A NEW CONVERGENCE CRITERION FOR FOURIER SERIES

GEN-ICHIRÔ SUNOUCHI

(Received September 8, 1953)

Let $\varphi(t)$ be an even periodic function which is integrable in the Lebesgue sense and let

$$\varphi(t) \sim \sum_{n=1}^{\infty} a_n \cos nt.$$

Then the author [2] has generalized Young's convergence criterion as follows:

THEOREM. *The Fourier series of $\varphi(t)$ converges at the point $t = 0$ to the value zero, provided that there is a $\Delta \geqq 1$ such that*

(1) $$\int_0^t \varphi(u)du = o(t^\Delta), \quad as\ t \to 0,$$

and

(2) $$\int_0^t |d\{u^\Delta \varphi(u)\}| = O(t), \quad 0 \leqq t \leqq \eta.$$

G. H. Hardy and J. E. Littlewood [1] have generalized the condition (1) for the case $\Delta = 1$ in the form:

(3) $$\Phi_\beta(t) = o(t^\beta), \quad \text{as } t \to 0$$

for any $\beta > 0$, where $\Phi_\beta(t)$ is the β-th integral of $\varphi(t)$. Corresponding to this result, we prove the following theorem which generalizes the above theorem in the Hardy-Littlewood type.

THEOREM. *The Fourier series of $\varphi(t)$ converges at the point $t = 0$ to the value zero, provided that there is a $\Delta \geqq 1$ such that*

(4) $$\Phi_\beta(t) = o(t^\gamma), \quad \gamma \geqq \beta$$

and

(5) $$\int_0^t |d\{u^\Delta \varphi(u)\}| = O(t), \quad 0 \leqq t \leqq \eta,$$

where

$$\Delta = \gamma/\beta \geqq 1.$$

For the proof of this theorem, we need following lemmas.

LEMMA 1. *If $0 < \alpha \leqq 1$ and C is a positive constant, for $C \geqq v > u \geqq 0$,*

$$\int_u^v \cos nt(t-u)^{\alpha-1}dt = O(n^{-\alpha}),$$

$$\int_u^v \sin nt(t-u)^{\alpha-1}dt = O(n^{-\alpha}).$$

PROOF. We have

$$n^\alpha \int_u^v \cos nt\,(t-u)^{\alpha-1}\,dt = n^\alpha \int_0^{v-u} \cos n(u+t)\,t^{\alpha-1}dt$$

$$= n^\alpha \int_0^{v-u} t^{\alpha-1} \cos nu \, \cos nt \, dt - n^\alpha \int_0^{v-u} t^{\alpha-1} \sin nu \, \sin nt \, dt = I - J,$$

say. We have

$$I = n^\alpha \cos nu \int_0^{v-u} t^{\alpha-1} \cos nt \, dt$$

$$= \cos nu \int_0^{n(v-u)} t^{\alpha-1} \cos t \, dt = \cos nu \left\{ \int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{n(v-u)} \right\} t^{\alpha-1} \cos t \, dt$$

$= K + L,$ say. Then

$$|K| \leqq \int_0^{\frac{\pi}{2}} t^{\alpha-1}\,dt = \frac{1}{\alpha}\left(\frac{\pi}{2}\right)^\alpha,$$

and

$$|L| = \left| \left(\frac{\pi}{2}\right)^{\alpha-1} \int_{\frac{\pi}{2}}^{\xi} \cos t \, dt \right| \leqq 2\left(\frac{\pi}{2}\right)^{\alpha-1}$$

Thus we have

$$|I| \leqq \frac{1}{\alpha}\left(\frac{\pi}{2}\right)^\alpha + 2\left(\frac{\pi}{2}\right)^{\alpha-1} \leqq (\pi+4)/\pi.$$

Since analogous estimations hold for J, we have

$$\left| n^\alpha \int_u^v \cos nt \, (t-u)^{\alpha-1} \right| \leqq 2(\pi+4)/\pi.$$

LEMMA 2. *For the Fourier series*

$$\varphi(t) \sim \sum_{n=1}^\infty a_n \cos nt,$$

if

(6) $$\Phi_\beta(t) = o(t^\gamma), \gamma \geqq \beta$$

and

(7) $$a_n = O(n^{-\frac{\beta}{\gamma}}),$$

then the Fourier series converges to zero at $t = 0$.

