NORMAL STATES OF COMMUTATIVE OPERATOR ALGEBRAS

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This note is a continuation of the studies [9, 15, 16] of the authors or the second named author on a class of states of a commutative operator algebra on a Hilbert space. Some theorems of these previous notes will be given new proofs or another applications.

In § 1, we shall state definitions and the notation. Most of them are due to I. E. Segal [11, 12]. The representation theorem of a commutative operator algebra due to Gelfand-Neumark will be explained, which plays an essential rôle in the below. A systematic use of the canonical spectral measure due to N. Dunford [3] may short-cut its development. Most of theorems of this sections are already known, whence we shall omit their proofs.

In § 2, as a supplement of R. Pallu de la Barrière's theorem (quoted in J. Dixmier [2]), the canonical representation of normal states by the wave functions is generalized, which is proved by the second named author [15] under the separability restriction. It may be worth to note, that the theorem is equivalent to the Radon-Nikodym Theorem of normal states (cf. [9]).

In § 3, we shall concern with the problem of unitary equivalence for operators or algebras under the existence of cyclic vectors. Our treatment will show that the Hellinger-Hahn Theorem for an operator with simple spectrum is a consequence of the deflation of the decomposition in the sense of I. E. Segal [12].

In § 4, the Wecken-Plessner-Rokhlin Theorem in the formulation of Segal is proved under Pallu de la Barrière's theorem. Our technics based on the fact that the weakly closed operator algebra generated by an operator is the weak* closure of the uniformly closed algebra in the conjugate space of the weighted spectrum of the operator. Although we shall discuss the unitary invariants, our treatment concerns mostly to the algebraic isomorphism problem. Unitary invariants appear as by-products. Therefore, we shall not enter the multiplicity theory, which is recently developed by Halmos, Kelley, and Segal.

In § 5, we shall reconstruct the materials of the previous sections from an another point of view. It will be shown that the Dunford integral representation can be generalized into the unitary mapping of the L_2-space to the Hilbert space under the existence of cyclic vector. This is known by F. Maeda [8] in the separable case and again gives a natural proof of the Hellinger-Hahn Theorem for an operator algebra. And the direct decomposition of the space into cyclic subspaces gives the Wecken-Plessner-Rokhlin Theorem.

The paper concludes in § 6 with a remark concerning the normality of states considered in the previous note [9].

1. After Segal [12], a self-adjoint algebra of operators will be called a *C^*-algebra* (*W^*-algebra*) according to that the algebra is closed in the uniform (weak) operator topology. We shall assume in the below that *operator algebras contain the identity operator.*

An extremely important theorem concerning a commutative C^*-algebra is the Gelfand-Neumark representation theorem: The commutative C^*-algebra is isometrically isomorphic to the algebra $C(\Omega)$ of all complex-valued continuous functions on a compact Hausdorff space Ω, which will be called the *spectrum* of the algebra. Recently, N. Dunford [3] refined that the above isomorphism is given by the integral representation:

$$x = \int x(\omega)\, de(\omega), \tag{1.1}$$

where $de(\omega)$ means the (regular) *spectral measure* (i. e., projection-valued set function defined for all Borel sets of Ω,[1] the integration is the abstract Stieltjes integral or in the sense that

$$(\xi x, \eta) = \int x(\omega)\, d(\xi e(\omega), \eta). \tag{1.2}$$

Conversely, a spectral measure defines a C^*-algebra. That is, a representation of a commutative C^*-algebra corresponds to a spectral measure of the spectrum in one-to-one fashion.

A linear functional σ on a C^*-algebra A is called *positive if*

$$\sigma(xx^*) \geqq 0 \tag{1.3}$$

for all x of A. A positive linear functional will be called a *state*, if it is normalized, i. e, $\sigma(1) = 1$. By the well-known theorem of Riesz-Markhoff-Kakutani (cf. [6]), a state gives a normalized measure $d\sigma$ on the spectrum Ω in the sense

$$\sigma(x) = \int x(\omega)\, d\sigma(\omega). \tag{1.4}$$

A typical example of a state is the *wave function*:

$$\sigma(x) = (\varphi x, \varphi), \qquad \|\varphi\| = 1, \tag{1.5}$$

A measure defined by the wave function will be called, according to J. L. Kelley [7], a *characteristic meaure*[2]. In a connection with the Dunford integral represention, it is not hard to see

$$\sigma(x) = \int x(\omega)\, d\sigma(e(\omega)), \tag{1.4'}$$

i.e., $d\sigma(\omega) = d\sigma(e(\omega))$ if it has a meaning, and in the case of a wave function

1) The notion of spectral measures in firstly (probably) introduced by F. Maeda [8]. He applied it to generalize the Hellinger-Hahn Theorem. Cf. Prop.6 of § 3.

