NOTES ON FOURIER ANALYSIS (XXVII) A THEOREM ON CESÀRO SUMMATION

SHIN-ICHI IZUMI

(Received February 24, 1951)

Let $\phi(t)$ denote an even periodic function with Fourier series

$$(1) \qquad \phi(t) \sim \sum_{n=0}^{\infty} a_n \cos nt, \qquad a_0 = 0.$$

The α-th integral of $\phi(t)$ is defined by

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \phi(u)(t-u)^{\alpha-1}\, du, \qquad \alpha > 0,$$

and the β-th Cesàro sum of (1) is given by

$$s_n^\beta = \sum_{\nu=0}^{n} \binom{\beta}{n-\nu} a_\nu \cos \nu t.$$

L. S. Bosanquet[1] has proved that

$$\Phi_\beta(t) = o(t^\beta) \qquad (t \to 0)$$

implies

$$s_n^\alpha = o(n^\alpha) \qquad (n \to \infty)$$

for $\alpha > \beta$, and conversely

$$s_n^\beta = o(n^\beta) \qquad (n \to \infty)$$

implies

$$\Phi_\alpha(t) = o(t^\alpha) \qquad (t \to 0)$$

for $\alpha > \beta + 1$.

Recently F. T. Wang[2] has proved that

$$\Phi_\beta(t) = o(t^\gamma) \qquad (t \to 0)$$

for $\beta > \gamma$ implies

$$s_n^\alpha = o(n^{\alpha+\beta-\gamma}) \qquad (n \to \infty)$$

for $\alpha > \beta + (\beta - \gamma)$. Analogously to the converse part of the Bosanquet theorem, we prove the following theorem[3]

THEOREM. $$s_n^\beta = o(n^\gamma) \qquad (n \to \infty)$$

for $\beta - \gamma > -1$, $\beta > -1$, $\gamma > -1$, *implies*

$$\Phi_\alpha(t) = o(t^{\alpha+\beta-\gamma}) \qquad (t \to 0)$$

for $\alpha > 1 + \gamma$.

This theorem is proved by J. M. Hyslop[4] for $0 \leqq \beta - \gamma < 1$ and $0 < \gamma < \beta$. In the case $\beta - \gamma \leqq -1$, Theorem becomes trivial.

---

1) L. S. Bosanquet, Proc. London Math. Soc. (2), 31(1930).

2) F. T. Wang, Annals of Math. (2), 44(1943). Cf. J. M. Hyslop, Journ. London Math. Soc. 24(1949).

3) The case $\beta < \gamma$ was suggested by G. Sunouchi.

4) J. M. Hyslop, loc. cit.

PROOF. We will first consider the case $0<\gamma<1$. Then we can suppose $1<\alpha<2$.

$$\Gamma(\alpha)\Phi_\alpha(t)=\int_0^t \phi(u)(t-u)^{\alpha-1}du \tag{2}$$
$$=\sum_{n=0}^{\infty} a_n \int_0^t \cos nu(t-u)^{\alpha-1}du.$$

By the definition

$$a_n=\sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{\beta+1}{n-\nu}s_\nu^\beta,$$

and then

$$\Gamma(\alpha)\,\Phi_\alpha(t)=\sum_{n=0}^{\infty}\int_0^t \cos nu(t-u)^{\alpha-1}du\sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{\beta+1}{n-\nu}s_\nu^\beta \tag{3}$$
$$=\sum_{\nu=0}^{\infty}s_\nu^\beta\sum_{n=\nu}^{\infty}(-1)^{n-\nu}\binom{\beta+1}{n-\nu}\int_0^t \cos nu(t-u)^{\alpha-1}du,$$

where the interchange of the order of summation is legitimate. For, since

$$\int_0^t \cos nu(t-u)^{\alpha-1}\,du=O(1/n^\alpha),$$

$$\sum_{\nu=0}^{M}s_\nu^\beta\sum_{n=M+1}^{\infty}(-1)^{n-\nu}\binom{\beta+1}{n-\nu}\int_0^t \cos nu\ (t-u)^{\alpha-1}du$$
$$=o\left(\sum_{\nu=0}^{M}\nu^\gamma\sum_{n=M+1}^{\infty}\frac{1}{(n-\nu)^{\beta+2}n^\alpha}\right)$$
$$=o\left(\frac{1}{M^\alpha}\sum_{\nu=0}^{M}\frac{\nu^\gamma}{(M+1-\nu)^{\beta+1}}\right)=o\left(\frac{1}{M^{\alpha-\gamma}}\right)=o(1).$$

