ON THE CESÀRO SUMMABILITY OF FOURIER SERIES II

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1. Introduction Let $f(t)$ be an integrable function with period 2π and let $\varphi(t) = \varphi_x(t) = f(x+t) + f(x-t) - 2s$.

J. J. Gergen's Cesàro summability criterion of Fourier series reads as follows [1]:

THEOREM A. *Let $\varphi_\beta(t)$ be the βth integral of $\varphi(t)$. If*

$$\varphi_\beta(t) = o(t^\beta) \qquad (t \to 0)$$

and

$$\lim_{k\to\infty} \limsup_{u\to 0} u^\rho \int_{ku}^{\pi} \frac{|\Delta_u^{(m)}\varphi(t)|}{t^{1+\rho}}\, dt = 0,$$

then the Fourier series of $f(t)$ is summable (C, ρ) to s at $t = x$, where $-1 < \rho$ and

$$\Delta_u^{(m)}\varphi(t) = \sum_{\nu=0}^{m} (-1)^{m+\nu} \binom{m}{\nu} \varphi(t + \nu u).$$

S. Izumi and G. Sunouchi [2], [7] proved the following theorems:

THEOREM B. *Let $\Delta = \gamma/\beta \geqq 1$. If $\varphi_\beta(t) = o(t^\gamma)$ $(t \to 0)$, and*

$$\int_0^t |d\{u^\Delta \varphi(u)\}| = O(t) \qquad (0 < t < \eta),$$

then the Fourier series of $f(t)$ converges to s at $t = x$.

THEOREM C. *Let $\Delta = \gamma/\beta \geqq 1$. If $\varphi_\beta(t) = o(t^\gamma)$ $(t \to 0)$ and*

$$\lim_{k\to\infty} \limsup_{u\to 0} \int_{(ku)^{1/\Delta}}^{\pi} \frac{|\varphi(t) - \varphi(t+u)|}{t}\, dt = 0,$$

then a Fourier series of $f(t)$ converges to s at $t = x$.

In the previous paper [5], we have proved the following:

THEOREM D. *Let $\Delta \geqq 1$, $-1 < \rho < 1$ and*

$$\gamma = \Delta - \rho(\Delta - 1).$$

If $\varphi_1(t) = o(t^\gamma)$, $(t \to 0)$ and

$$\lim_{k\to\infty} \limsup_{u\to 0} u^\rho \int_{(ku)^{1/\Delta}}^{\pi} \frac{|\Delta_u^{(m)}\varphi(t)|}{t^{1+\rho}}\, dt = 0,$$

then the Fourier series of $f(t)$ is summable (C, ρ) to s at $t = x$.

THEOREM E. *Let $\Delta \geqq 1$, $-1 < \rho < 1$ and*

$$\gamma = \Delta - \frac{2\rho(\Delta - 1)}{1 + \rho}$$

If

$$\varphi_1(u) = o(t^\gamma)$$

and

$$\int_0^t |d\{u^\Delta \varphi(u)\}| = O(t), \tag{1.1}$$

then the Fourier series of $f(t)$ is (C, ρ) summable to s at $t = x$.

Concerning Theorems B and E, recently K. Kanno [4] has proved the following theorem.

THEOREM F. *If $\varphi_\beta(t) = o(t^\gamma)$, $\gamma > \beta > 0$, and the condition (1.1) holds, then the Fourier series of $f(t)$ is (C, ρ) summable to s at $t = x$, where*

$$\Delta \geqq \gamma/\beta$$

and

$$\rho = \frac{\Delta\beta - \gamma}{\Delta + \gamma - \beta - 1},$$

that is,

$$\gamma = \Delta\beta - \frac{\rho(\beta + 1)(\Delta - 1)}{1 + \rho} \quad \text{and} \quad \rho \geqq 0.$$

In this paper we shall prove the following theorems.

THEOREM 1. *Let $\Delta \geqq 1, 1 > \rho \geqq 0$, $\gamma \geqq \beta > 0$ and*

$$\gamma = \Delta\beta - \rho(\Delta - 1).$$

If

$$\varphi_\beta(t) = o(t^\gamma) \qquad (t \to 0)$$

and

$$\lim_{u \to 0} u^\rho \int_{u^{1/\Delta}}^\pi \frac{|\Delta_u^{(m)} \varphi(t)|}{t^{1+\rho}} dt = 0, \tag{1.2}$$

then the Fourier series of $f(t)$ is summable (C, ρ) to s at $t = x$.
If $\beta = \rho$ (i.e. $\gamma = \beta = \rho$), then we suppose $\Delta = 1$.