2) The second named author called it as spectre measure in the previous [16].

$$(1.5') \qquad \sigma(x) = \int x(\omega)\, d\|\varphi e(\omega)\|^2.$$

If A is a commutative W*-algebra, the spectrum Ω of A possesses some special properties. Let A^r be the real algebra of all self-adjoint members of A. Then A is ordered with respect to the usual ordering of non-negative definitness, i. e., $x \geqq y$ if and only if $x-y$ is non-negative definite. Concerning this ordering, it is not hard to see that A^r forms a vector lattice[3] in the sense of G. Birkhoff [1]. Moreover, we have[4]

PROPOSITION 1. *A metrically bounded Moore-Smith set of self-adjoint elements of a commutative W*-algebra converges strongly to its bound, i.e., the order convergence implies the strong convergence.*

The proposition can be proved with a few modification of a proof due to F. Riesz and B. von Sz. Nagy [14; p. 15] in the case of the monotone sequential convergence, whence we shall omit its detail.

From Proposition 1 and a theorem of T. Ogasawara [10; p. 20], the spectrum of a commutative W*-algebra has the following propeties : (a) the closure of an open set is open, (b) open-closed sets form a complete lattice and a basis for open sets, (c) a Borel set is congruent to an open-closed set modulo maigre set[5], (d) a Borel measurable function coincides with a continuous function except points of a maigre set, etc. After J. Dixmier [2], such space will be called a *Stone space.* That is, the spectrum of a commutative W*-algebra is a Stone space.

A state σ of a commutative W*-algebra A is called normal[6] in the sense of J. Dixmier [2], if σ satisfies

$$(1.6) \qquad x_a \downarrow 0 \text{ implies } \sigma(x_a) \rightarrow 0.$$

It is clear by Proposition 1, the wave function is normal. A linear functional ρ will be called normal if it is a complex linear combination of normal states. A measure on the spectrum will be called normal provided that it is defined by a normal linear functional. Such measure annihilates maigre sets, and

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3) This is true for all C*-algedras. Actually, it is an abstract (M) space in the sense of S.Kakutani [6].

4) This proposition is also obtained by Fell-Kelley [4]. Compare with G.Birkhoff [1;p.118].

5) A maigre set is a set of the first category in the usual definition. The term due to Bourbaki.

6) This notion originally due to G.Birkhoff [1;p.73]. He used it for a decomposition of complemented modular lattices. The notion is generalized to vector lattices by T.Ogasawara [10]. Ogasawara, Nakano and the first named author examined this notion in some details. Application of the notion to states of W*-algebras is introduced by J.Dixmier [2] and the authors [9]. The present term due to the former. The authors called "order-continuous" in the previous note.

conversely a measure annihilating maigre sets is normal[7] if it is finite.

An important example of commutative W*-algebras is maximal abelian self-adjoint (shortly masa, according to Segal [12]) algebra. A fundamental theorem for masa algebra on which we shall base read as follows[8]:

THEOREM 1 (SEGAL). *A masa algebra is unitarily equivalent to the multiplication algebra of L_2-space of a measure space. Consequently, two masa algebras are unitarily equivalent if and only if they are algebraically isomorphic.*

In the theorem, the *multiplication algebra* of a measure space (X, ν) means $L_\infty(X, \nu)$ considering as the operators on $L_2(X, \nu)$ such that

(1.7) $\xi x(\omega) = \xi(\omega)x(\omega)$ where $x \in L_\infty(X, \nu)$, $\xi \in L_2(X, \nu)$.

