SOME TRIGONOMETRICAL SERIES VI.

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The object of this paper is to treat the convergence of the series

$$\sum_{n=1}^{\infty} | s_n^*(x) - f(x)|^2 n^\alpha \tag{1}$$

for $\alpha > 0$, where $s_n^*(x)$ is the nth modified partial sum of the Fourier series of $f(x)$, and is to derive an approximation theorem of infinitely differentiable functions.

1. THEOREM 1. *If the function*

$$g(t) = \varphi_x(t)/(2\tan(t/2)) \tag{2}$$

where $\varphi_x(t) = f(x+t) + f(x-t) - 2f(x)$, *is* L^2-*integrable and*

$$\int_0^{2\pi}\int_0^{2\pi} \frac{|g(t+h) - g(t)|^2}{h^p}\, dt\, dh < \infty \tag{3}$$

for a $p > \alpha + 1 \geqq 1$, *then the series* (1) *converges for* α $(1 \geqq \alpha \geqq 0)$. *If* (3) *holds with* p $(2 < p < 3)$, *then* (1) *converges for* $\alpha = p - 1$.

The L^2-integrability of (2) is stronger than the Dini's condition and (3) holds when

$$\int_0^{2\pi} \left| \frac{\varphi_x(t+h)}{t+h} - \frac{\varphi_x(t)}{t} \right|^2 dt = O(h^\beta)$$

for a $\beta > \alpha$. This is stronger than the convergence criterion due to Pollard.

We shall now prove the theorem,

$$s_n^*(x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\varphi_x(t)}{2\tan(t/2)} \sin nt\, dt$$

$$= \frac{2}{\pi}\int_0^{2\pi/n} \left\{ \sum_{k=0}^{n-1} \frac{(t - \pi + 2k\pi/n)}{2(\tan(t - \pi + 2k\pi/n)/2)} \right\} \sin nt\, dt.$$

Let us put

$$F_n(t) = \frac{2\pi}{n} \sum_{k=0}^{n-1} \frac{\varphi_x(t - \pi + 2k\pi/n)}{2(\tan(t - \pi + 2\pi/kn)/2)},$$

which is the Riemann sum of $g(t) = \varphi_x(t)/(2\tan(t/2))$ in the interval $(-\pi, \pi)$. By the assumption, $g(t)$ is integrable. If $g(t)$ is continued periodically and is expanded in Fourier series such that

$$g(t) \sim \sum_{\nu=-\infty}^{\infty} c_\nu e^{i\nu t},$$

then

$$F_n(t) \sim \sum_{\nu=-\infty}^{\infty} c_{\nu n} e^{i\nu nt}.$$

Hence

$$s_n^*(x) - f(x) = \frac{n}{\pi^2} \int_0^{2\pi/n} F_n(t) \sin nt\, dt$$

$$= \frac{n}{\pi^2} \int_0^{2\pi/n} (F_n(t) - c_0) \sin nt\, dt,$$

$$|s_n^*(x) - f(x)|^2 \leqq \frac{n^2}{\pi^4} \cdot \frac{2\pi}{n} \int_0^{2\pi/n} (F_n(t) - c_0)^2\, dt$$

$$= \frac{2}{\pi^3} \int_{-\pi}^{\pi} (F_n(t) - c_0)^2\, dt$$

$$= \frac{2}{\pi^2} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} c_{kn}^2,$$

since $F_n(t)$ has the period $2\pi/n$. Thus[1]

$$\sum_{n=1}^{\infty} n^\alpha |s_n^*(x) - f(x)|^2 \leqq \frac{2}{\pi^2} \sum_{n=1}^{\infty} n^\alpha \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} c_{kn}^2$$

$$= \frac{2}{\pi^2} \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} \sigma_\alpha(|\nu|) c_\nu^2,$$

where

$$\sigma_\alpha(\nu) = \sum_{\alpha | \nu} d^\alpha.$$

It is known that

(4) $\qquad \sigma_\alpha(\nu) = O(\nu^\alpha) \qquad (\alpha > 1),$

(5) $\qquad \sigma_\alpha(\nu) = O(\nu^{\alpha+\varepsilon}) \qquad (1 \geqq \alpha \geqq 0)$

for any $\varepsilon > 0$.[2]

If $0 \leqq \alpha \leqq 1$, we put $p = (\alpha + \varepsilon) + 1 < 2$. Then, by (5), we have

1) Cf. Marcinkiewicz and R. Salem, Fund. Math, 30(1949).