THEOREM 2. *In Theorem 1, if $-1 < \rho \leqq 0$, then (1.2) may be replaced by*

$$\lim_{k \to \infty} \limsup_{u \to 0} u^\rho \int_{(ku)^{1/\Delta}}^\pi \frac{|\Delta_u^{(m)} \varphi(t)|}{t^{1+\rho}} dt = 0. \tag{1.2*}$$

THEOREM 3. *Let $\Delta \geqq 1$, $\rho > -1$, $\gamma \geqq \beta > 0$ and*

$$\gamma = \Delta\beta - \frac{\rho(\beta+1)(\Delta-1)}{1+\rho}.$$

If

$$\varphi_\beta(t) = o(t^\gamma) \qquad (t \to 0)$$

and

$$\int_0^t |d\{u^\Delta \varphi(u)\}| = O(t),$$

then the Fourier series of $f(t)$ is summable (C, ρ) to s at $t = x$.

2. **Proof of Theorem 1.** In our theorem, if we put $\Delta = \gamma/\beta$, we have $\rho = 0$. Hence, this case is Theorem C. The case $\Delta = 1$ and the case $\gamma = \beta$ are Theorem A. Therefore it is sufficient to prove the theorem in case of $\gamma > \beta$, $\Delta > 1$, $1 > \rho > 0$. The method of proof is analogous to those of Gergen [1] and Izumi and Sunouchi [3].

For the proof of our theorem, we need several lemmas.

Let us donote by $K_n^\rho(t)$ the n-th Cesàro mean of order ρ of the series $\frac{1}{2} + \sum_{k=1}^\infty \cos kt$. Then we have

LEMMA 1 (*cf.* GERGEN [1], LEMMA 6). *If we suppose $-1 < \rho \leqq 1$, then*

$$K_n^\rho(t) = S_n^\rho(t) + R_n^\rho(t), \tag{2.1}$$

where

$$S_n^\rho(t) = \frac{\cos(A_n t + A)}{A_n^\rho (2\sin t/2)^{1+\rho}}, \quad A_n = n + (\rho+1)/2, \quad A = -(\rho+1)\pi\ 2, \tag{2.2}$$

$$|R_n^\rho(t)| \leqq \frac{M}{nt^2}, \quad \left|\frac{d}{dt} R_n^\rho(t)\right| < \frac{M}{nt^3} + \frac{M}{n^2 t^4}; \tag{2.3}$$

and

$$\left|\left(\frac{d}{dt}\right)^h K_n^\rho(t)\right| \begin{cases} \leqq M n^{h+1}, & \text{for } h \geqq 0, \\ \leqq M n^{h-\rho} t^{-1-\rho}, & \text{for } nt \geqq 1,\ h \geqq 0 \text{ and } 0 < \rho \leqq 1. \end{cases} \tag{2.4}$$

LEMMA 2 (*cf.* GERGEN [1], LEMMA 7). *If $x^{1/\Delta} \leqq v$, then*

$$\int_{x^{1/\Delta}}^v |\Delta_x^{(r+m)} \varphi_r(t)|\, dt \leqq x^r (v + rx)^{1+\rho} \int_{x^{1/\Delta}}^{v+rx} \frac{|\Delta_x^{(m)} \varphi(t)|}{t^{1+\rho}}\, dt$$

for every pair of integers $r \geqq 0$ and $m \geqq 1$.

LEMMA 3. *Under the assumption of the theorem, we have*

$$\varphi_r(t) = o(t^{1+(r-1)\Delta - \rho(\Delta-1)}), \qquad (t \to 0),$$

where r is an integer such that $1 \leqq r \leqq [\beta] + 1$.

PROOF. Let β be non-integral and $\mu = [\beta] + 1$. Then, by the assumption, we have

$$\varphi_\mu(t) = o(t^{\gamma + (\mu - \beta)}),$$

hence

$$\varphi_\mu(t) = o(t^{1+(\mu-1)\Delta-\rho(\Delta-1)}),$$

since

$$\gamma + (\mu - \beta) - \{1 + (\mu - 1)\Delta - \rho(\Delta - 1)\}$$
$$= (\beta - [\beta])(\Delta - 1) > 0.$$

Therefore it is sufficient to prove that $\varphi_{r+1}(t) = o(t^{\xi})$ imply $\varphi_r(t) = o(t^{\xi-\Delta})$, where $r \geqq 1$ and $\xi = 1 + r\Delta - \rho(\Delta - 1)$. Let us put $R = m + r - 1$, $h = 1/(R+1)^{\Delta}$, and $h_1 = 1/\{(R+1)^{\Delta} + 1\}$. We shall consider the integral

$$\int_{h_1 x^\Delta}^{hx^\Delta} dt \int_{t^{1/\Delta}}^{x-Rt} \Delta_t^{(R)}\varphi_{r-1}(u)\, du = \eta.$$

By the definition, we have

$$\eta = \int_{h_1 x^\Delta}^{hx^\Delta} dt \int_{t^{1/\Delta}}^{x-Rt} \left\{\sum_{\nu=0}^{R} (-1)^\nu \binom{R}{\nu} \varphi_{r-1}(u+\nu t)\right\} du$$
$$= (-1)^R (h - h_1) x^\Delta \varphi_r(x) + \eta^*,$$

where η^* is the linear combination of φ_{r+1}.

