EXPECTATIONS IN AN OPERATOR ALGEBRA

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Introduction. Let A be a C^*-algebra having the identity. A mapping $\theta(x) = x^{\varepsilon}$ will be called an *expectation* of A if it satisfies

(0.1) $$(\alpha x + \beta y)^{\varepsilon} = \alpha x^{\varepsilon} + \beta y^{\varepsilon},$$

(0.2) $$x^{*\varepsilon} = x^{\varepsilon *},$$

(0.3) $$x \geq 0 \text{ implies } x^{\varepsilon} \geq 0,$$

(0.4) $$(x^{\varepsilon} y)^{\varepsilon} = x^{\varepsilon} y^{\varepsilon} = (x y^{\varepsilon})^{\varepsilon},$$

(0.5) $$1^{\varepsilon} = 1;$$

and will be called *abelian* if it satisfies moreover

(0.6) $$(xy)^{\varepsilon} = (yx)^{\varepsilon}.$$

Many known operations on C^*-algebras can be considered as expections:

EXAMPLE 1. If σ is a *state* (in the sense of I. E. Segal), i. e., a linear functional on A which is positive and normalized, then σ can be considered as an expectation of A which maps A into the field of scalar multiples of the identity: For, (0.1)-(0.3) and (0.5) are obvious and (0.4) follows from $\sigma(x\sigma(y)) = \sigma(x)\sigma(y)$. The trace of A is a *scalar* valued expectation which is abelian on A.

EXAMPLE 2. J. Dixmier's *centering* $\natural$ can be generalized in a C^*-algebra as an expectation of A into the center Z, which is abelian and

(0.7) $$x \in Z \text{ implies } x^{\natural} = x.$$

A (bounded) trace τ on a finite W^*-algebra can be considered the expectation of A which is the combination of a state and the centering, since

(0.8) $$\tau(x) = \tau(x^{\natural})$$

for any x. (Cf. also [5]).

For spaces of functions, the following examples exist:

EXAMPLE 3. Let A be the space of all continuous functions defined on $S \times T$ where S and T are compact spaces. Put

(0.9) $$x^{\varepsilon}(s, t) = \int x(s', t) ds'.$$

Then it is not hard to show that x^{ε} is an expectation of A, since

$$(x^{\varepsilon} y)^{\varepsilon} = \int y(s, t) \int x(s', t) ds' ds = \int x(s', t) ds' \int y(s, t) ds.$$

EXAMPLE 4. Let A be the space of bounded random variables on a probability space (S, F, μ), and let

$$x^{\mathcal{E}} = E\{x|F'\} \tag{0.10}$$

be the *conditional expectation* of x conditioned by a certain fixed subfield F' of F. As in Shuh-Teh Chen Moy [1; Introduction], $\mathcal{E}$ is an expectation of A, considering A as a W^*-algebra on $L^2(S, \mu)$, and moreover $\mathcal{E}$ is sequentially order-continuous. Shuh-Teh Chen Moy proved conversely, that a sequentially order-continuous expectation of A is the conditional expectation under a certain additional condition [1; Thm. 2.2]. Furthermore, she proved, relaxing (0.5), a sequentially order-continuous quasi-expectation $\mathcal{E}$ allows the expression $x^{\mathcal{E}} = E\{xg|F'\}$ where $g \geqq 0$ is an integrable function on S [1; Thm. 1.1].

Thus, it may be expected that the theory of expectation conform several notions of operator algebras. Moreover, recently J. Dixmier [2; Thm. 8] proved the following important theorem: In a finite W^*-algebra A having a faithful normal trace τ, for a given W^*-subalgebra B there exists an expectation $\mathcal{E}$ of A *onto* B such that $\tau(x^{\mathcal{E}}y) = \tau(xy)$ for all $y \in B$. Inspired by Shuh-Teh Chen Moy's results, H. Umegaki [7] found that Dixmier's Theorem is a non-commutative extension of conditional expectations, and he developed his non-commutative probability theory on the basis of this fact. Therefore, the importance of the study of expectations is not only the conformization but also to cut open a wide field for the theory of operator algebras.

