NOTES ON BANACH SPACE (XI): BANACH LATTICES WITH POSITIVE BASES*)

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After G. Birkhoff [2], in the Banach lattices the lattice operations are uniformly cotinuous with respect to the strong (norm) topology. This theorem does not hold for the weak topology. For, let $x_n(t)$ $(n=1,2,...)$ be the sequence of Rademacher functions, considering in the Banach lattice L^p $(p \geqq 1)$, its positive part converges weakly to 1/2, however $x_n(t)$ converges weakly to zero. The first part of this paper is devoted to investigate Banach lattices in which the lattice operations are weakly continuous. This is given by the character of the intervals. In the second part, we give characteristic properties of the k-space, whose definition is given in Definition 3.

Throughout this paper, we shall use the technical terms and notations in Birkhoff's book [2] without any explanation.

1. We will prove firstly the following

Theorem 1. *The lattice operations of the conjugate space of a separable Banach lattice are continuous with respect to the weak topology as functionals if and only if the interval of the Banach lattice is strongly compact.*

Proof: Suppose that the lattice operations are weakly continuous in the conjugate space E^* of a separable Banach lattice E. Then

$$f_n^+(x) = \sup \{f(y)\,;\, 0 \leqq y \leqq x\}$$

converges to zero whenever f_n converges weakly to zero. Hence it holds $|f_n(y)| \leqq f_n^+(y) - f_n^-(y) < \varepsilon$ for any positive number ε, sufficiently large n and y with $0 \leqq y \leqq x$. That is, $\{f_n(y)\}$ converges uniformly on the interval $(0, x)$, and so the interval is strongly compact by a compactness theorem due to I. Gelfand [5] and R. S. Phillips [9].

Conversely, if each interval $(0, x)$ is compact in the strong topology, then $\{f_n(y)\}$ converges uniformly on it whenever f_n converges weakly to zero, whence $\sup_y f(y)$ converges to zero for any y belonging to the interval, and

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so f_n^+ converges weakly to zero. This completes the proof of the theorem.

We shall now introduce the following definitions.

DEFINITION 1. *A complete Banach lattice is said to be a* K'-space *if it satisfies the following:*

CONDITION F: $x_n \downarrow 0$ *implies* $\| x_n \| \to 0$.

DEFINITION 2. *A separable K'-space is said to be a* k'-space *if it has a positive orthogonal base.*

Under these definition we will prove nextly the following

THEOREM 2. *A separable Banach lattice is a k'-space if and only if its each interval is strongly compact.*

We need several lemmas.

LEMMA 1. *If each interval of a Banach lattice is compact, then it is a complete lattice.*

Proof: Let S be a metrically closed Moore-Smith subset of an interval $(0, a)$ and L be a metrically closed simply ordered subset of S. Then S and L are compact by the hypothesis. Let x_n be the supremum of the centers of the $1/n$-net of L. We may assume without loss of generalities that $x_n \leqq x_{n+1}$ holds for any n. Then $\{x_n\}$ converges metrically and so it has a limit x in L. Evidently by the construction x becomes the supremum of L, that is, S is inductively ordered, whence it contains at least one maximal element by the Zorn lemma. On the other hand, since S has the Moore-Smith property, it has at most one maximal element, and so it becomes supremum of S. This proves the lemma.

LEMMA 2. *If each interval of a Banach lattice is compact, then it holds the Condition F.*

PROOF: Suppose $x_n \downarrow 0$. Then by the compactness of interval a suitable subsequence $\{x_n'\}$ converges metrically to some x with $0 \leqq x \leqq x_n'$, whence $x = 0$ by the assumption, and so $\| x_n' \| \to 0$ holds. Thus the proof follows from the monotonity of the norm.

Here we state a known theorem as a lemma, which will be used later.