On the other hand, by Lemma 2, η is majorated by

$$\int_{h_1 x^\Delta}^{hx^\Delta} t^{r-1}(x - mt)^{1+\rho}\, dt \int_{t^{1/\Delta}}^{x-mt} \frac{|\Delta_t^{(m)}\varphi(u)|}{u^{1+\rho}}\, du$$
$$\leqq \bar{\eta}_{\rho,\Delta}^{(m)} \cdot (h^{r-\rho} - h_1^{r-\rho})\, x^{1+r\Delta-\rho(\Delta-1)},$$

where

$$\bar{\eta}_{\rho,\Delta}^{(m)} = \operatorname*{least\ upper\ bd.}_{h_1 x^\Delta \leqq t \leqq h x^\Delta} \left\{ t^\rho \int_{t^{1/\Delta}}^{\pi} \frac{|\Delta_t^{(m)}\varphi(u)|}{u^{1+\rho}}\, du \right\}.$$

Hence we have

$$\varphi_r(x) = o(x^{\xi-\Delta}),$$

which is the required result.

In what follows, we put $y = \pi/A_n = \pi/\{n + (\rho+1)/2\}$. Then

LEMMA 4. *Under the assumption of the theorem, we have*

$$I = \int_0^{y^{1/\Delta}+\nu y} \varphi(t) K_n^\rho(t)\, dt = o(1), \qquad (n \to \infty),$$

where ν is a positive integer.

PROOF. We may replace by $y^{1/\Delta}$ the upper limit of the above integral. There is an integer μ such that $\mu - 1 < \beta \leqq \mu$. We may suppose that $\mu - 1 < \beta < \mu$, since the case $\mu = \beta$ can be easily deduced by the following argument. By μ times application of integration by parts, we get

$$I=\sum_{h=1}^{\mu}(-1)^h\Big[\varphi_h(t)\Big(\frac{d}{dt}\Big)^{h-1}K_n^{\rho}(t)\Big]_0^{y^{1/\Delta}}+(-1)^{\mu}\int_0^{y^{1/\Delta}}\varphi_\mu(t)\Big(\frac{d}{dt}\Big)^{\mu}K_n^{\rho}(t)\,dt$$

$$=\sum_{h=1}^{\mu}(-1)^h I_h+(-1)^{\mu}I_{\mu+1}.$$

By Lemma 3 and (2.5),

$$I_h=o\{n^{h-1-\rho}[t^{(h-1)\Delta-\rho\Delta}]_{t=y^{1/\Delta}}\}=o(1).$$

On the other hand,

$$\Gamma(\mu-\beta)I_{\mu+1}=\Gamma(\mu-\beta)\int_0^{y^{1/\Delta}}\varphi_\mu(t)\Big(\frac{d}{dt}\Big)^{\mu}K_n^{\rho}(t)\,dt$$

$$=\int_0^{y^{1/\Delta}}\Big(\frac{d}{dt}\Big)^{\mu}K_n^{\rho}(t)\,dt\int_0^{t}\varphi_\beta(u)(t-u)^{\mu-\beta-1}\,du$$

$$=\int_0^{y^{1/\Delta}}\varphi_\beta(u)\,du\int_u^{y^{1/\Delta}}\Big(\frac{d}{dt}\Big)^{\mu}K_n^{\rho}(t)(t-u)^{\mu-\beta-1}\,dt$$

$$=\int_0^{y}du\int_u^{u+y}dt+\int_y^{y^{1/\Delta}}du\int_u^{u+y}dt+\int_0^{y^{1/\Delta}-y}du\int_{u+y}^{y^{1/\Delta}}dt-\int_{y^{1/\Delta}-y}^{y^{1/\Delta}}du\int_{y^{1/\Delta}}^{u+y}dt$$

$$=J_1+J_2+J_3-J_4,$$

say, where

$$J_1=O\Big\{n^{\mu+1}\int_0^{y}|\varphi_\beta(u)|\Big[(t-u)^{\mu-\beta}\Big]_u^{u+y}du\Big\}=o(n^{\beta-\gamma})=o(1),$$

$$J_2=\int_y^{y^{1/\Delta}}\varphi_\beta(u)\,du\int_u^{u+y}\Big(\frac{d}{dt}\Big)^{\mu}K_n^{\rho}(t)(t-u)^{\mu-\beta-1}\,dt$$

$$=o\Big\{n^{\mu-\rho}\int_y^{y^{1/\Delta}}u^{\gamma-1-\rho}\,du\int_u^{u+y}(t-u)^{\mu-\beta-1}\,dt\Big\}$$

$$=o(n^{\beta-\gamma})+o(n^{\beta-\rho-\frac{1}{\Delta}(\gamma-\rho)})=o(1),\ \text{since } \gamma=\Delta\beta-\rho(\Delta-1).$$