In the present note, analogously to Shuh-Teh Chen Moy and contrary to J. Dixmier and H. Umegaki, we shall consider an expectation as a given operation of an operator algebra (whence we do not concern with the existence problem of such operation). In §§1–2, some fundamental properties of an expectation of a C^*-algebra are discussed, among them the notion of the associated subalgebra of an expectation, originally due to Shuh-Teh Chen Moy, is central. In § 3, the effect of the weak closedness of the algebra is observed and as its consequence a generalization of theorems of Chen Moy and Umegaki is obtained. In § 4, dropping (0.5), quasi-expectations will be observed, and a non-commutative version of a theorem of Chen Moy is proved.

1. Associate subalgebra. Let $\mathcal{E}$ be an expectation of a C^*-algebra A. For $\mathcal{E}$, we shall firstly introduce the ***right*** (*left*) ***associate modul*** by

$$D_r = \{a;\ (ax)^{\mathcal{E}} = ax^{\mathcal{E}} \text{ for any } x \in A\}, \tag{1.1}$$

$$D_l = \{a;\ (xa)^{\mathcal{E}} = x^{\mathcal{E}}a \text{ for and } x \in A\}. \tag{1.2}$$

It is not hard to see that D_r and D_l are vector spaces of A, and algebraical (non-self-adjoint) subalgebras of A since $((ab)x)^{\mathcal{E}} = a(bx)^{\mathcal{E}} = (ab)x^{\mathcal{E}}$ for any pair a and b of D_r. By (0.2), D_r and D_l are mutually adjointed, since $(ax)^{\mathcal{E}*} = (x^*a^*)^{\mathcal{E}} = x^{*\mathcal{E}}a^* = (ax^{\mathcal{E}})^*$if $a^* \in D_l$ shows that $a \in D_r$ implies $a^* \in D_l$, and conversely. Moreover, 1 belongs to both of D_r and D_l because $(1x)^{\mathcal{E}} = (1^{\mathcal{E}}x)^{\mathcal{E}} = 1^{\mathcal{E}}x^{\mathcal{E}} = 1x^{\mathcal{E}}$ and dually. Hence $D = D_r \cap D_l$ is a self-adjoint subalgebra of A, which will be called the *associate subalgebra* of the given expectation.

PROPOSITION 1. *The associate subalgebra of an expectation of a C^*-algebra is a C^*-subalgebra containing the identity, which is the totality of fix points under the expectation.*

PROOF. By the uniform continuity of the expectation, it suffices to show that $D = A^{\varepsilon}$ where A^{ε} means the range of the expectation: $A^{\varepsilon} = \{x^{\varepsilon};\ x \in A\}$, since $x^{\varepsilon\varepsilon} = 1^{\varepsilon}x^{\varepsilon} = x^{\varepsilon}$ implies the invariance of elements of D under the expectation by $D = A^{\varepsilon}$.

To see this, if $a \in D$ then $a^{\varepsilon} = (a1)^{\varepsilon} = a1^{\varepsilon} = a$ shows $D \subseteq A^{\varepsilon}$ and if $a^{\varepsilon} \in A^{\varepsilon}$ then $a^{\varepsilon}x^{\varepsilon} = (a^{\varepsilon}x)^{\varepsilon}$ shows $A^{\varepsilon} \subseteq D$. Therefore the Proposition is proved.

The following corollary is immediate by (0. 4) and (0. 6).

COROLLARY 1. 1. *If an expectation is abelian, then the associate subalgebra is commutative.*

Let A^0 be the set of all elements x such that $x^{\varepsilon} = 0$. A^0 will be called *dispersive* subspace. It is obvious that A^0 is a closed linear subspace of A and A is the direct sum (as Banach space) of A and A^0, therefore the expectation is determined on A and A^0. This observation implies obviously the following

COROLLARY 1. 2. *If a C^*-subalgebra B of A becomes associate algebra of an expectation then B has the complementary closed subspace in A.*

The converse implication follows if there exists a projection of A (as a Banach space) onto B which commutes with multiplication by elements of B.