Thus (3) is proved. Since

$$\sum_{n=\nu}^{\infty}(-1)^{n-\nu}\binom{\beta+1}{n-\nu}\cos nu=\Re\left\{\sum_{m=0}^{\infty}(-1)^m\binom{\beta+1}{m}e^{imu}\cdot e^{i\nu u}\right\}$$
$$=2^{\beta+1}\left(\sin\frac{u}{2}\right)^{\beta+1}\cos\left(\left(\nu+\frac{\beta+1}{2}\right)u-\frac{\beta+1}{2}\pi\right);$$

$$\int_0^t\left\{\sum_{n=\nu}^{\infty}(-1)^{n-\nu}\binom{\beta+1}{n-\nu}\cos nu\right\}(t-u)^{\alpha-1}\,du \tag{4}$$
$$=2^{\beta+1}\int_0^t\left(\sin\frac{u}{2}\right)^{\beta+1}\cos\left(\left(\nu+\frac{\beta+1}{2}\right)u-\frac{\beta+1}{2}\pi\right)(t-u)^{\alpha-1}\,du$$
$$=O(t^{\beta+1}/\nu^\alpha)$$

for $\nu t \geqq 1$,[5] we have

$$\sum_{\nu=M}^{\infty} s_\nu^\beta \sum_{n=\nu}^{\infty} (-1)^{n-\nu} \binom{\beta+1}{n-\nu} \int_0^t \cos nu(t-u)^{\alpha-1}\, du$$

(5) $$= o\left(\sum_{\nu=M}^{\infty} \nu^\gamma \frac{t^{\beta-1}}{\nu^\alpha}\right) = o(1/M^{\alpha-\gamma-1}) .$$

Let $N \equiv [1/t]$ and

$$\Gamma(\alpha)\, \Phi_\alpha(t) = \sum_{\nu=0}^{\infty} = \sum_{\nu=0}^{N} + \sum_{\nu=N+1}^{\infty} \equiv I + J.$$

By the second mean value theorem

$$\int_0^t \left(\sin \frac{u}{2}\right)^{\beta+1} \cos\left(\left(\nu + \frac{\beta+1}{2}\right)u + \frac{\beta+1}{2}\pi\right)(t-u)^{\alpha-1} du = O\left(\frac{t^{\alpha+\beta}}{\nu}\right),$$

and then

$$I = o\left(\sum_{\nu=0}^{N} \nu^{\gamma-1} \cdot t^{\alpha+\beta}\right) = o\left(t^{\alpha+\beta} N^\gamma\right) = o(t^{\alpha+\beta-\gamma}) .$$

By (5)

$$J = o\,(t^{\beta+1}/N^{\alpha-\gamma-1}) = o(t^{\alpha+\beta-\gamma}).$$

Thus the theorem is proved for $0 < \gamma < 1$.

For $-1 < \gamma < 0$ above estimations hold[6] and theorem is also true.

Next consider the case $1 \leqq \gamma < 2$, whence we can suppose $2 < \alpha < 3$. Using Abel's lemma in (2), we have

(6) $$\Gamma(\alpha)\Phi_\alpha(t) = \sum_{n=0}^{\infty} s_n \int_0 \Delta \cos nu(t-u)^{\alpha-1}\, du,$$