2. The following theorem has the deteremining importance in the theory of normal states: [9]

THEORAM 2 (PALLU DE LA BARRIÈRE) *Each normal linear functional σ of a commutative W^*-algebra allows the canonical representation*:

(2.1) $$\sigma(x) = (\xi x, \eta).$$

Then we get the following[10]

PROPOSITION 2. *A commutative W^*-algebra is the conjugate space of all measures of the form* (2.1).

PROOF. By the above theorem, all measures of the form (2.1) constitue a linear space and an element of the algebra A gives a linear functional on such measures. Conversely, if $\phi(\sigma)$ is a linear functional on that measures, then

$$|< \xi, \eta >| = |\phi(\sigma)| \leqq \|\phi\|\cdot\|\sigma\| \leqq \|\phi\|\cdot\|\xi\|\cdot\|\eta\|$$

implies the bilinearity of $< \xi, \eta >$, where $< \xi, \eta > = \phi(\sigma)$ if $\sigma(x) = (\xi x, \eta)$. Hence by Riesz' Lemma there is a linear operator a such as $<\xi, \eta> = (\xi a, \eta)$. Let b be an element of the commutator A' of A, then the equation

$$(\xi ab, \eta) = (\xi a, \eta b^*) = \phi(\sigma') = \phi(\sigma'') = (\xi b a, \eta)$$

implies $a \in A$, where $\sigma'(x) = (\xi x, \eta b^*)$ and $\sigma''(x) = (\xi bx, \eta)$, since $\sigma' = \sigma''$.

Proposition 2 implies at once

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7) It will be proved similar to Lemma 1 of the previous note, by the help of Ogasawara's theorem.

8) Theorem 1 is due to Segal [11,12]. A similar result is also obtained by J.Dixmier [2].

9) The theorem is stated without proof in Dixmier [2]. Two proofs of the theorem are contained in the second named author's [15].

10) Proposition 2 is obtained by the second named author independently to Dixmier [2] and Kelley [7]. It is published (in Japanese) Zitukansû Geppo 6, No.2(1952).

PROPOSTION 3. *In a commutative W^*-algetra, the weak operator topology coincides with the weak* topology as the conjugate Banach space. Or, the weak topology for a commutative W^*-algebra is purely algebraical.*

Our chief object of the present section is to prove the following theorem, which may supply the preceding theorem[11]

THEOREM 3. *Concerning a state of a commutative W^*-algebra, the following three statements are mutually equivalent;* (i) *it is strongly continuous,* (ii) *it is normal,* (iii) *it is a wave function.*

Since obviously (i) implies (ii) and (iii) implies (i) by Proposition 1 and the definition of the strong convergence, we shall prove only that (ii) implies (iii). To prove this, we may assume that the implication is true for masa algebras since it is deducible directly from Theorem 1.

Let σ be a normal state of a commutative W^*-algebra A and let B be a masa extension of A. By Theorem 2, σ is extensible to a normal linear functional on B with $\sigma(x) = (\varsigma x, \varsigma')$ for all $x \in B$, whence

$$\sigma(x) = (\varsigma x, \varsigma') = (\xi x, \xi) - (\eta x, \eta).$$

If $0 \leqq x \in A$, then $(\xi x, \xi) \geqq (\eta x, \eta) \geqq 0$, whence by the Radon-Nikodym Theorem of [9], there exists an a in A with $0 \leqq a \leqq 1$ and $(\xi x a, \xi) = (\eta x, \eta)$. Hence $\sigma(x) = (\xi(1-a)x, \xi)$. Putting $b^2 = 1 - a$ and $0 \leqq b \in A$, we have

$$\sigma(x) = (\xi b x, \xi b) = (\varphi x, \varphi)$$

where $\varphi = \xi b$, as desired.

