COINCIDENCE POINTS OF A CURVE

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1. Halphen [1] defined a coincidence point of a plane curve in the following way. There is a pencil of cubics having 8-point contact with the curve and with each other at a given point P. These cubics all pass through a further point H If H coincides with P, then P is called a *coincidence point* of the curve. The cubics therefore have 9-point contact with each other at P, but only 8-point contact with the curve. Halphen [2] also shows that a necessary and sufficient condition that a simple point P is a coincidence point is that there is a cubic with a double point at P such that P counts nine times as an intersection of the curve and the cubic: in general, such a cubic will have a node at P, one branch having 8-point contact with the curve.

The object of this note is to determine the number of coincidence points of a curve in terms of its Plücker numbers. We shall assume that the curve has no singularities other than nodes or cusps, and we shall not regard a cusp or an inflexion as a coincidence point.

2. We first consider the case where the curve has no cusps. As a preliminary result, we find the number of simple points P such that a cubic through 7 fixed points of the curve, $O_1, \dots . O_7$, has a node at P. The cubics through $O_1, \dots . O_7$ form a net, and the nodes lie on the Jacobian of the net, which is of order 6 and has nodes at $O_1, \dots, O_7$ [3]. Hence the number of points is $6n - 14$.

We now consider the (a, a') correspondence between a point P of the curve and a point Q where the cubic through 6 fixed points of the curve, $O_1, \dots, O_6$, and having a node at P meets the curve again. If γ is the valency of the correspondence, then we have $a = 3n - 8$, $a' = 6n - 14$ (from the last paragraph), $\gamma = 2$. By the Cayley-Brill formula [4], the number of coincidences is $9n - 22 + 4p$. This is therefore the number of points P such that there is a cubic through $O_1, \dots, O_6$ which has a node at P, one branch having 2-point contact at P.

We now consider the (a, a') correspondence between a point P of the curve and a point Q where the cubic through 5 fixed points of the curve, $O_1, \dots, O_5$ and having a node at P, one branch having 2-point contact, meets the curve again. We now have $a = 3n - 8$, $a' = 9n - 22 + 4p$, $\gamma = 3$. Hence the number of coincidences is $12n - 30 + 10p$. This is therefore the number of points P such that there is a cubic through $O_1, \dots, O_5$ which has a node at P, one branch having 3-point contact at P.

Proceeding in this way, we arrive at the result that the number of points P such that there is a cubic with a node at P, one branch having 8-point contact at P is $27n - 70 + 70p$. This would include the inflexions, however, since the inflexional tangent counted three times would be a degenerate cubic satisfying the conditions. The number of coincidence points is therefore

$$27 n - 70 + 70 p - \iota = 32 m - 40 n.$$

3. We must now find the modification necessary if the curve has cusps. If the process described in section 2 is applied to the problem of finding the number of sextactic points of a curve, it is easy to allow for the number of cusps: in fact, at each stage the number of cusps has to be subtracted. In finding the number of coincidence points, however, this method presents difficulties. We therefore adopt another method.

We shall show that the dual of a coincidence point is the tangent at a coincidence point, thus showing that the number of coincidence points is self-dual. Now the dual of a coincidence point P of a curve S is a tangent p to the dual curve Σ such that there is a 3-cusped quartic with p as the bitangent, which has 8-line contact with Σ at one point of contact.

Now it may be shown (I omit the details, which are rather laborious) that the points of contact of the bitangent of a 3-cusped quartic are coincidence points. Hence there is a cubic with a node at such a point, one branch having 8-point contact with the 3-cusped quartic there, and therefore 8-point contact with Σ there (since 8-line contact is equivalent to 8-point contact). It follows that the dual of a coincidence point is the tangent at a coincidence point.

Let the number of coincidence points be $32 m - 40 n + f(\kappa)$. Then it is also $32 n - 40 m + f(\kappa)$. Hence $72(m - n) = f(\iota) - f(\kappa)$. But, from one of Plücker's equations, we have $3(m - n) = \iota - \kappa$. Hence $f(\kappa) = 24 \kappa$, and the number of coincidence points is $32 m - 40 n + 24 \kappa$.

4. With care, the above formula may be applied even for cubic curves. In this case, of course, the cubic is itself one of the family of cubics having 8-point contact at a point P, so that, at a coincidence point, the family of cubics have 9-point with the given cubic.

Special mention should be made of the nodal cubic. The formula gives 8 coincidence points. Now if the curve is expressed in the standard form $x = t$, $y = t^2$, $z = 1 + t^3$, the curve meets any other cubic in nine points t_i such that $\Pi t_i = -1$. If the 9 points coincide, we have $t^9 = -1$. Ignoring the 3 roots given by $t^3 = -1$, which give the inflexions, we have, apparently, only 6 coincidence points. However, the cubics having 8-point contact with one branch at the node meet the given cubic 9 times altogether at the node, so the node counts as two coincidence points, one for each branch. For a general curve, however, a node does not count as a coincidence point.

The work on coincidence points in Hilton's Plane Algebraic Curves (2nd edition) needs some modification. The definition of a coincidence point on p. 254 only applies for a cubic curve, not for curves in general. The result given in ex. 23, p. 257 is incorrect: the error was presumably caused by taking the number of coincidence points of a nodal cubic as 6, but in any case the method used there is suspect.

I wish to thank Professor Simpson for his great help in connection with this paper. He has verified the number of coincidence points for a *rational* curve, using an entirely different method.

REFERENCES

[1] HALPHEN, G. H. Oeuvres 2, p 198.
[2] ibid., p. 205.
[3] SEMPLE, J. G. and ROTH, L. Algebraic Geometry, p. 116.
[4] ibid., p. 372.

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