COROLLARY 1. 3. *The commutor D', the set of all elements which commute with D in elementwise, is invariant under the expectation, i. e.,*

$$(1, 3) \qquad a \in D' \text{ implies } a^{\varepsilon} \in D \cap D'.$$

PROOF. If $x \in D$ and $a \in D'$, then $xa^{\varepsilon} = (xa)^{\varepsilon} = (ax^{\varepsilon}) = a^{\varepsilon}x$ implies $a^{\varepsilon} \in D'$, whence Proposition 1 implies the corollary.

By Corollary 1. 3, it is obvious that the expectations of central elements is in the center of the associated algebra; whereas the converse is not true in general.

2. The conjugate expectation. Since an expectation of a C^*-algebra A is a linear transformation of the Banach space A into itself, there exists the conjugate of the expectation which maps A^*, the conjugate space of A, into itself. The conjugate transformation, which is defined

$$(2. 1) \qquad \rho^{\varepsilon}(x) = \rho(x^{\varepsilon})$$

will be called the *conjugate expectation* of A. It is not hard to see that the conjugate expectation is positive and idempotent in the sense that

$$(2. 2) \qquad \rho^{\varepsilon}(x) \geq 0 \text{ if } x \geq 0,$$

$$(2. 3) \qquad \rho^{\varepsilon\varepsilon} = \rho^{\varepsilon}.$$

Since $\rho^{\varepsilon}(1) = \rho(1)$ by the definition, the property that ρ is a state is preserved by the conjugate expectation, whereas the trace property is not preserved.

Since the conjugate expectation is linear, positive and idempotent, naturally A^* is decomposed into the direct sum of $A^{*\mathcal{E}}$ and A^{*0} where

$$A^{*\mathcal{E}} = \{\rho^{\mathcal{E}};\ \rho \in A^*\}, \tag{2.4}$$

$$A^{*0} = \{\rho;\ \rho^{\mathcal{E}} = 0\}. \tag{2.5}$$

Clearly $A^{*\mathcal{E}}$ is the set of all invariant functionals defined on A under the conjugate expectation, whence the existence of such functionals is obvious. Moreover, we shall show that A^* contains at least one state:

PROPOSITION 2. *For any expectation of a C^*-algebra, there exists at least one state which is invariant under the conjugate expectation, i.e.,*

$$\sigma(x^{\mathcal{E}}) = \sigma(x) \tag{2.6}$$

for all x of the algebra.

PROOF. Let σ_0 be a state on D. Clearly by the representation theory for C^*-algebras σ_0 exists. By a theorem of extensions of states due to I. E. Segal σ_0 has an extension σ, which is a state of A, whence $\sigma^{\mathcal{E}}$ is a state of A which is invariant under the conjugate expectation by (2.3). This proves the Proposition.

As a consequence of Proposition 2 and Corollary 1.1, we have easily

COROLLARY 2.1. *For an abelian expectation of a C^*-algebra, there exists a trace which is invariant under the conjugate expectation.*

One of the central problem of the theory of expectations, is to find the conditions for the invariance of the given state under the conjugate expectation. As an answer for this problem, we shall show the following corollary which is a C^*-algebra extension of theorem of Shuh-Teh Chen Moy [1; Thm. 2.2] and H. Umegaki [7].

COROLLARY 2.2. *If σ is a state and*

$$\sigma(x^{\mathcal{E}}) \leq \sigma(x) \tag{2.7}$$

for any $x \geq 0$, then the state σ is invariant.

PROOF. Since σ and $\sigma^{\mathcal{E}}$ are states on the C^*-algebra A, the hypothesis $\sigma^{\mathcal{E}} \leq \sigma$ can not be true unless $\sigma^{\mathcal{E}} = \sigma$. This proves the Corollary.

REMARK. For an abelian expectation, a Bochner type integral representation of the value of expectation is possible. Let X be the spectrum of the associate algebra D, and let $d\tau$ be the induced measure on X by a trace τ on A, where τ is invariant. Then

$$\tau(x) = \int \chi(x) d\tau(\chi). \tag{2.8}$$

This generalizes an integral representation of trace by the centering given in [5].