Integrating by parts,

$$J_3=\int_0^{y^{1/\Delta}-y}\varphi_\beta(u)\,du\int_{u+y}^{y^{1/\Delta}}\Big(\frac{d}{dt}\Big)^{\mu}K_n^{\rho}(t)(t-u)^{\mu-\beta-1}\,dt$$

$$=\int_0^{y^{1/\Delta}-y}\varphi_\beta(u)\,du\Big\{\Big[\Big(\frac{d}{dt}\Big)^{\mu-1}K_n^{\rho}(t)(t-u)^{\mu-\beta-1}\Big]_{u+y}^{y^{1/\Delta}}$$

$$-(\mu-\beta-1)\int_{u+y}^{y^{1/\Delta}}\Big(\frac{d}{dt}\Big)^{\mu-1}K_n^{\rho}(t)(t-u)^{\mu-\beta-2}\,dt\Big\}=J_3^1-(\mu-\beta-1)J_3^2,$$

say. We have

$$J_3^1 = O\left\{\int_0^{y^{1/\Delta}-y} |\varphi_\beta(u)|\, du\left(n^{\mu-1-\rho}[t^{-1-\rho}(t-u)^{\mu-\beta-1}]_{t=y^{1/\Delta}}\right)\right\}$$

$$+ O\left\{\int_0^{y^{1/\Delta}-y} |\varphi_\beta(u)| du\left[\left(\frac{d}{dt}\right)^{\mu-1} K_n^\rho(t)(t-u)^{\mu-\beta-1}\right]_{t=u+y}\right\}$$

$$= J_3^{1,1} + J_3^{1,2}.$$

$$J_3^{1,1} = o\left\{n^{\mu-1-\rho+\frac{1}{\Delta}(1+\rho)}(y^{1/\Delta} - y^{1/\Delta} + y)^{\mu-\beta-1}\int_0^{y^{1/\Delta}} u^\gamma\, du\right\}$$

$$= o(n^{-\rho+\beta+\frac{1}{\Delta}(\rho-\gamma)}) = o(1).$$

$$J_3^{1,2} = \int_0^{1/n} + \int_{1/n}^{y^{1/\Delta}-y} = J_3^{1,2,1} + J_3^{1,2,2},$$

say, where

$$J_3^{1,2,1} = O\left\{n^{\mu-(\mu-\beta-1)}\int_0^{1/n} |\varphi_\beta(u)|\, du\right\} = o(n^{\beta-\gamma}) = o(1)$$

and

$$J_3^{1,2,2} = O\left\{n^{\mu-1-\rho}\int_{1/n}^{y^{1/\Delta}-y}\left(|\varphi_\beta(u)|\, [t^{-1-\rho}(t-u)^{\mu-\beta-1}]_{t=u+y}\right) du\right\}$$

$$= o\left\{n^{\mu-1-\rho-(\mu-\beta-1)}\int_{1/n}^{y^{1/\Delta}} u^{\gamma-1-\rho}\, du\right\} = o(n^{\beta-\gamma}) + o(n^{\beta-\rho-\frac{1}{\Delta}(\gamma-\rho)}) = o(1).$$

Thus we get $J_3^{1,2} = o(1)$ and hence $J_3^1 = o(1)$.

We shall now estimate J_3^2.

$$J_3^2 = \int_0^{y^{1/\Delta}-y} \varphi_\beta(u)\, du \int_{u+y}^{y^{1/\Delta}} \left(\frac{d}{dt}\right)^{\mu-1} K_n^\rho(t)(t-u)^{\mu-\beta-2}\, dt$$

$$= \int_0^{1/n} du + \int_{1/n}^{y^{1/\Delta}-y} du = J_3^{2,1} + J_3^{2,2},$$

say, where

$$J_3^{2,2} = O\left\{n^{\mu-1-\rho}\int_{1/n}^{y^{1/\Delta}-y} |\varphi_\beta(u)| du \int_{u+y}^{y^{1/\Delta}} t^{-1-\rho}(t-u)^{\mu-\beta-2}\, dt\right\}$$

$$= o\left\{n^{\mu-1-\rho}\int_{1/n}^{y^{1/\Delta}-y} u^{\gamma-1-\rho}\left[(t-u)^{\mu-\beta-1}\right]_{u+y}^{y^{1/\Delta}} du\right\}$$

$$= o\left\{ n^{\beta-\rho} \int_{1/n}^{y^{1/\Delta}-y} u^{\gamma-1-\rho}\, du \right\} = o(n^{\beta-\gamma}) + o(n^{\beta-\rho-\frac{1}{\Delta}(\gamma-\rho)}) = o(1).$$

and

$$J_3^{2,1} = O\left\{ n^{\mu} \int_0^{1/n} |\varphi_\beta(u)| \Big[(t-u)^{\mu-\beta-1} \Big]_{u+y}^{y^{1/\Delta}} du \right\}$$

$$= o\left\{ n^{\mu}(y^{1/\Delta} - 1/n)^{\mu-\beta-1} \int_0^{1/n} u^{\gamma}\, du \right\} + o(n^{\beta-\gamma})$$

$$= o\left\{ n^{\mu-\frac{1}{\Delta}(\mu-\beta-1)-(\gamma+1)} \right\} + o(n^{\beta-\gamma}) = o(1),$$

since the exponent of the first term is less than

$$([\beta] - \beta)\,(\Delta - 1)/\Delta \leqq 0.$$

Thus we get $J_3^2 = o(1)$. Accordingly we have $J_3 = o(1)$. By the similar way, we get $J_4 = o(1)$.