REMARK 2.1. Although Theorem 3 is proved using the Radon-Nikodym Theorem, the latter can be proved by virtue of the former, that is, *Theorem* 3 *is equivalent to the Radon-Nikodym Theorem for normal states.* Since by a Theorem[12] of Segal [12; II, Th. 5] each abelian W^*-algebra is algebraically isomorphic to a masa algebra, we may assume A is itself masa. If σ and τ are two normal states with $\sigma \leqq \alpha\tau$, and $\tau(x) = (\varphi x, \varphi)$, then it is not less general to assume φA is dense in the Hilbert space H and $\tau(xy^*) = (\varphi x, \varphi y)$. Put $<\xi, \eta> = \sigma(xy^*)$ for $\xi = \varphi x$ and $\eta = \varphi y$, then $<\xi, \eta>$ is bilinear by $|<\varphi x, \varphi y>|^2 = |\sigma(xy^*)|^2 \leqq \sigma(xx^*)\sigma(yy^*) \leqq \alpha^2 \tau(xx^*)\,\tau(yy^*) \leqq \alpha^2 \|\varphi x\|^2 \ \|\varphi y^2\|$. Hence by Riesz' Lemma we have an hermitian non-negative a with $a \leqq \alpha$ such as $\sigma(xy^*) = (\varphi x a, \varphi y)$. It is not hard to see by the masa character of A, a belongs to A. Therefore $\sigma(xy^*) = \tau(xay^*)$. This is desired.

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11) Theorem 3 is clearly a generalization of a theorem of the previous note [15] of the second named author, in which the separability of the underlying space is assumed. A similar result is proved by Dye, Trans. Amer. Math. Soc.,72 (1952), 243-280 who proved for σ-finite W^*-algebras of finite type. Cf. Proposition 10.

12) A proof of this theorem without using the multiplicity theory is contained in [15].

3. In this section, we shall assume that A is a C^*-algebra having a *cyclic vector* in the sense that φA spans the underlying space H. In this case, we shall say also that A is *cyclic over* H. The purpose of this section is to show that the deflation of decomposition implies the Hellinger-Hahn Theorem. For this purpose, the following proposition is basic for us[13].

PROPOSITION 4. *If A is a commutative C^*-algebra having a cyclic vector φ, and if C is the weak closure of A. Then C is algebraically isomorphic with the multplication algebra of the measure space (Ω, σ), where Ω is the spectrum of A and $\sigma(x) = (\varphi x, \varphi)$.*

In short, a commutative W^*-algebra cyclic over H is determined up to isomorphism by the spectrum of a weakly dense C^*-algebra and its *normalizing measure*, where the normalizing measure of an algebra is the measure on the spectrum induced by $(\varphi x, \varphi)$ in which φ in cyclic.

PROOF. Let $de(\omega)$ be the canonical spectral measure on the spectrum. If $x(\omega)$ is an essentially bounded measurable function on Ω with respect to $d\sigma = d\sigma(e(\omega))$, then

$$x = \int x(\omega)\, de(\omega) \tag{3.1}$$

exists as an operator on H and belong to $A'' = C$. Clearly the mapping from $L_\infty(\Omega, \sigma)$ to C is one-to-one, linear, multiplicative and norm-preserving. Hence it remains to show that each x of C is expressible in (3.1) with a suitable $x(\omega)$ of $L_\infty(\Omega, \sigma)$.

Since A is weakly dense in C, we can select a phalanx $\{x_a;\ x_a \in A\}$ converging (bounded) weakly to the given x of C. For any y of A,

$$\sigma(z_a y) = (\varphi z_a, \varphi y^*) = \int z_a(\omega) y(\omega)\, d\sigma(\omega) \to (\varphi x, \varphi y^*) = (\varphi x y, \varphi).$$

Since $C(\Omega)$ is dense in $L_1(\Omega, \sigma)$ by the strong topology, the representation $z_a(\omega)$ of z_a in $C(\Omega)$ converges weak* in $L_\infty(\Omega, \sigma)$ to an essentially bounded measurable $x(\omega)$:

$$(\varphi z_a, \varphi y^*) = \int z_a(\omega) y(\omega)\, d\sigma \to \int x(\omega) y(\omega)\, d\sigma .$$

This shows each x of C is expressible in (3.1).