This lemma is a special case of Wang's convergence criterion [4].

PROOF OF THEOREM. From Hardy-Littlewood's theorem, we should prove the case $\Delta > 1$. In view of Lemma 2, it is sufficient to prove

$$a_n = O(n^{-\frac{1}{\Delta}}), \quad \Delta = \frac{\gamma}{\beta} > 1.$$

Since the convergence of Fourier series is the local property, we may suppose that (5) is true for $0 \leqq t \leqq \pi$.

Spliting up the integral at $\alpha = n^{-1/\Delta}$, we have

$$a_n = \int_0^\alpha \varphi(t) \cos nt \, dt + \int_\alpha^\pi \varphi(t) \cos nt \, dt = I + J,$$

say. Let us put

$$\theta(t) = t_\Delta \varphi(t), \quad \Theta(t) = \int_0^t |d\theta(u)|,$$

then

$$\Theta(t) = O(t), \quad \theta(t) = O(t).$$

Our concerning integral becomes

$$J = \int_\alpha^\pi \varphi(t) \cos nt \, dt = \int_\alpha^\pi \theta(t) \frac{\cos nt}{t^\Delta} dt$$

$$= -\int_\alpha^\pi \theta(t) dA(t),$$

where

$$A(t) = \int_t^\pi \frac{\cos nt}{t^\Delta} dt = \frac{1}{t^\Delta} \int_t^\xi \cos nt \, dt = O(n^{-1} t^{-\Delta}).$$

Then

$$-J = \int_\alpha^\pi \theta(t) dA(t) = [\theta(t) A(t)]_\alpha^\pi + \int_\alpha^\pi A(t) \, d\theta(t)$$

$$= J_1 + J_2,$$

say. Since $\alpha = n^{-\frac{1}{\Delta}}$ we have

$$J_1 = O(n^{-1/\Delta}),$$

and

$$|J_2| \leqq \int_\alpha^\pi |A(t)| \, |d\theta(t)| = O\left\{ n^{-1} \int_\alpha^\pi t^{-\Delta} |d\theta(t)| \right\}$$

$$= O(n^{-1} [t^{-\Delta} \Theta(t)]_\alpha^\pi) + O\left\{ n^{-1} \int_\alpha^\pi \Theta(t) \, t^{-(\Delta+1)} dt \right\}$$

$$= O(n^{-1}) + O(n^{-1} \alpha^{-\Delta+1}) + O(n^{-1} \int_\alpha^\pi t^{-\Delta} \, dt)$$

$$= O(n^{-1} \alpha^{-\Delta+1}) = O(n^{-\frac{1}{\Delta}}).$$

If β is integral, then the estimation of I is easy, so we suppose that β is fractional. If $0 < \beta < 1$, then

$$\Phi(t) = o(t^{1+\gamma-\beta}) = o(t)$$

and

$$\begin{aligned} I &= \int_0^\alpha \varphi(t) \cos nt \, dt \\ &= [\Phi(t) \cos nt]_0^\alpha + n \int_0^\alpha \Phi(t) \sin nt \, dt \\ &= I_1 + I_2, \end{aligned}$$

say, where

$$I_1 = O(\alpha) = O(n^{-\frac{1}{\Delta}})$$

and

$$\begin{aligned} I_2 &= n \int_0^\alpha \sin nt \left\{ \int_0^t \Phi_\beta(u) (t-u)^{-\beta} du \right\} dt \\ &= n \int_0^\alpha \Phi_\beta(u) \, du \int_u^\alpha \sin nt \, (t-u)^{-\beta} dt. \end{aligned}$$

