AN ELEMENTARY AND PURELY SYNTHETIC PROOF FOR THE DOUBLE SIX THEOREM OF SCHLÄFLI

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In one of my previous papers [43][1] I have given four elementary, purely projective and synthetic proofs of the double six theorem of Schläfli carried on within the scope of points, lines and planes only. In the present paper I will give a new, again purely synthetic proof of the same theorem, using only the Lemma 1 given below.

LEMMA 1. *Every straight line, which meets three straight lines out of given four all meeting three straight lines in skew positions, always meets the remaining fourth.*

The Lemma 2, which will also be used, follows immediately from the Lemma 1:

LEMMA 2. *When four straight lines a_1, a_2, a_3 and a_4 in skew positions are met by a straight line b_6, there exists necessarily another straight line b_5 different from b_6, which meets all of a_1, a_2, a_3 and a_4, provided that the row of points $b_6(a_1a_2a_3a_4)$ and the sheaf of planes $b_6(a_1a_2a_3a_4)$ are not projective.*

In the axiomatic theory of the projective geometry, the Lemma 1 is equivalent to the fundamental theorem of the projective geometry: "the rows of points $ABCD$ and $ABCD'$ are projective, when and only when D and D' are one and the same"; so that these are equivalent to the theorem of Pascal-Pappus as well as to the law of commutativity of the field (Körper), whereon the coordinates are defined. Hence it will be worth while to prove the double six theorem of Schläfli and the Lemma 2 using only the Lemma 1.

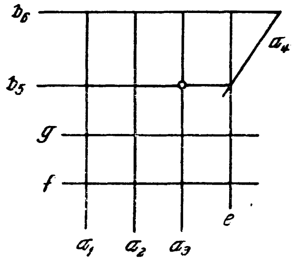
PROOF FOR LEMMA 1. Let the four straight lines meeting all of three straight lines b_1, b_5 and b_6 be a_2, a_3, a_4 and c respectively. I will show that a straight line d which meets all of a_2, a_3 and a_4 also meets c. If we denote, namely, the intersection points (b_5c) and (b_6c) by X and Y respectively, then among the sheaves of planes, the following relations hold:

$$d(a_2a_3a_4X) \overset{b_5}{\overline{\wedge}} b_1(a_2a_3a_4c) \overset{b_6}{\overline{\wedge}} d(a_2a_3a_4Y).$$

Hence the planes Xd and Yd are one and the same, so that d meets the joining line c of X and Y.

PROOF FOR LEMMA 2. Take any two straight lines f and g, which meet all of a_1, a_2 and a_3 and are different from b_6. Let the joining line of the

1) The number in the square brackets refers to the references at the end.



two points, in which the plane a_4b_6 is cut by f and g, be e. Then the e meets all of a_4, b_6, f and g. By the assumption it may easily be shown that e, a_4 and b_6 do not meet in a single point. We can draw the straight line b_5 from the intersection point (ea_4), to meet all of e, a_1 and a_2, and then the b_5 meets also a_3 by the Lemma 1, because e, a_1, a_2 and a_3 meet all of b_6, f and g.

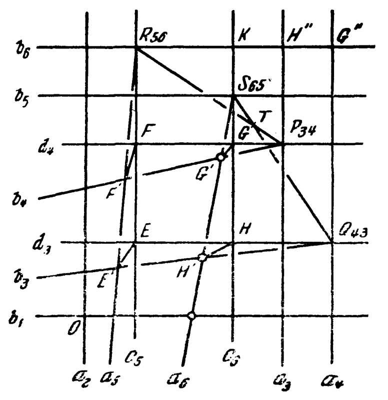
Thus the common intersector b_5 different from b_6 of a_1, a_2, a_3 and a_4 is obtained. It may easily be proved that there exists no other common intersector of a_1, a_2, a_3 and a_4.

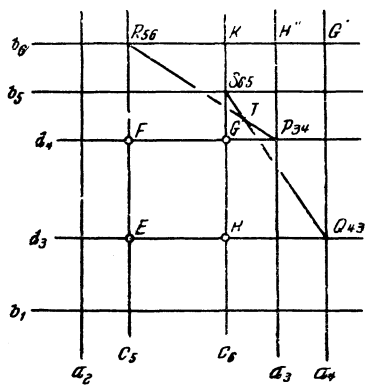
Double Six Theorem of Schläfli. *Let five straight lines a_1, a_2, a_3, a_4 and a_5 in space, no two of which intersect, meet a straight line b_6 in such a way that no quadruple of elements of the row of points $b_6(a_1a_2a_3a_4a_5)$ are projective to the corresponding quadruple of elements of the sheaf of planes $b_6(a_1a_2a_3a_4a_5)$. Let the common intersector*

of a_2, a_3, a_4 and a_5 other than b_6 be b_1,
that of a_1, a_3, a_4 and a_5 other than b_6 be b_2,
that of $a_1; a_2, a_4$ and a_5 other than b_6 be b_3,
that of a_1, a_2, a_3 and a_5 other than b_6 be b_4, and let
that of a_1, a_2, a_3 and a_4 other than b_6 be b_5.

