdots, q_n) = G(\sigma_1, \cdots, \sigma^n)
\end{aligned}\right\}. \tag{15}$$

Then

THEOREM 8. $g(\sigma_1, \cdots, \sigma_n) = m(\sigma_1, \cdots, \sigma_n)$, $G(\sigma_1, \cdots, \sigma_n) = M(\sigma_1, \cdots, \sigma_n)$.

PROOF. Let $q_k(t)$ be defined by (25) of § 3, where we take $\delta_k = \delta$ $(k = 1, 2, \cdots, n)$, then

$$\int_{-\infty}^{\infty} |q_k(t)|^2 dt = 2\delta.$$

Hence from (42) of § 3,

$$(2\delta)^n g(\sigma_1, \cdots, \sigma_n) \leqq H^{\sigma_1 \cdots \sigma_n}(q_1, \cdots, q_n) \leqq (2\delta)^n u(\sigma_1 + i\lambda'_1, \cdots, \sigma_n + i\lambda'_n),$$

or $g(\sigma_1, \cdots, \sigma_n) \leqq u(\sigma_1 + i\lambda'_1, \cdots, \sigma_n + i\lambda'_n)$, so that for $\delta \to 0$, $g(\sigma_1, \cdots, \sigma_n) \leqq u(\sigma_1 + i\lambda_1^0, \cdots, \sigma_n + i\lambda_n^0)$. Since $(\lambda_1^0, \cdots, \lambda_n^0)$ is arbitrary,

$$g(\sigma_1, \cdots, \sigma_n) \leqq m(\sigma_1, \cdots, \sigma_n). \tag{16}$$

On the other hand, let $q_k(t)$ be any function of class (L^0), such that $\int_{-\infty}^{\infty} |q_k(t)|^2 dt = 1$ and let $q_k(t) = 0$ for $|t| \geqq T'$. We define $v_k(\lambda)$ by

$$v_k(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\lambda} q_k(t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-T'}^{T'} e^{it\lambda} q_k(t)\, dt, \tag{17}$$

then

$$\int_{-\infty}^{\infty} |v_k(\lambda)|^2 d\lambda = \int_{-\infty}^{\infty} |q_k(t)|^2 dt = 1. \tag{18}$$

Writing σ_k instead of σ_k^0 in (38) of § 3, we have

$$\begin{aligned} &\int_{-T'}^{T'}\!\!\cdots\!\int J(T, t_1 - s_1, \cdots, t_n - s_n) \left(\prod_{k=1}^{n} q_k(t_k) \overline{q_k(s_k)}\right) dt_1, \cdots, ds_n \\ &= \frac{1}{T^n} \int_0^T\!\!\cdots\!\int K(a_1, \cdots, a_n)\, da_1 \cdots da_n, \end{aligned} \tag{19}$$

where

$$\begin{aligned} K(a_1, \cdots, a_n) &= \frac{1}{(2\pi)^n} \int_{-a_k}^{a_k}\!\!\cdots\!\int u(\sigma_1 + i\lambda_1, \cdots, \sigma_n + i\lambda_n) \\ &\qquad \left(\prod_{k=1}^{n} \left| \int_{-T}^{T} \exp(it_k \lambda_k) q_k(t_k)\, dt_k \right|^2\right) d\lambda_1 \cdots d\lambda_n \\ &= \int_{-a_k}^{a_k}\!\!\cdots\!\int u(\sigma_1 + i\lambda_1, \cdots, \sigma_n + i\lambda_n) \left(\prod_{k=1}^{n} |v_k(\lambda_k)|^2\right) d\lambda_1 \cdots d\lambda_n. \end{aligned}$$

By (4) of § 3,

$$\left| K(a_1, \cdots, a_n) \right| \leqq \frac{2^n}{\sigma_1 \cdots \sigma_n} \prod_{k=1}^{n} \left(\int_{-\infty}^{\infty} | v_k(\lambda_k)|^2 d\lambda_k \right) = \frac{2^n M}{\sigma_1 \cdots \sigma_n},$$

so that $K(a_1, \cdots, a_n)$ is uniformly bounded, hence

$$\lim_{T\to\infty} \frac{1}{T^n} \int_0^T \cdots \int K(a_1, \cdots, a_n)\, da_1 \cdots da_n = \lim_{a_1 \ \ a_n \to \infty} K(a_1, \cdots, a_n)$$

$$= \int_{-\infty}^{\infty} \cdots \int u(\sigma_1 + i\lambda_1, \cdots, \sigma_n + i\lambda_n) \left(\prod_{k=1}^{n} | v_k(\lambda_k)|^2 \right) d\lambda_1 \cdots d\lambda_n$$

$$\geqq m(\sigma_1, \cdots, \sigma_n) \left(\prod_{k=1}^{n} \int_{-\infty}^{\infty} | v_k(\lambda_k)|^2 d\lambda_k \right) = m(\sigma_1, \cdots, \sigma_n). \tag{20}$$

Hence from (35) of § 3 and (19),

$$H^{\sigma_1 \cdots \sigma_n}(q_1, \cdots, q_n)$$

$$= \int_{-T'}^{T'} \cdots \int f(t_1 - s_1, \cdots, t_n - s_n) \exp\left(-\sum_k |t_k - s_k|\, \sigma_k\right) \left(\prod_{k=1}^{n} q_k(t_k) \overline{q_k(s_k)} \right) dt_1 \cdots ds_n$$

$$= \lim_{T\to\infty} \int_{-T'}^{T'} \cdots \int J(T, t - s_1, \cdots, t_n - s_n) \prod_{k=1}^{n} \left(q_k(t_k) \overline{q_k(s_k)} \right) dt_1 \cdots ds_n$$

$$= \lim_{T\to\infty} \frac{1}{T^n} \int_0^T \cdots \int K(a_1, \cdots, a_n)\, da_1 \cdots da_n \geqq m(\sigma_1, \cdots, \sigma_n),$$

so that

$$g(\sigma_1, \cdots, \sigma_n) \geqq m(\sigma_1, \cdots, \sigma_n). \tag{21}$$

From (16), (21), we have $g(\sigma_1, \cdots, \sigma_n) = m(\sigma_1, \cdots, \sigma_n)$. Similarly we can prove $G(\sigma_1, \cdots, \sigma_n) = M(\sigma_1, \cdots, \sigma_n)$.

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