2) The author learned these relations from J. Uchiyama. He proved more precise results than (5).

$$\Sigma|s_n^*(x) - f(x)|^2 n^\alpha \leqq \text{const.} \sum_{\nu=-\infty}^{\infty} |\nu|^{p-1}c_\nu^2. \tag{6}$$

On the other hand,

$$\int_0^{2\pi} |g(t+h) - g(t)|^2 dt = 4\sum_{-\infty}^{\infty} c_\nu^2 \sin^2 \nu h,$$

$$\int_0^{2\pi}\int_0^{2\pi} \frac{|g(t+h) - g(t)|^2}{h^p} dt\, dh = 4\sum_{-\infty}^{\infty} c_\nu^2 \int_0^{2\pi} \frac{\sin^2 \nu h}{h^p} dh$$

$$\geqq \text{const.} \sum_{-\infty}^{\infty} |\nu|^{p-1}c_\nu^2.$$

Hence

$$\Sigma n^\alpha |s_n^*(x) - f(x)|^2 \leqq \text{const.} \int_0^{2\pi}\int_0^{2\pi} \frac{|g(t+h) - g(t)|^2}{h^p} dt\, dh.$$

Thus we get the first part of the theorem. The second part may be proved similarly, using (4) instead of (5).

2. THEOREM 2. *If the function $g(t)$ is k-times differentiable and $g^{(k)}(t)$ belongs to L^2, and if further*

$$\int_0^{2\pi}\int_0^{2\pi} \frac{|g^{(k)}(t+h) - g^{(k)}(t)|^2}{h^p} dt\, dh < \infty \tag{7}$$

for a $p > \alpha - 2k + 1 \geqq 1$, then the series (1) *converges for α $(2k \leqq \alpha \leqq 2k + 1)$. If* (7) *holds for $p = \alpha + 1$, $2k + 1 < \alpha < 2k + 2$, then* (1) *converges for such α.*

For the proof we use the notation of the proof of Theorem 1, then we have (6). Further we have

$$\int_0^{2\pi} |g^{(k)}(t+h) - g^{(k)}(t)|^2 dt = 4\sum_{-\infty}^{\infty} \nu^{2k} c_\nu^2 \sin^2 \nu h,$$

$$\int_0^{2\pi}\int_0^{2\pi} \frac{|g^{(k)}(t+h) - g^{(k)}(t)|^2}{h^p} dt\, dh$$

$$= 4\sum_{-\infty}^{\infty} \nu^{2k} c_\nu^2 \int_0^{2\pi} \frac{\sin^2 \nu h}{h^p} dh$$

$$\geqq \text{const.} \sum_{-\infty}^{\infty} |\nu|^{2k+p-1} c_\nu^2.$$

Hence, for α and p in the theorem,

$$(8)\qquad \sum_{n=1}^{\infty} n^{\alpha}|s_n^*(x)-f(x)|^2 \leqq \text{const.} \int_0^{2\pi}\int_{\pi}^{2\pi} \frac{|g^{(k)}(t+h)-g^{(k)}(t)|^2}{h^p}\,dt\,dh.$$

Thus the theorem is proved.

3. THEOREM 3. *If $f(x)$ is differentiable infinitely many times and*

$$A_k = \max_{0\leqq x\leqq 2\pi} |f^{(k)}(x)| \quad (k=0,1,2,\dots),$$

$$(9)\qquad \sum_{k=1}^{\infty} \frac{A_{k+2}^2}{k^2}\left\{\frac{1}{\psi(2k)}+\frac{1}{\psi(2k+1)}\right\}<\infty,$$

then the series

$$(10)\qquad \sum_{n=1}^{\infty} |s_n^{**}(x)-f(x)|^2\,\varphi(n)$$

converges uniformly, where

$$s_n^{**}(x) = \int_{-\pi}^{\pi} f(x+t)\,\frac{\sin nt}{t}\,dt \Big/ \int_{-n\pi}^{n\pi}\frac{\sin t}{t}\,dt$$

and

$$\varphi(n) = \sum_{k=1}^{\infty} n^k/\psi(k).$$

Especially there is a trigonometrical polynomial $t_n(x)$ of order n such that

$$(11)\qquad t_n(x)-f(x) = O\left(1/\sqrt{\varphi(n)}\right)$$

uniformly.