A commutative C^*-algebra acting on H is called having *simple spectrum* if it is weakly dense in a masa algebra, and an hermitean operator a has *simple*

13) Proposition 4 is a direct generalization of a result of I.E.Segal [12; I, §10]. Our proof is essentially a verbal change of his proof. It may be remarked that the spectrum of C is a Kakutani space of the perfection of the measure space (Ω, σ) in the sense of I.E. Segal [11]. Hence Proposition 4 is a direct generalization of a result of the second named author's [16] Cf. Proposition 8.

spectrum if a and 1 generate a C*-algebra having simple spectrum. Using this term, the preceeding proposition will be rewritten by Theorem 1 in the following

PROPOSITION 5. *A commutative C*-algebra having simple spectrum which is cyclic over the Hilbert space determines its weak closure up to unitary equivalence.*

In the proposition, the assumption of simple spectrum is superfluous since a commutative W*-algebra is masa if it has a cyclic vector by Segal [12; II, Cor. 1.1]. Conversely, on a separable space, the masa character of an algebra implies the existence of a cyclic vector by Segal [12; II, Cor. 1.2] (compare Stone [13; p. 274]). Hence, *on a separable space a commutative C*-algebra having simple spectrum is uniquely determined within the unitary equivalence by the spectrum and the normalizing measure.*

If two commutative C*-algebras A and B both having simple spectrum on a separable H are algebraically isomorphic, let φ and ψ be cyclic vectors for A and B respectively. Then their weak closure A'' and B'' are isomorphic if their normalizing measures $d\sigma$ and $d\tau$(induced by $\sigma(x) = (\varphi x, \varphi)$ and $\tau(y) = (\psi y, \psi)$ respectively) are mutually equivalent by the preceeding propositions. Hence, by Theorem 1, A'' and B'' are unitarily equivalent, and also A and B. Conversely, if A and B are unitarily equivalent, it is obvious there exists two cyclic vectors which define mutually equivalent normalizing measures. This shows

PROPOSITION 6 (MAEDA). *Two C*-algebras, both having the identity and simple spectrum on a separable Hilbert space, are unitarily equivalent if and only if they have same spectrum and mutually equivalent normlizing measures.*

Now, consider a C*-algebra A generated by an hermitean operator a and 1. By the well-known theorem of I. Gelfand [5], the spectrum Ω of A coincides with the usual spectrum of a (within homeomorphisms). Hence we shall consider Ω as a compact set in the real axis. Therefore, if a has simple spectrum and acts on a separable space, then the normalizing measure $d\sigma$ can be considered as a usual Stieltjes measure on the real axis. The preceeding proposition implies at once the following[14]

THEOREM 4 (HELLINGER-HAHN). *Suppose that two hermitean operators*

$$a = \int_{-\infty}^{\infty} \lambda \, dp(\lambda) \quad \text{and} \quad b = \int_{-\infty}^{\infty} \lambda \, dq(\lambda) \tag{3.2}$$

have simple spectra on a separable Hilbert space. If φ and ψ are cyclic vectors for a and b respectively, then a and b are unitarily equivalent if and only if they have same spectrum in common and the normalizing measures

14) Compare with Stone [13; VII, §§2-3] and Segal [12; II, The. 4 esp. Lemma. 4. 2].

(3.3) $d\sigma(\omega) = d(\varphi p(\omega), \varphi)$ *and* $d\tau(\omega) = d(\psi q(\omega), \psi)$
are mutually equivalent.

Before concluding this section, we may note that Proposition 4 and a theorem of J. von Neumann give a proof of the well-known von Neumann-Riesz Theorem[15] concerning the functions of an hermitean operators under the support of Segal's Theorem [12; II, Th. 5]. The converse implication is obvious.

4. In this section, we shall prove the Wecken-Plessner-Rohklin Theorem in the form of Segal [12; II, Th. 4]. Before stating the theorem, we shall introduce a notion of *weighted spectrum* $E(A)$ of a C^*-algebra A ($E(a)$ of an hermitean operator a) as follows : If A is a C*-algebra (C*-algebra generated by a and 1) acting on a Hilbert space H, then $E(A)$ ($E(a)$) is the set of all positive measures on the spectrum of A induced by

(4.1) $\rho(x) = (\xi x, \xi), \qquad \xi \in H.$

By Theorem 3, the weighted spectrum of a W*-algebra is the set of all finite positive normal measures on the spectrum, whence it is determined within the algebra.