Collecting above estimations, we get Lemma 4.

LEMMA 5. *Under the assumption of the theorem, we have*

$$\int_{y^{1/\Delta}}^{\pi+\xi y} \varphi(t)\, R_n^{\rho}(t)\, dt = o(1), \qquad (n \to \infty),$$

where ξ is an integer.

PROOF. By Lemma 3, we have $\varphi_1(t) = o(t^{1-\rho(\Delta-1)})$. Using this and integration by parts, we get

$$\int_{y^{1/\Delta}}^{\pi+\xi y} \varphi(t) R_n^{\rho}(t)\, dt = \Big[\varphi_1(t) R_n^{\rho}(t) \Big]_{y^{1/\Delta}}^{\pi+\xi y} - \int_{y^{1/\Delta}}^{\pi+\xi y} \varphi_1(t) \frac{d}{dt} R_n^{\rho}(t)\, dt$$

$$= R_1 - R_2,$$

say, where by (2. 3)

$$R_1 = o(1) + o\{n^{-1}[t^{1-\rho(\Delta-1)-2}]_{t=y^{1/\Delta}}\} = o(n^{-(\Delta-1)(1-\rho)/\Delta}) = o(1)$$

and

$$R_2 = o\left\{ \int_{y^{1/\Delta}}^{\pi} t^{1-\rho(\Delta-1)}[n^{-1}t^{-3} + n^{-2}t^{-4}]\, dt \right\}$$

$$= o\{n^{-1}[t^{-1-\rho(\Delta-1)}]_{t=y^{1/\Delta}}\} + o\{n^{-2}[t^{-2-\rho(\Delta-1)}]_{t=y^{1/\Delta}}\}$$

$$= o(n^{-(\Delta-1)(1-\rho)/\Delta}) + o(n^{-(\Delta-1)(2-\rho)/\Delta}) = o(1).$$

LEMMA 6. *Under the assumption of the theorem, we have*

$$T = \frac{1}{A_n^{\rho}} \int_{y^{1/\Delta}}^{\pi-my} \varphi(t+\nu y)\, \omega(t,y) \cos(A_n t + A)\, dt = o(1),$$

as $n \to \infty$, *where* m *and* ν *are integers such that* $1 \leqq \nu \leqq m$, *and*

$$\omega(t,y) = \frac{2m}{\{\sin(t+\nu y)/2\}^{1+\rho}} - \frac{2m-\nu}{(\sin t/2)^{1+\rho}} - \frac{\nu}{\{\sin(t+2my)/2\}^{1+\rho}}.$$

Proof. We need the following inequalities

$$\omega(t,y) = O(y^2 t^{-3-\rho}), \qquad \frac{\partial\omega}{\partial t} = O(y^2 t^{-4-\rho}),$$

which is Lemma 13 in Gergen [1].

Integrating by parts, we get

$$T = \frac{1}{A_n^\rho}\Big[\varphi_1(t+\nu y)\,\omega(t,y)\cos(A_n t + A)\Big]_{y^{1/\Delta}}^{\pi - my}$$

$$- \frac{1}{A_n^\rho}\int_{y^{1/\Delta}}^{\pi-my} \varphi_1(t+\nu y)\frac{\partial\omega}{\partial t}\cos(A_n t + A)\,dt$$

$$+ \frac{A_n}{A_n^\rho}\int_{y^{1/\Delta}}^{\pi-my} \varphi_1(t+\nu y)\,\omega(t,y)\sin(A_n t + A)\,dt$$

$$= T_1 - T_2 + T_3,$$

say, where

$$T_1 = o\{n^{-\rho-2}[t^{1-\rho(\Delta-1)-3-\rho}]_{t=y^{1/\Delta}}\} = o\{n^{-2(\Delta-1)/\Delta}\} = o(1),$$

$$T_2 = o\left\{ n^{-\rho-2}\int_{y^{1/\Delta}}^{\pi} t^{1-\rho(\Delta-1)-4-\rho}\,dt \right\} = o\left\{ n^{-\rho-2}[t^{-2-\rho\Delta}]_{t=y^{1/\Delta}} \right\} = o(1)$$

and

$$T_3 = o\left\{ n^{1-\rho-2}\int_{y^{1/\Delta}}^{\pi} t^{1-\rho(\Delta-1)-3-\rho}\,dt \right\} = o\left\{ n^{-(\Delta-1)/\Delta} \right\} = o(1).$$

Thus we get the lemma.

LEMMA 7. *If* (1.2) *holds for an integer* $m \geqq 1$, *then the relation* (1.2) *is still valid when* m *is replaced by* m' $(m' \geqq m)$.

Proof runs similarly as Lemma 14 in Gergen [1].

Using above lemmas, we shall now prove the Theorem 1.