For, since we can verify that (8) holds for

$$\delta_n(x) = \frac{1}{\pi}\int_0^{\pi} \varphi_x(t)\frac{\sin nt}{t}\,dt,\quad g_1(t) = \frac{\varphi_x(t)}{t}$$

instead of $s_n^*(x)-f(x)$ and $g(t)$, we have

$$(12)\qquad \sum_{n=1}^{\infty}\delta_n^2(x)\varphi(n) = \sum_{n=1}^{\infty}\delta_n^2(x)\sum_{k=1}^{\infty}\frac{n^k}{\psi(k)}$$

$$= \sum_{k=1}^{\infty}\frac{1}{\psi(k)}\sum_{n=1}^{\infty}\delta_n^2(x)\,n^k$$

$$\leqq \text{const.} \sum_{k=1}^{\infty}\left\{\frac{1}{\psi(2k)}+\frac{1}{\psi(2k+1)}\right\}\int_0^{2\pi}\int_0^{2\pi}\frac{|g_1^{(k)}(x+h)-g_1^{(k)}(x)|^2}{h^p}\,dt\,dh.$$

If we put

$$\max_{0\leqq x\leqq 2\pi}|g_1^{(k)}(x)| = B_k \qquad (k=0,1,\dots),$$

then

$$\sum_{n=1}^{\infty} \delta_n^2(x)\varphi(n) \leqq \text{const.} \sum_{k=1}^{\infty} B_{k+1}^2 \left\{ \frac{1}{\psi(2k)} + \frac{1}{\psi(2k+1)} \right\}.$$

Now,

$$g_1^{(k)}(t) = \sum_{\nu=0}^{k} \frac{k(k-1)\ldots(k-\nu-1)}{\nu!} \varphi_x^{(k-\nu)}(t) \frac{d^\nu}{dt^\nu}\left(\frac{1}{t}\right)$$

$$= \sum_{\nu=0}^{k}(-1)^\nu k(k-1)\ldots(k-\nu-1)\, t^{-\nu-1}\varphi_x^{(k-\nu)}(t)$$

$$= (-1)^{k+1}\frac{k!}{t^{k+1}}\left\{ f(x) - \sum_{\mu=0}^{k}(-1)^\mu f^{(\mu)}(x+t)\frac{t_\mu}{\mu!} \right.$$

$$\left. + f(x) - \sum_{\mu=0}^{k} f^{(\mu)}(x-t)\frac{t^\mu}{\mu!} \right\}$$

$$= \frac{1}{k+1}\{f^{(k+1)}(x+\theta t) + f^{(k+1)}(x-\theta' t)\},$$

where $0 < \theta < 1$, $0 < \theta' < 1$. Hence

$$B_k \leqq 2A_{k+1}/(k+1).$$

Thus we have

$$\sum_{n=1}^{\infty} \delta_n^2(x)\varphi(n) \leqq \text{const.}\sum_{k=1}^{\infty} \frac{A_{k+2}^2}{k^2} \left\{ \frac{1}{\psi(2k)} + \frac{1}{\psi(2k+1)} \right\},$$

which is finite by the assumption.

For the proof of (11), it is sufficient to put

$$t_n(x) = \int_{-\pi}^{\pi} f(x+t)\frac{\sin nt}{t}dt \Big/ \int_{-n\pi}^{n\pi} \frac{\sin t}{t}\, dt.$$

For example, let us consider the function

$$(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{e^n}$$

and let $s_n(x)$ be the nth partial sum of the series.

Then

(13) $$f(x) - s_n(x) = O(1/e^n),$$

which is the best approximation. A little weak estimation is derived from our theorem. For $A_k = \max |t^{(k)}(x)| = O\left(\sum_{n=1}^{\infty}\frac{n^k}{e^n}\right) = O(k!)$, and the series

$$\sum \frac{A_{k+2}^2}{k^2\psi(2k)} = \sum \frac{((k+2)!)^2}{k^2\psi(2k)}$$

converges when $\psi(2k) = \{(k+2)!\}^2 k^{-\alpha}\ (0 < \alpha < 1)$ and then it is sufficient to take

$$\psi(k) = k^{4+\epsilon}k^k e^{-k}/2^k.$$

Hence $\varphi(n) = \Sigma 2^k n^k/k^k e^{-k} k^{4+\epsilon} \sim e^{2n}/n^{2+\epsilon}$.
Thus (11) becomes, for any $\varepsilon > 0$,

$$t_n(x) - f(x) = o(n^{2+\epsilon}/e^n),$$

which is weaker than (13) a little.

Secondly, let us take $\psi(k) = k!$, then $\varphi(n) = e^n$. In this case, Theorem 3 becomes:

If $A_k \leqq \text{const.}\, 2^k k!/k^2$ $(k = 1, 2, \ldots.)$, then there is a trigonometrical polynomial $t_n(x)$ of order n such that

$$\sum_{n=1}^{\infty} e^n |f(x) - t_n(x)|^2 < \infty.$$

Further, if

$$A_k \leqq \text{const.}\ (2k)!,$$

then there is a trigonometrical polynomial $t_n(x)$ of order n such that

$$\sum_{n=1}^{\infty} \frac{e^n}{n^\epsilon} |f(x) - t_n(x)|^2 < \infty$$

for any $\varepsilon > 0$, and then

$$f(x) - t_n(x) = o(n^\epsilon/e^{\sqrt{n}}).$$

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