LEMMA 3 (Ogasawara [7]). *If in a separable Banach lattice (hence it has a principal unit* 1 *by a theorem of H. Freudenthal* [4]) *each interval is compact, then the structure lattice (and so the unit lattice* (cf. 6; Def. 3)) *is an atomic Boolean algebra with enumerable atoms.*

LEMMA 4. *A separable K'-space is a k'-space if and only the structure lattice* (*i. e. the lattice of all ideals or the complemented normal subspaces of G. Birkhoff*) *is atomic.*

PROOF: Since the necessity is obvious, it is sufficient to prove the converse. The atomicity of the unit lattice implies the existence of a orthogonal positive maximal independent set $\{a_i\}$ with the norm unity in virtue of Zorn's lamma. Let e_i be the components of 1 in the atomic ideal including a_i then $V_{i=1}^{n} e_i$ order-converges to 1, whence $1 - V_{i=1}^{n} e_i = V_{i=n+1}^{\infty} e_i$ order-converges to zero, and so by Conditon F it converges metrically. On the other hand $e_i = \alpha_i a_i$ for some α_i, whence $\{a_i\}$ is a base.

LEMMA 5. *A separable Banach lattice, whose each interval is compact, is a k'-space.*

PROOF: By Lemmas 1 and 2 such Banach lattice becomes a K'-space which becomes a k'-space by Lemmas 3 and 4.

LEMMA 6. *If a Banch lattice has an orthogonal positive base $\{a_i\}$, then $x \leqq y$ holds if and only if $\xi_i \leqq \eta_i$ for any i, ξ_i and η_i being the coefficients of x and y with respect to a_i respectively.*

PROOF: It suffices to show that $x \geqq 0$ implies $\xi_i \geqq 0$. Suppose the contrary and $\xi_i < 0$ for some i. Then it holds by the distributivity

$$x = x \vee 0 = \sum_{i=1}^{\infty} \xi_i a_i \vee 0 = \sum_{i=1}^{\infty} \xi_i^{+} a_i > \sum_{i=1}^{\infty} \xi_i a_i = x,$$

where $\xi_i^{+} = \xi_i \vee 0$, which is a contradiction.

PROOF OF THEOREM 2: Necessity of the condition follows from Lemma 5. In order to prove the sufficiency, let us suppose

$$0 \leqq x = \sum_{i=1}^{\infty} \leqq \xi_i a_i \leqq \sum_{i=1}^{\infty} \alpha_i a_i = a.$$

By Lemma 6 we have

$$\left| x - \sum_{i=1}^{n} \xi_i a_i \right| \leqq \left| \sum_{i=n+1}^{\infty} \xi_i a_i \right| \leqq \left| \sum_{i=n+1}^{\infty} \alpha_i a_i \right| \leqq \left| \sum_{i=1}^{\infty} \alpha_i a_i - a \right| < \varepsilon$$

for any positve ε and all n with $n \geqq n(\varepsilon)$, namely, $\Sigma_i \xi_i a_i$ converges uniformly on the interval $(0, a)$. Hence by the compactness theorem of R. S. Phillips [9], the interval is stronly compact. Thus the theorem is proved.

Combing Theorems 1 and 2 we have

THEOREM. 3 *The lattice operations of the conjugate space of a separable*

Banach lattice are weakly continuous if an only if the Banach lattice is a k'-space.

2. We will now introduce the following definitions;

DEFINITION 3. *A complete Banach lattice is said to be a* k-space *if it is a k'-space satisfying the following*

CONDITION L: $0 \leqq x_n \leqq x_{n+1}$ *and* $\| x_n \| \leqq \lambda$ *for all n imply the existence of* $\bigvee_n x_n$.

DEFINITION 4 (Dunford-Morse [3]). *A Banach space is said to have a* Dunford-Morse base *if it has a base a_i such that*

CONDITION D: *if* $\| \sum_{i=1}^n \xi_i a_i \| \leqq \lambda$ *holds for a sequence of real numbers* $\{\xi_i\}$ *then the series* $\sum_i \xi_i a_i$ *converges mterically.*

Then we have

THEOREM 4. *A K'-space is a k-space if and only if it has an orthogonal positive Dunford-Morse base.*