3. Application to W^*-algebras. In this section we shall assume that A is a W^*-algebra acting on a Hilbert space H. An expectation of A will be called *normal* provided that

(3.1) $$x_\alpha \uparrow x \text{ implies } x_\alpha^\varepsilon \uparrow x^\varepsilon,$$

where $x_\alpha \uparrow x$ means that (x_α) is a directed set of non-decreasing elements of A having x as supremum. The normality of expectations is essential by the following

PROPOSITION 3. *The associate algebra of a normal expectation of W^*-algebra is a W^*-subalgebra.*

PROOF. By a theorem of J. Dixmier [2; Cor. 1] the normality implies the σ-weak continuity of the expectation, where the σ-weak topology of A is defined by the semi-norms $|\Sigma < \xi_i x, \eta_i > |$ for $\Sigma \|\xi_i\|^2 < \infty$ and $\Sigma \|\eta_i\|^2 < \infty$. Hence its fix points forms a σ-weakly closed set which is nothing but the associate algebra by Proposition 1, and so it is a W^*-subalgebra.

COROLLARY 3.1. *The conjugate of a normal expectation of a W^*-algebra preserves the σ-weak continuity of linear functionals, whence σ^ε is a normal state if σ is a normal state.*

PROOF. By Proposition 3, the normality of the expectation preserves the normality of a state because $x_\alpha \uparrow x$ implies $x_\alpha^\varepsilon \uparrow x^\varepsilon$ and the latter implies $\sigma(x_\alpha^3) \uparrow \sigma(\varepsilon x)$. Since any σ-weakly continuous linear functional is the linear combination of normal states and conversely by a theorem of J. Dixmier [2; Thm. 3], the above argument implies our Corollary.

After J. Dixmier, let A_* be the set of all σ-weakly continuous linear functionals of a W^*-algebra A. By a theorem due to him, A is the conjugate space of A_* as a Banach space. Proposition 3 shows that the conjugate expectation maps A_* into itself. This fact shows the following .

COROLLARY 3.2. *An normal expectation of a W^*-algebra is the conjugate transformation of the restriction of the conjugate expectation on the space of all σ-weak continuous linear functionals.*

Basing on Corollary 3.2, we can decompose the space A_* into the fix point subspace and the "dispersive" subspace, and analogously to § 2 the existence of normal states under the expectation. We do not enter this point.

COROLLARY 3.3. *If a W^*-algebra A acting on a Hilbert space H has a separating vector, then for any ϕ in H there exists a vector ψ such that*

(3.2) $$< \phi x^\varepsilon, \phi > = < \psi x, \psi >.$$

PROOF. The hypothesis implies by virtue of J. Dixmier's Theorem [3; Prop. 6] that

(3.3) $$\sigma(x) = < \phi x^\varepsilon, \phi >$$

is expressible in (3.2) because σ is normal by Corollary 3.1.

REMARK. An appropriate application of the Radon-Nikodym Theorem of

Dye [4; Thm. 4] in Corollary 3. 3 implies Umegaki's theorem [7; Thm 3]. For an example, if A is a finite W^*-algebra with a faithful (bounded) normal trace τ, and if A is standard in the sense of I. E. Segal [6] on a Hilbert space H, then for any expectation of A there exists a non-negative "integrable" operator h with the dense domain such that

(3. 4) $$\tau(x^{\mathcal{E}}) = \tau(xh)$$

for all $x \in A$. The h is the Radon-Nikodym derivative of $\tau^{\mathcal{E}}$ with respect to τ. The space and algebra restrictions are used to enjure the Radon-Nikodym Theorem.

Corollary 2. 2 and Proposition 3 implies at once the following theorem, which is a slight generalization of the corresponding theorems of Shuh-Teh Chen Moy [1; Thm. 2. 2] and H. Umegaki [7].

THEOREM. *If an normal expectation $\mathcal{E}$ of a W^*-algebra, and if σ is a normal state with*

(3. 4) $$\sigma(x^{\mathcal{E}}) \leq \sigma(x) \quad \text{for } x \geq 0,$$

then σ is invariant.