Applying Lemma 1,

$$\begin{aligned} I_2 &= O\left(n \int_0^\alpha \Phi_\beta(u) n^{\beta-1} du \right) \\ &= O\left(n^\beta \int_0^\alpha u^\gamma \, du \right) \\ &= O(n^\beta [u^{\gamma+1}]_0^\alpha) = O(n^\beta n^{-\frac{\beta}{\gamma}(\gamma+1)}) \\ &= O(n^{-\frac{\beta}{\gamma}}) = O(n^{-\frac{1}{\Delta}}). \end{aligned}$$

Thus we get the desired formula, in the case $0 < \beta < 1$,

$$a_n = O(n^{-\frac{1}{\Delta}}).$$

If $1 < \beta < 2$, then

$$\begin{aligned} I &= [\Phi(t) \cos nt]_0^\alpha + n \int_0^\alpha \Phi(t) \sin nt \, dt \\ &= [\Phi(t) \cos nt]_0^\alpha + n \, [\Phi_2(t) \sin nt]_0^\alpha - n^2 \int_0^\alpha \Phi_2(t) \cos nt \, dt \\ &= I_1 + I_2 - I_3, \end{aligned}$$

say. From (5), we have $\varphi(t) = O(t^{-\Delta+1})$.

Applying a generalized convexity theorem of M. Riesz (see Sunouchi [3]) to

$$\varphi(t) = O(t^{-\Delta+1}) \text{ and } \Phi_\beta(t) = o(t^\gamma),$$

we get

$$\Phi(t) = o(t^{1+\frac{\gamma-\beta}{\beta^2}}) = o(t), \quad \Phi_2(t) = o(t^{2+\gamma-\beta}).$$

Therefore

$$I_1 = o(\alpha) = O(n^{-\frac{1}{\Delta}})$$

and

$$I_2 = o(n\,\alpha^{2+\gamma-\beta}) = o(\alpha^{-\frac{\gamma}{\beta}+2+\gamma-\beta})$$
$$= o(\alpha^{1+(\gamma-\beta\,(1-\frac{1}{\beta}))}) = o(\alpha) = o(n^{-\frac{1}{\Delta}}),$$

for $1 < \beta < 2$, and $\gamma > \beta$. Concerning to I_3, we have

$$I_3 = n^2 \int_0^\alpha \Phi_2(t) \cos nt\, dt = n^2 \int_0^\alpha \cos nt\, dt \int_0^t \Phi_\beta(u)(t-u)^{2-\beta-1}\, du$$
$$= n^2 \int_0^\alpha \Phi_\beta(u)\, du \int_u^\alpha \cos nt\, (t-u)^{2-\beta-1}\, dt$$
$$= O\left(n^2 \int_0^\alpha u^\gamma \cdot n^{\beta-2}\, du\right) = O(n^\beta [u^{\gamma+1}]_0^\alpha)$$
$$= O(n^{-\frac{\beta}{\gamma}}) = O(n^{-\frac{1}{\Delta}}).$$

Thus proceeding, the proof of the case $n < \beta < n+1$ ($n = 0, 1, 2, 3, \dots$) is now in hand. Since the proof for integral β is easy, we have completed the proof of the Theorem.

LITERATURES

[1] G. H. HARDY AND J. E. LITTLEWOOD, Notes on the theory of Series (VII); On Young's convergence criterion for Fourier series, Proc. London Math. Soc., 28(1928), 301–311.

[2] G. SUNOUCHI, Notes on Fourier analysis (XLVI); A convergence criterion for Fourier series, Tôhoku Math. Journ., 3(1951), 216–219.

[3] G. SUNOUCHI, Notes on Fourier analysis (XLVIII); On convexity theorem and its applications to Fourier series, Journal of Math., 1(1952), 104–109.

[4] F. T. WANG, On Riesz summability of Fourier series (II), Journ. London Math. Soc., 17(1942), 98–107.

MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY, SENDAI.