Now the theorem will be read as follows :

THEOREM 5 (WECKEN-PLESSNER-ROKHLIN). *Two hermitean operators with simple spectra are unitarily equivalent if and only if they have same weighted spectrum.*

Naturally, we are assuming that two operators have same spectrum. From this, it is easy to deduce (with the Gelfand-Neumark Theorem), two operators generate C*-algebras which are algebraically isomorphic and have the same spectrum in common. Hence, as in the preceeding section, it is sufficient to show that the weak closure of a commutative C*-algebra A, acting on H, having simple spectrum is determined by the weighted spectrum up to unitary equivalence. On the other hand, two masa algebras are unitarily equivalent if they are isomorphic, whence the proof is further reduced into the following

PROPOSITION 7. *A commutative W*-algebra is determined up to isomorphism by the weighted spectrum of a weakly dense C*-subalgebra.*

PROOF. Let A be a commutative C*-algebra having the spectrum Ω and the weak closure C. If E is the set of all (complex) linear combinations of $E(A)$, then, by Theorem 2 and 3, E forms a Banach space and its conjugate space E^* coincincides with C. Therefore, C is determined by $E(A)$ within isometric isomorphisms as a Banach space[16].

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15) J. von Neumann, Math. Ann., 102 (1929), 370-427, Satz 10. B. von Sz. Nagy [14, p.93]. Segal [12; II, Cor.5.1] gives a proof basing on the realization of separable measure rings.

16) Proposition 7 is deducible from this and a theorem due to R.Kadison, Ann. of Math, 54 (1951), 325-338, Theorem 14.

Now, let E^r be the set of all real linear combinations of $E(A)$. It is the set of all real normal measures on the spectrum Γ of C, and consequently it forms an (AL)-space in the sense of S. Kakutani [6]. If C^r is the set of all self-adjoint members of C, then C^r is isometrically isomorphic with the space $C^r(\Gamma)$ of all real-valued continuous functions on Γ, whence C^r is the conjugate space of $L_1{}^r(x, \mu)$, the Banach space of all integrable real-valued functions on a measure space (X, μ) by a theorem due to I. E. Segal [11] and J. Dixmier [2]. On the other hand, C^r is the conjugate space of E^r by Pallu de la Barriére's theorem, whence E^r is equivalent to $L_r{}^1(X, \mu)$. Therefore, C^r is determined E^r up to vector lattice isomorphisms as an (AM)-space, whence the spectrum of C, which is coincides with the spectrum of C^r, is determined by E^r up to homeomorphisms by a theorem of S. Kakutani [6], and so C is determined by the weighted spectrum.

REMARK 4.1. Since the Wecken-Plessner-Rokhlin Theorem is an inseparable extension of the Hellinger-Hahn Theorem, it is natural to try to find a proof of the former which is an extension of the proof of the latter. The proof in this section shows that the unitary equivalence problem (of masa algebras) is a consequence of the representation, whence it differs from the preceeding section which indicates that the problem of the unitary equivalence is a deflation phenomema. Hence it will be given a sketch concerning this attempt.

Let A, E and Ω be as in the theorem, A linear function f on E will be called an E-function. Although an E-function f is not a function in general, we shall denote by $f(\lambda)$ for the convenience, and put

$$f(a) = \int_{-\infty}^{\infty} f(\lambda)\, de(\lambda), \tag{4.2}$$

where a is the element which generates A and

$$a = \int_{-\infty}^{\infty} \lambda\, de(\lambda). \tag{4.3}$$

The equation (4.2) has a meaning in the sense that[17]

$$(\xi f(a), \eta) = \int_{-\infty}^{\infty} f(\lambda)\, d(\xi e(\lambda), \eta) = f((\xi e(.), \eta)). \tag{4.4}$$

This defines an E-function of an operator a, and gives a generalization of the operator calculus.

To prove the theorem, we must prove (4.2) gives an algebraic isomorphism between C and E-functions. Main difficulty lies in the definition of multiplication of E-functions, which the authors do not succeed without using the representation of (AM)-spaces.

17) Proof of the existence of $f(a)$ as a linear operator on H is similar to that of Proposition 2, whence we shall omit it.