PROOF: Let E be a k-space and partial sums of the series $\sum_i \xi_i a_i$ be uniformly bounded. If the coefficients are non-negative, then the partial sum $x_n = \sum_{i=1}^n \xi_i a_i$ forms an increasing sequence with the unifomly bounded norm, which by Conditions F and L converges metrically to the supremum of x_n, and so E has a Dunford-Morse base. In the general case, if we put

$$x_n^+ = \sum_{i=1}^n \xi_i^+ a_i, \quad x_n^- = \sum_{i=1}^n \xi_i^- a_i$$

where $\xi_i^+ = \xi_i \vee 0$ and $\xi_i^- = \xi_i \wedge 0$, then $\| x_n \| \leqq \lambda$ implies $\| x_n^+ \| \leqq \lambda$ and $\| x_n^- \| \leqq \lambda$ by the montonity of norm, whence $\{x_n^+\}$ and $\{x_n^-\}$ converge metrically to x^+ and x^- respectively, and so $x_n = x_n^+ + x_n^-$ converges metrically to $x^+ + x^-$.

Conversely, let E has a Dunford-Morse base $\{a_i\}$ and $x^n = \sum_i \xi_i^n a_i$ be an increasing seqnence of elements with $\| x^n \| \leqq \lambda$, and moreover x_m^n be the m-th partial sum of the series x^n. Then $\{\xi_i^n\}$ is a bounded decreasing sequence of real numbers for each i, whence $\xi_i = \lim_n \xi_i^n$ exist. Let us now put $x_m = \sum_{i=1}^m \xi_i a_i$. Then $\{x_m^n\}$ order-converges, and so metrically converges to x_m. Hence $\| x_m^n \| \leqq \| x_n \| \leqq \lambda$ implies $\| x_m \| \leqq \lambda$ for any m, and so x_m converges metrically to x by the hypothesis. Since the metric convergence preserves the order, we have $x^n \leqq x$ for all n, and consequently $x = \vee_i x^i$. This complete the proof of the theorem.

As L.Alaoglu [1] proved, a Banch space having a Dunford-Morse base is isomorphic to a conjugate space of a certain Banach space. Hence it is also conjectured in the case of the Banach lattices. That is, we have

THEOREM 5. *A Banch lattice is a k-space if and only if it is isomorphic with a separble conjugate space of a k'-space.*

PROOF OF SUFFICIENCY: If E is isomorphic to F^*, the conjugate space of a k'-space F, then F^* (and also E) is a k-space by a theorem of T. Ogasawara (cf. [6; Thm. 11]). On the other hand, the structure lattices of a K'-space and its conjugate space are isomorphic, by a theorem also due to him (cf. [9; Thm. 6]), whence Lemma 4 implies that F^* (and so E) is a k-space.

To prove the converse here we state without proof a known theorem as a convenient lemma;

LEMMA 7 (Ogasawara [8]). *If f and g are linear functionals on a Banach lattice E satisfying $f \wedge g = 0$, then for a positive element x there exists a pair of elements y and z such that $y \wedge z = 0$, $y \vee z \leqq x$ and*

$$0 \leqq f(x) - f(y) < \varepsilon, \qquad 0 \leqq g(x) - g(y) < \varepsilon.$$

PROOF OF THE NECESSITY OF THE THEOREM: By a generalized Radon-Nikodym Theorem [6; Thm. 9], the second conjugate space E^{**} of a k-space E is decomposed into the direct union of E and E' where E' is the complement of E in E^{**}. Suppose

$$F = \{f\,;\, 0 < \ddot{x} \in E' \rightarrow \ddot{x}(|f|) = 0\}.$$

Then evidently F is a normal subspace of E^*. Since $|f - g| < \varepsilon$ implies $0 \leqq \ddot{x}(|g|) < \varepsilon\, |x|$ for $\ddot{x} \in E'$, F is also metrically closed. Hence F is a Banach lattice. Since each linear functional $\hat{x}$ of F can be extended to a linear functional $\ddot{x}$ on the whole space E^*, $\ddot{x}$ is decomposed into $\ddot{x} = x + \ddot{x}'$ where $\ddot{x}' \in E'$ and $\ddot{x}'$ vanishes on F, whence there corresponds an element x of E for each $\hat{x}$ of F^*.