For the deduction to Theorems of Chen-Moy and Umegaki, the following equality will be used:

(3. 5) $$\sigma(ex) = \sigma^{\mathcal{E}}(ex) = \sigma((ex)^{\mathcal{E}}) = \sigma(ex^{\mathcal{E}}) \text{ for } e \in D = A^{\mathcal{E}}$$

if σ is invariant.

4. Quasi-expectation. To cover Shuh-Teh Chen Moy's theorem [1; Thm 1. 1], our hypothesis on expectations is too restrictive. For this purpose we need to relax as follows: A mapping $x \to x^{\mathcal{E}}$ on a C^*-algebra A will be called a *quasi-expectation* if it satisfies (0. 1)-(0. 4). In this section we shall study the properties of quasi-expectations. Although the notion of quasi-expectation can be introducing the algebra without the identity, we shall assume the existence of the identity in the algebra for the conveniences.

In the case of a quasi-expectation, the associate subalgebra D can be defined as in §1. As in §1 (§3), the associated subalgebra of a (normal) quasi-expectation of a C^*-(W^*-) algebra is a C^*-(W^*-) subalgebra, by a suitable change of the proof of Proposition 1(3). It is not difficult to see that $A^{\mathcal{E}} \subseteqq D$.

LEMMA 1. *For a quasi-expectation of a C^*-algebra A, the following statements are equivalent:*

(4. 1) $$D = A,$$

(4. 2) $$yx^{\mathcal{E}} = (yx)^{\mathcal{E}} = y^{\mathcal{E}}x,$$

(4. 3) *there exists a central element c such that $x^{\mathcal{E}} = xc = cx$.*

PROOF. Clearly (4. 1) implies (4. 2) by the definition (2. 1)-(2. 2). If (4. 2) is true, then the quasi-expectation becomes an A-endomorphism of the A-modul A, whence $1^{\mathcal{E}} = c$ is required in (4. 3). It is to be noticed that $1^{\mathcal{E}}$ belongs to the center of the associate algebra, which follows from a similar argument to Corollary 1.3. Finally, if (4.3) is true, then $yx^{\mathcal{E}} = y(xc) = (yx)c = (yx)^{\mathcal{E}}$ implies (4.1). This proves the Lemma.

LEMMA 2. *If a quasi-expectation is idempotent, then* $1^{\mathcal{E}}$ *is a central projection.*

PROOF. It remains only to show the idempotency of $1^{\mathcal{E}}$ since (0.2) implies that $1^{\mathcal{E}}$ is hermitean $1^{\mathcal{E}} = 1^{\mathcal{E}\mathcal{E}} = 1^{\mathcal{E}}1^{\mathcal{E}} = 1^{\mathcal{E}^2}$ implies the desired conclusion.

The existence of a non-idempotent quasi-expectation is obviously follows from Lemma 1, if $c^2 \neq c$. Also an example of $A^{\mathcal{E}} \neq D$ for quasi-expectations can be given if c is not regular in Lemma 1. It is not hard to see that $A^{\mathcal{E}} = D$ if and only if $1 \in A^{\mathcal{E}}$.

The following proposition can be seen as a generalization of Chen Moy's Theorem [1, Thm. 1.1]:

PROPOSITION 4. *If A is a* W^**-algebra acting on a Hilbert space having separating vector and if* $\mathcal{E}$ *is normal quasiexpectation on A, then there exists a vector* ψ *such that*

$$< \phi x^{\mathcal{E}}, \phi > \; = \; < \psi x, \psi > \tag{4.4}$$

where ϕ *is an arbitrary given vector in H.*

PROOF. Put $\sigma(x) = < \phi x^{\mathcal{E}}, \phi >$. Since the quasi-expectation is positive and linear, it is easily seen that σ is a positive linear functional which is normal by the normality of the quasi-expectation. Therefore again by J. Dixmier's theorem [3; Prop. 6], there exists a vector ψ with $\sigma(x) = < \psi x, \psi >$. This proves the Proposition.

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