We denote by $\sigma_n^\rho(x)$ the nth Cesàro mean of order ρ of the Fourier series of $f(t)$ at the point x. After Gergen, we have

$$2^{2m-1}\pi[\sigma_n^\rho(x) - s]$$

$$= \sum_{\nu=0}^{2m}\binom{2m}{\nu}\left\{\int_0^{y^{1/\Delta}+\nu y} + \int_{y^{1/\Delta}+\nu y}^{\pi+(\nu-m)y} + \int_{\Delta+\pi(\nu-m)y}^{\pi}\right\}\varphi(t)\,K_n^\rho(t)\,dt$$

$$= Q_1 + Q_2 + Q_3,$$

say, where $Q_1 = o(1)$ by Lemma 4 and $Q_3 = 0$, since $\varphi(t)K_n^\rho(t)$ is an even periodic function. Accordingly it is sufficient for the proof to show that

$Q_2 = o(1)$:

$$Q_2 = \sum_{\nu=0}^{2m}\binom{2m}{\nu}\int_{y^{1/\Delta}+\nu y}^{\pi+(\nu-m)y} \varphi(t)\, S_n^\rho(t)\, dt + \sum_{\nu=0}^{2m}\binom{2m}{\nu}\int_{y^{1/\Delta}+\nu y}^{\pi+(\nu-m)y} \varphi(t)\, R_n^\rho(t)\, dt$$

$$= Q_4 + Q_5,$$

say. By Lemma 5, we have $Q_5 = o(1)$. Concerning Q_4, we get

$$Q_4 = \frac{1}{2^{1+\rho}A_n^\rho}\left\{\int_{y^{1/\Delta}}^{\pi-my} \frac{\Delta_y^{(2m-1)}\varphi(t+y)}{\{\sin(t+2my)/2\}^{1+\rho}}\cos(A_n t + A)\, dt\right.$$

$$-\int_{y^{1/\Delta}}^{\pi-my} \frac{\Delta_y^{(2m-1)}\varphi(t)}{(\sin t/2)^{1+\rho}}\cos(A_n t + A)\, dt$$

$$\left.+\sum_{\nu=1}^{2m-1}\frac{(-1)^\nu}{2m}\binom{2m}{\nu}\int_{y^{1/\Delta}}^{\pi-my} \varphi(t+\nu y)\,\omega(t,y)\cos(A_n t + A)\, dt\right\}.$$

Hence, by the assumption of the theorem and Lemmas 6 and 7, we get $Q_4 = o(1)$. Thus the theorem is completely proved.

3. Proof of Theorem 2. It is sufficient to consider the case $-1 < \rho < 0$. For this purpose we need some lemmas.

LEMMA 8.

$$\left(\frac{d}{dt}\right)^r S_n^\rho(t) = O(n^{r-\rho}t^{-1-\rho}), \qquad \textit{for } nt \geqq 1.$$

Proof is easy.

LEMMA 9. *If* $(kx)^{1/\Delta} \leqq v$, *then*

$$\int_{(kx)^{1/\Delta}}^{v} |\Delta_x^{(r+m)}\varphi_r(t)|\, dt \leqq x^r(v+rx)^{1+\rho}\int_{(kx)^{1/\Delta}}^{v+rx} \frac{|\Delta_v^{(m)}\varphi(t)|}{t^{1+\rho}}\, dt$$

for every pair of integers $r \geqq 0$ *and* $m \geqq 1$.

LEMMA 10. *If* $\varphi_\beta(t) = o(t^\gamma)$ *and*

$$\lim_{k\to\infty}\limsup_{x\to 0}\eta_{\rho,\Delta}^{(m)}(x,k) = \lim_{k\to\infty}\limsup_{x\to 0} x^\rho\int_{(kx)^{1/\Delta}}^{\pi} \frac{|\Delta_x^{(m)}\varphi(t)|}{t}\, dt = 0$$

for $0 > \rho > -1$, *then*

$$\varphi_r(t) = o(t^{1+(r-1)\Delta-\rho(\Delta-1)}),$$

where $1 \leqq r \leqq [\beta]+1$.

PROOF. It is sufficient to prove that if $\varphi_{r+1}(t) = o(t^\xi)$ for $r \geqq 1$, then $\varphi_r(t) = o(t^{\xi-\Delta})$, where $\xi = 1 + r\Delta - \rho(\Delta-1)$.

Let us put $R = m + r - 1$, $h = 1/(R + k^{1/\Delta})^\Delta$ and $h_1 = 1/\{(R+k^{1/\Delta})^\Delta + 1\}$. By the method of the proof of Lemma 3, we have

$$\frac{|\varphi_r(x)|}{x^{\xi-\Delta}} \leqq o(1) + \frac{h^{r-\rho} - h_1^{r-\rho}}{h - h_1}\Big(\underset{h_1 x^\Delta \leqq t \leqq h x^\Delta}{\text{least upper bd.}} \ \eta^{(m)}_{\rho,\Delta}(t,k)\Big).$$
$$= \breve{o}(1).$$

Thus we have $\varphi_r(x) = o(x^{\xi-\Delta})$.