5. In this section, we shall give an another proof of Theorem 5 from a different point of view. It is a closed analogy to the classical theory due to Hellinger, Hahn, Stone and Maeda. Firstly we shall show[18]

PROPOSITION 8. *In Proposition 4, A is unitarily equivaleut to $C(\Omega)$ as a subalgebra of the multiplication algebra on $L_2(\Omega, \sigma)$. More precisely, $L_2(\Omega, \sigma)$ is mapped onto H in one-to-one manner by*

(5.1) $$\xi = \int \xi(\omega)\ d\varphi e(\omega),$$

where $\xi(.) \in L_2(\Omega, \sigma)$. Therefore,

(5.2) $$\xi x = \int \xi(\omega)\ x(\omega)\ d\varphi e(\omega)$$

where $x(.) \in C(\Omega)$ is the reprerentation of x of A.

PROOF. It is not hard to see by the definition of the integral, (5.1) gives an element ξ since

(5.3) $$\| \xi(.) \|^2 = \int |\xi(\omega)|^2 d \| \varphi e(\omega) \|^2 = \int |\xi(\omega)|^2 d\sigma$$

is finite. Hence it is sufficient to show that each ξ has the representation of (5.1). On the other hand, φA (and so $\varphi e(.)$) spans H, the mapping (5.1) is additive and homogeneous, and

(5.4) $$\| \xi \|^2 = (\xi, \xi) = \int |\xi(\omega)|^2 d(\varphi e(\omega), \varphi) = \| \xi(.) \|^2$$

by the property of the spectral measure, whence (5.1) gives a linear isometry between $L_2(\Omega, \sigma)$ and H. This shows the second part. (5.2) follows easily from the property of the spectral measure and (1.1), whence the first half is proved.

Since the direct sum of L_2-spaces of measure spaces is the L_2-space of the direct sum of measure space, the Wecken-Plessner-Rokhlin Theorem follows from the following[19]

PROPOSITION 9. *If A is a commutative C^*-algebra with simple spectrum on H, if $\sigma(x) = (\varphi x, \varphi)$ and $\tau(x) = (\psi x, \psi)$ are states of A, and moreover if K and L are closures of φA and ψA respectively. Then the following two statement are true :*

(5.5) *$\psi \in K$ if and only if $d\tau < d\sigma$;*

(5.6) *$K \perp L$ if and only if $d\sigma \perp d\sigma$.*

In the proposition, $d\tau < d\sigma$ means that $d\tau$ is absolutely continuous with respect to $d\sigma$, and $d\sigma \perp d\tau$ means that $d\rho < d\sigma$ and $d\rho < d\tau$ imply $d\rho = 0$, where $d\rho$ is a measure on the spectrum of the algebra induced by $\rho(x) = (\zeta x, \zeta)$.

18) Cf. Kelley [7], Maeda [8], Segal [12; II. Lem. 4.2] and Stone [13 ; p.226 & 243]. Identifying A and $C(\Omega)$, we can say that H and $L_2(\Omega, \sigma)$ are isometrically isomorphic as A-modul.

19) The essential part of the proposition is proved by Kelley [7], Maeda [8] and Segal [12 ; II, Lemma. 4.1].

Before entering the proof, we need a definition: A projection p of a W*-algebra C is a *carrier projection* (in the sense of Dye) of a normal state σ of C if $\sigma(p) = 1$ and if $\sigma(e) = 0$ implies $e \leqq 1-p$ for any projection e of C. It is known, in a commutative W*-algebra, the ortho-complement $1-p$ of the carrier projection of a normal state σ generates a principal ideal I which is the kernel of σ, i.e., $x \in I$ if and only if $\sigma(xx^*) = 0$. If σ has the form of (1.5), then I is the set of all operators of the algebra which vanish φ.

LEMMA. *In Proposition* 9, *if p is the projection belonging to K, then it is the carrier projection of σ, where σ will be understood as a state of C which is the weak closure of A.*

PROOF. If q is the carrier projection of σ, then $\varphi(1-p)=\varphi(1-q)=0$ implies $q \leqq p$, since it is easy to deduce by the masa character of C that p and q belong to C. On the other hand, $\varphi Cq = \varphi C$ implies $q \geqq p$. Hence $p = q$.

PROOF OF (5.5). If ψ belongs to K, then there exists a square σ-integrable function $\psi(\omega)$ with $d\tau(\omega) = \psi(\omega)\, d\sigma(\omega)$ by the preceeding proposition, whence the condition is necessary.