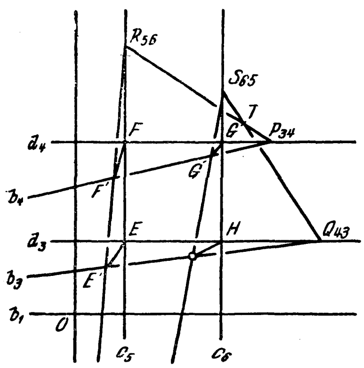
Then the five straight lines b_1, b_2, b_3, b_4 and b_5 are necessarily met by another straight line a_6.

Proof. We can draw from the point $S = S_{65}$ of intersection of b_5 with the plane π determined by the three intersection points $(a_3b_4) = P_{34} = P$, $(a_4b_3) = Q_{43} = Q$ and $(a_5b_6) = R_{56} = R$, a straight line a_6 to meet both b_4 and b_1.





Now we can draw a straight line d_3 from Q and a straight line d_4 from P respectively, to meet all of a_2, a_3 and a_4. We can draw also a straight



line c_5 from R and a straight line c_6 from S respectively, to meet all of b_1, b_5 and b_6. By the Lemma 1 d_3 meets c_5. We denote the intersection point (c_5d_3) with E. Similarly we get the intersection points $(c_5d_4) = F$, $(c_6d_4) = G$ and $(c_6d_3) = H$. Let us set for the respective intersection points as follows: $(a_5b_3) = E'$, $(a_5b_4) = F'$, $(a_6b_4) = G'$, $(a_2b_1) = O$ and $(PR, SQ) = T$. The point O lies on the planes ERE' and EQE', so that the straight line EE' passes through O. Similarly the straight line GG' passes through O. The three points E, G and T lie on the plane PRF and also on the plane SQH, so that E, G and T are collinear. Hence the three points E', G' and T lie on the plane determined by the two straight lines $EE'O$ and $GG'O$. These three points E', G' and T lie also on the plane PRF, so that they are collinear. Therefore on the plane determined by the two straight lines $E'G'T$ and STQ, the straight line $G'S = a_6$ meets the straight line $E'Q = b_3$.

Thus, in other words, it is proved that, if we draw from the point S_{65} which lies on b_5, a straight line a_6 to meet both b_4 and b_3, the a_6 meets b_1. Similarly, replacing a_2 and b_1, by a_1 and b_2 respectively, it may be shown that the a_6 also meets b_2.

Thus the proof is accomplished.

Now, if we denote the intersection points (a_6b_3), (a_3b_6), (a_4b_6) and (c_6b_6) with H', H'', G'' and K respectively, then for the sheaves of planes we have

$$a_2(b_3b_4b_5b_6) = a_2(d_3d_4b_5b_6) = a_2(HGSK) \overset{c_6}{\overline{\wedge}} PQ(HGSK)$$
$$= PQ(H'G''RK) \overset{b_6}{\overline{\wedge}} b_1(H'G''RK) = b_1(a_3a_4c_5c_6) = b_1(a_3a_4a_5a_6),$$

so that we have the following well known corollary.

COROLLARY. *In the double six*

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6, \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6, \end{matrix}$$

the following relations hold:

(i) *the four intersection points* (a_3b_4), (a_4b_3), (a_5b_6) *and* (a_6b_5) *are coplanar.*

(ii) *row of points* $a_1(b_3b_4b_5b_6)$
$\overline{\wedge}$ *sheaf of planes* $a_2(b_3b_4b_5b_6)$
$\wedge$ *sheaf of planes* $b_1(a_3a_4a_5a_6)$
$\overline{\wedge}$ *row of points* $b_2(a_3a_4a_5a_6)$.

(iii) *row of points* $a_1(b_2b_3b_4b_5b_6)$
$\wedge$ *sheaf of planes* $b_1(a_2a_3a_4a_5a_6)$.

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