LEMMA 11. *Under the assumption of the theorem, we have*

$$\int_{ky}^{(ky)^{1/\Delta}+\nu y} \varphi(t)\, S^\rho_n(t)\, dt = o(1), \qquad (n \to \infty),$$

where $-1 < \rho < 0$.

By using Lemma 10 instead of Lemma 1, the proof runs similarly as in the proof of Lemma 4.

LEMMA 12. *If* $\varphi_1(t) = o(t)$, *then*

$$\lim_{n\to\infty} \int_0^{ky} \varphi(t)\, K^\rho_n(t)\ dt = 0,$$

for $-1 < \rho \leqq 1$.

LEMMA 13. *If* $\varphi_1(t) = o(t)$, *then*

$$\lim_{n\to\infty} \int_{ky}^{\pi+\xi y} \varphi(t)\, R^\rho_n(t)\ dt = 0.$$

LEMMA 14. *If* $\varphi_1(t) = o(t)$, *then*

$$\lim_{n\to\infty} \frac{1}{A^\rho_n} \int_{(ky)^{1/\Delta}}^{\pi-my} \varphi(t+\nu y)\, \omega(t,y) \cos(A_n t + A)\ dt = 0.$$

LEMMA 15. *If* $(1.2)^*$ *holds for an integer* $m \geqq 1$, *then the relation* $(1.2)^*$ *is still valid when* m *is replaced by* m' $(m' \geqq m)$.

We shall now prove Theorem 2. We have

$$2^{2m-1}\pi\{\sigma^\rho_n(x) - s\} = \sum_{\nu=0}^{2m} \binom{2m}{\nu} \int_0^{ky} \varphi(t)\, K^\rho_n(t)\ dt$$
$$+ \sum_{\nu=0}^{2m} \binom{2m}{\nu} \int_{ky}^{\pi+(\nu-m)y} \varphi(t) R^\rho_n(t)\, dt + \sum_{\nu=0}^{2m} \binom{2m}{\nu} \int_{ky}^{(ky)^{1/\Delta}+\nu y} \varphi(t)\, S^\rho_n(t)\, dt$$
$$+ \sum_{\nu=0}^{2m} \binom{2m}{\nu} \int_{(ky)^{1/\Delta}+\nu y}^{\pi-(\nu-m)y} \varphi(t)\, S^\rho_n(t)\, dt + \sum_{\nu=0}^{2m} \binom{2m}{\nu} \int_{\pi+(\nu-m)y}^{\pi} \varphi(t)\, K^\rho_n(t)\, dt$$
$$= Q_1 + Q_2 + Q_3 + Q_4 + Q_5,$$

say, where $Q_1 = o(1)$ by Lemma 12, $Q_2 = o(1)$ by Lemma 13, $Q_3 = o(1)$ by Lemma 11 and $Q_5 = 0$. By the same method as in the proof of Theorem 1, we get $Q_4 = \breve{o}(1)$.

4. **Proof of Theorem 3.** It is sufficient to prove the theorem for the case $-1<\rho<0$, because the case $\rho \geqq 0$ is the Sunouchi–Kanno theorem.

Since $\varphi(t)=O(t^{1-\Delta})$ by (1. 1'), we have by the convexity theorem due to Sunouchi [8],

$$
\begin{aligned}
&\varphi_h(t)=o(t^{\{(\beta-h)(1-\Delta)+h\gamma\}/\beta}), \qquad (h=1,2,\ldots,\mu-1)\\
(4.1)\qquad &\\
&\varphi_\mu(t)=o(t^{\gamma-\beta+\mu}),
\end{aligned}
$$

where μ is an integer such that $\mu-1<\beta<\mu$. If $\Delta=1$, then we have $\varphi_1(t)=o(t^{\gamma/\beta})=o(t)$. Hence the case $\Delta=1$ is the Hardy-Littlewood theorem. Therefore we may suppose $\Delta>1$. Under the these assumptions we shall now prove Theorem 3.

We have

$$
\begin{aligned}
\pi\{\sigma_n^\rho(x)-s\} &= \int_0^\pi \varphi(t)K_n^\rho(t)\,dt\\
&= \int_0^{k/n}\varphi(t)K_n^\rho(t)\,dt+\int_{k/n}^\pi \varphi(t)R_n^\rho(t)\,dt+\int_{k/n}^\pi \varphi(t)S_n^\rho(t)\,dt\\
&= J_1+J_2+J_3.
\end{aligned}
$$

Since $\varphi_1(t)=o(t)$, by Lemmas 12 and 13 we get $J_1=o(1)$, $J_2=o(1)$. Concerning J_3, we put

$$
J_3=\int_{k/n}^{(k/n)^\delta}\varphi(t)S_n^\rho(t)\,dt+\int_{(k/n)^\delta}^\pi \varphi(t)S_n^\rho(t)\,dt=J_4+J_5,
$$

where

$$
\delta=\frac{1+\rho}{\Delta+\rho}=\frac{1+\beta}{\Delta+\gamma}=\frac{\beta-\rho'}{\gamma-\rho}<1
$$

Similarly as in the proof of Theorem 2 in the author's paper [5], we have $J_5=o(1)$.