Conversely, if $d\tau$ is absolutely continuous with respect to $d\sigma$, then there exists σ-integrable function $\theta(\omega)$ which is the Radon-Nikodym derivative of $d\tau$ and non-negative almost everywhere with respect to $d\sigma$, whence $\phi(\omega) = \theta(\omega)^{\frac{1}{2}}$ exists almost everywhere and is square integrable with respect to $d\sigma$. By Proposition 8,

$$\phi = \int \phi(\omega)\, d\varphi e(\omega)$$

exists in K. Put $\rho(x) = (\phi x, \phi)$. Then

$$\rho(x) = \int x(\omega)\, \phi(\omega)^2\, d(\varphi e(\omega), \varphi) = \int x(\omega)\, \theta(\omega)\, d\sigma = \int x(\omega)\, d\tau = \tau(x)$$

implies $\rho = \tau$ on C, whence their carrier projections coincide, and so the closure of ϕA coincides with L by the above Lemma. Clearly, ϕA is a subset of K by the definition, L is included in K, and which shows the sufficiency.

PROOF OF (5.6). If $\rho(x) = (\zeta x, \zeta)$ and if $d\rho$ is absolutely continuous with respect to both $d\sigma$ and $d\tau$, then ζ belongs to $K \cap L$ by the above. If K is orthogonal to L, then this shows $d\rho = 0$, or $d\sigma \perp d\tau$, and so the sufficiency follows.

Conversely, suppose that $d\sigma \perp d\tau$. If e is the projection belonging to $K \cap L$, then $\rho(x) = (\xi ex, \xi)$ defines a measure $d\rho$ which is absolutely continuous with respect to both $d\sigma$ and $d\tau$ by (5.5), whence $\xi e = 0$ for any ξ in H, and so $e = 0$ or $K \cap L = 0$. If p and q are projections belonging to K and L respectively, then $\pi(x) = (\xi px, \xi)$ defines $d\pi$ such that $d\pi < d\sigma$ by (5.5). Moreover, if ξ is an element of L, then $d\pi$ is also absolutely continuous with respect to $d\tau$. Hence $\xi p = 0$ for all $\xi \in L$. This shows $K \perp L$.

Let A be a commutative W*-algebra over H. Then it is not hard to see by Proposition 8 that two cyclic subspaces are operator-isomorphic (by a unitary operator) as A-modul if and only if two normalizing measures are equivalent. From this and (5.5), no pair of mutually orthogonal cyclic subspaces are A-operator-isomorphic if A is masa. Hence, the direct decomposition of the underlying space into orthogonal subspaces by (5.6) under a masa algebra has no component which is A operator-isomorphic to another.

6. In this section, we shall add a remark on the notion of the normality of traces of a W*-algebra of finite type in the sense of J.Dixmier. As in the before, a trace of a W*-algebra will be called normal if it satisfies (1.6). In the previous note [9], the authors stated without proof that *a trace of a W*-algebra of finite type is normal if and only if it is normal in the center of the algebra.* Here, we shall give a proof of this.

Before entering the proof, we shall make a definitions: A property of a C*-algebra A will be called *purely central* if it is determined within the center Z of A. Under this definition we shall prove the above statement in the following

PROPOSITION 10. *The normality of traces in a W*-algebra of finite type is purely central.*

The proof is essentially a version of the second named author's [15]. It is sufficient to show that a trace τ is normal if it is normal in Z. If $x_a \downarrow 0$, then we can assume without loss of generality $\|x_a\| \leqq 1$, whence by Proposition 1 the phalanx $\{x_a\}$ converges bounded strongly to zero, and so by a thoerem of J. Dixmier[20] $\{x_a^\natural\}$ converges strongly in Z. Therefore by the hypothesis $\tau(x_a^\natural) = \tau(x_a)$ converges to zero, and so the proposition is proved.

ADDED IN PROOF : After the note presented to the editors, the authors find that Propositions 1 and 10 are established by J.Dixmier in the following papers : (1) C. R.,230 (1950), 267-269; (2) Comp.Math.,10 (1952), 1-55, Prop.2, respectively.

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