---

5) This is proved by the Lebesgue's device. That is, putting $\lambda \equiv (\beta+1)/2$, we have

$$\int_0^t \left(\sin \frac{u}{2}\right)^{\beta+1} \cos((\nu+\lambda)u - \lambda\pi)(t-u)^{\alpha-1}\, du$$

$$= \frac{1}{2}\left\{\int_0^t \left(\sin\frac{u}{2}\right)^{\beta+1} \cos\,((\nu+\lambda)u-\lambda\pi)(t-u)^{\alpha-1}\, du\right.$$

$$\left. - \int_{-\pi/(\nu+\lambda)}^{t-\pi/(\nu+\lambda)} \sin \frac{1}{2}\left(u + \frac{\pi}{\nu+\lambda}\right)^{\beta+1} \cos\,((\nu+\lambda)u - \lambda\pi)\left(t-u-\frac{\pi}{\nu+\lambda}\right)^{\alpha-1} du\right\}$$

$$= \frac{1}{2}\left\{\int_0^t + \int_{-\pi/(\nu+\lambda)}^{0} + \int_{t-\pi/(\nu+\lambda)}^{t}\right\}$$

$$= O\left(\frac{t^{\beta+1}}{\nu^\alpha} + \frac{t^{\alpha+\beta-1}}{\nu^2} + \frac{t^{\alpha-1}}{\nu^{\beta+2}}\right) = O\left(\frac{t^{\beta+1}}{\nu^\alpha}\right),$$

for $\nu t \geqq 1$.

6) The difference is to take Cauchy limit in (2). That is,

$$\Gamma(\alpha)\Phi_\alpha(t) = \lim_{\epsilon\to 0} \int_0^{t-\epsilon} \phi(u)(t-u)\, du = \lim_{\epsilon \to 0} \sum_{n=0}^{\infty} a_n \int_0^{t-\epsilon} \cos nu(t-u)^{\alpha-1}\, du.$$

Cf. L. S. Bosanquet, Loc. cit. and Proc. London Math. Soc. 33(1932).

where s_n denotes the n-th partial sum of (1) and $\Delta \cos nu \equiv \cos nu - \cos(n+1)u$. For,

$$\int_0^t \cos nu\,(t-u)^{\alpha-1}\,du = O(1/n^2)$$

for $\alpha \geqq 2$ and $s_n = O(n^{\gamma/(\beta+1)})$ by $a_n = o(1)$ and $s_n^\beta = o(n^\gamma)$.
Substituting

$$s_n = \sum_{\nu=0}^{n} (-1)^{n-\nu}\binom{\beta}{n-\nu} s_\nu^\beta$$

into (6), we get

$$\begin{aligned}\Gamma(\alpha)\,\Phi_\alpha(t) &= \sum_{n=0}^{\infty}\int_0^t \Delta \cos nu\,(t-u)^{\alpha-1}\,du \sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{\beta}{n-\nu} s_\nu^\beta \\ &= \sum_{\nu=0}^{\infty} s_\nu^\beta \int_0^t \left\{\sum_{n=\nu}^{\infty}(-1)^{n-\nu}\binom{\beta}{n-\nu}\Delta \cos nu\right\}(t-u)^{\alpha-1}\,du.\end{aligned} \tag{7}$$

The interchange of the order of summation is proved as in the former case, using that

$$\sum_{n=\nu}^{\infty}(-1)^{n-\nu}\binom{\beta}{n-\nu}\Delta \cos nu = 2^{\beta+1}\left(\sin\frac{u}{2}\right)^{\beta+1}\cos\left(\left(\nu+\frac{\beta+1}{2}\right)u + \frac{\beta+1}{2}\pi\right).$$

(4) holds in this case[7] and

$$\int_0^t \Delta \cos nu (t-u)^{\alpha-1}\,du = O(t/\nu^\alpha).$$

Dividing (7) into I and J as in the former case, we can get

$$\Gamma(\alpha)\Phi_\alpha(t) = o(t^{\alpha+\beta-\gamma}).$$

Similarly proceeding we can complete the proof of the theorem.

Finally I have to express my hearty thanks to Dr. L. S. Bosanquet, who gave me some valuable remarks and advice. Originally the theorem was proved for $\beta > 0$ and $\gamma > 0$. He let me know that the theorem holds for $\beta > -1$ and $\gamma > -1$ and it is proved by his method in the paper cited in [1]. I have to mention that our method is like his method and order estimations are, explicitely or implicitely, contained in his paper.

MATHEMATICAL DEPARTMENT, TOKYO TORITSU UNIVERSITY, TOKYO.

---

7) (4) is proved by the twice use of the Lebesgue's device in this case. Generally, use it n times in the case $n<\alpha<n+1$.