By μ times application of integration by parts,

$$
\begin{aligned}
J_4 &= \sum_{h=1}^{\mu}(-1)^{h+1}\left[\varphi_h(t)\left(\frac{d}{dt}\right)^{h-1}S_n^\rho(t)\right]_{k/n}^{(k/n)^\delta}+(-1)^\mu\int_{k/n}^{(k/n)^\delta}\varphi_\mu(t)\left(\frac{d}{dt}\right)^\mu S_n^\rho(t)\,dt\\
&= \sum_{h=1}^{\mu}(-1)^{h+1}I_h+(-1)^\mu I_{\mu+1},
\end{aligned}
$$

say. By (4. 1) and Lemma 8, we get, for $h\leqq\mu-1$,

$$
\begin{aligned}
I_h &= o\left\{n^{h-1-\rho}\left[t^{\{(\beta-h)(1-\Delta)+h\gamma\}/\beta-1-\rho}\right]_{k/n}^{(k/n)^\delta}\right\}\\
&= o(n^{h-1-\rho-\delta[\{(\beta-h)(1-\Delta)+h\gamma\}/\beta-1-\rho]})+o(n^{h-1-\rho-\{(\beta-h)(1-\Delta)+h\gamma\}/\beta+1+\rho}),
\end{aligned}
$$

where the exponent of the first term is

$$
-h(\gamma-\beta+\Delta-1)/\beta(\Delta+\gamma)<0
$$

and the exponent of the second term is

$$-h(\gamma-\beta\Delta+\Delta-1)/\beta<0,$$

since $\gamma-\beta\Delta>0$. Hence we get $I_h=o(1)$ for $h\leqq\mu-1$. Concerning I_μ, we have

$$I_\mu=\left[\varphi_\mu(t)\left(\frac{d}{dt}\right)^{\mu-1}S_n^\rho(t)\right]_{k/n}^{(k/n)^\delta}=o\left\{n^{\mu-1-\rho}\left[t^{\gamma-\beta+\mu-1-\rho}\right]_{k/n}^{(k/n)^\delta}\right\}$$
$$=o(n^{\mu-1-\rho-\delta[\gamma+\mu-\beta-1-\rho]}),$$

where the exponent of n is

$$(\Delta-1)(\mu-\beta-1)/(\Delta+\rho)<0,$$

since $1+\rho=(\beta+1)(\Delta-1)/(\gamma+\Delta-\beta-1)$. Thus we have $I_\mu=o(1)$.

Concerning $I_{\mu+1}$, we devide it in four parts;

$$I_{\mu+1}=\int_{k/n}^{(k/n)^\delta}\left(\frac{d}{dt}\right)^\mu S_n^\rho(t)\,dt\int_0^t\varphi_\beta(t)(t-u)^{\mu-\beta-1}\,du$$
$$=\int_0^{k/n}\varphi_\beta(u)\,du\int_{k/n}^{u+k/n}\left(\frac{d}{dt}\right)^\mu S_n^\rho(t)(t-u)^{\mu-\beta-1}\,dt+\int_{k/n}^{(k/n)^\delta}du\int_u^{u+k/n}dt$$
$$+\int_0^{(k/n)^\delta-k/n}du\int_{u+k/n}^{(k/n)^\delta}dt-\int_{(k/n)^\delta-k/n}^{(k/n)^\delta}du\int_{(k/n)^\delta}^{u+k/n}dt$$
$$=J_1+J_2+J_3-J_4,$$

The method of the estimation of J_i is similar to one of the proof of Theorem 1. For example, we shall show that $J_2=o(1)$;

$$J_2=\int_{k/n}^{(k/n)^\delta}\varphi_\beta(u)\,du\int_u^{u+k/n}\left(\frac{d}{dt}\right)^\mu S_n^\rho(t)(t-u)^{\mu-\beta-1}\,dt$$
$$=o\left\{n^{\mu-\rho}\int_{k/n}^{(k/n)^\delta}u^{\gamma-1-\rho}\,du\int_u^{u+k/n}(t-u)^{\mu-\beta-1}\,dt\right\}=o\left\{n^{\mu-\rho-(\mu-\beta)}\left[u^{\gamma-\rho}\right]_{k/n}^{(k/n)^\delta}\right\}$$
$$=o(n^{\beta-\rho-\delta(\gamma-\rho)}),$$

where the exponent of n is

$$\beta-\rho-\delta(\gamma-\rho)=\beta-\rho-\frac{(\beta-\rho)}{(\gamma-\rho)}(\gamma-\rho)=0$$

Thus we have $J_2=o(1)$.

5. **Remark.** As we remarked in our previous paper [5], Theorem 1 in case of $\rho>0$ has the meaning when

$$0<\rho<1/(\Delta-1)$$

and Theorem 3, in case of $\rho,>0$ has the meaning when

$$0<\rho<1/(\Delta-2).$$

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