On the other hand, if f_i is a coefficient functional of the base a_i in E, then f_i belongs to F. For, if we assume the contrary and $\ddot{x}'(f_i) > 0$ for some $0 < \ddot{x} \in E'$, then $\ddot{x}' \wedge a_i = 0$ in E^{**} implies by Lemma 7 the existence of g and h with $g \wedge h = 0$, $g \vee h \leqq f_i$ and

$$0 \leqq a_i(f_i) - a_i(g) < \varepsilon, \qquad 0 \leqq \ddot{x}'(f_i) - \ddot{x}'(h) < \varepsilon$$

for sufficiently small ε. But this is impossible unless $f_i(a_i) = 0$, since, by a theorem due to T. Ogasawara (cf., [6; Thm. 6]), the structure lattice is atomic as a consequence of Lemma 4, and so either g or h equals to zero. Hence $x \neq 0$ implies $f(x) \neq 0$ for some $f \in F$. Therefore the correspondence from F^* to E is one-to-one. Moreover, if $\hat{x}$ corresponds to x, then it holds

$$\begin{aligned} \| \hat{x} \| &= \sup \{ |\hat{x}(f)| / \|f\| ; f \in F \} = \sup \{ |f(x)| / \|f\| ; f \in F \} \\ &\leqq \sup \{ |f(x)| / \|f\| ; f \in E \} = \|x\|, \end{aligned}$$

whence E and F^* are isomorphic as Banach spaces by a well-known theorem of S. Banach. Finally, each coefficient functional belongs to F, whence an element x is positive if and only if the corressonding $\hat{x}$ is positive, and so the correspondence is also a lattice isomorphism.

It remains to prove that F is a k'-space. Since the structure lattice of F is atomic by Lemma 4 and a theorem of T. Ogasawara (cf., [6; Thm. 6]), using Lemma 4 twice it suffices to show that F is a K'-space. But this is an immediate consequence of the following

THEOREM 6. *As a sublattice of the second conjugate space, a Banch lattice is normal subspace if and only if it is a K'-space.*

PROOF: Since the sufficiency is already proved in [6; Lemma 4], it remains to show the converse implication. Since a conjugate space of a Banch lattice is complete by a theorem of G. Brikhoff [2. p. 111] and the normality of E. Hence it suffices to prove condition F. Let $x_n \downarrow 0$, then $x_n(f)$ oonverges monotonically to zero for each $f \geqq 0$, whence the weak compactness of the intersection of the unit sphere and the positive cone of the conjugate space implies the uniform convergence of $\{x_n(f)\}$ as continuous functions on it by the Dini Theorem, and so it converges strongly to zero. This proves the theorem.

In the connection of Theorem 6, we will make a remark on the K-spaces in the following form:

THEOREM 7. A *Banach lattice is a K-space if and only if it is an ideal in the second conjugate space.*

PROOF: The necessity is already proved in Theorem 9 of [6], we will prove the converse. To prove this it is sufficient to show that E satisfies the condition L. Let $0 \leqq x_n \leqq x_{n+1}$ and $\|x_n\| \leqq \lambda$, then $\{x_n\}$ converges to $\ddot{x}$ in E^{**} which is the supremum of x_n. Hence by a remark of G. Birkhoff [3; p. 121] and the hypothesis, $\ddot{x}$ belongs to E, which is required.

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References

1. L. Alaoglu, *Weak topologies of normed linear spaces*, Ann. of Math., 41 (1940), 252-267.
2. G. Birkhoff, Lattice Theory, N. Y., 1940.
3. N. Dunford and A. P. Morse, *Remarks on the preceding paper of James A. Clarkson*, Trans. A. M.S., 40 (1936), 415-420.
4. H. Freudenthal, *Teilweise geordnete Moduln*, Proc. Ned. Acad. Amsterdam, 39(1936), 641-651.
5. I. Gelfand, *Abstrakte Funktionen und lineare Operationen*, Mat. Sbornik, 4 (1938), 235-284.
6. M. Nakamura, *Notes on Banach space* (IX), this Journal, 1 (1949), 100-108.
7. T. Ogasawara, *Compact metric Boolean algebras and vector lattices*, Hirosima J., 11 (1942), 125-128.
8. T. Ogasawara, Sokuron II (Lattice Theory in Jap.), Tôkyô, 1947.
9. R.S.Phillips, *Linear Transformations*, Trans. A. M. S., 48 (1940), 516-541.