AN ELEMENTARY PROOF OF BROUWER'S FIXED POINT THEOREM

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The well-known classical Brouwer's fixed point theorem reads:

If f maps continuously an n dimensional sphere $\|X\| \leqq 1$ into itself, there exists a fixed point X such that $f(X) = X$.

Here in this brief note an alternative proof of the theorem will be presented: this will be carried out by appealing to some elementary results on analytic functions rather than to a combinatoric lemma regarding a simplex on which the customary proof is based.

In § 2 the proof for the general case will be offered. We should like to point out, however, that the case for $n = 2$ allows us to obtain an extremely simple proof, which will be first described in § 1.

1. Case $n = 2$. We designate a point by a complex number $z = x + yi$ in a Gaussian plane. Without loss of generality we assume that f maps continuously a square K: $|x| \leqq 1$, $|y| \leqq 1$ into itself.

We assume f has no fixed point. Then $w = z - f(z)$ is continuous on K and does not vanish, and therefore $\mathrm{Amp}\, w$ is defined everywhere in K.

Take an arbitrary square M in K. If z runs around the boundary of M once in positive direction, the increment of $\mathrm{Amp}\, w$ is evidently a multiple of 2π, which we denote by $\rho(M)$.

On the boundary of K

$$-\pi < \mathrm{Amp}\, w - \mathrm{Amp}\, z < \pi$$

holds, as is easily shown by graphical consideration; and so, if z runs around the boundary of K, the increment of $\mathrm{Amp}\, w - \mathrm{Amp}\, z$ is zero. Since $\mathrm{Amp}\, z$ is increased by 2π when z runs around the boundary of K, the corresponding increment of $\mathrm{Amp}\, w$ is also 2π. Therefore we have $\rho(K) = 2\pi$.

Now, if we subdivide K into m^2 squares $K_1, \ldots, K_{m^2}$, each with edge of length $2/m$, the following relation holds as is easily seen:

$$\rho(K) = \rho(K_1) + \ldots + \rho(K_{m^2}) \tag{1}$$

Since $c = \mathrm{Min}|w|$ is positive by our assumption, there is, by the uniform continuity of w, a positive number ε such that

$$|z_1 - z_2| < \varepsilon \text{ implies } |w_1 - w_2| < c/2$$

where $w_i = z_i - f(z_i)$ $(i = 1, 2)$.

If we take in such a way $m > 2\sqrt{2}/\varepsilon$, then for any z and z' in K_i we have $|w - w'| < c/2$; therefore w lies in the circle with center w' and radius $c/2$, which does not involve the origin. Therefore for such a number m we have $\rho(K_i) = 0$ $(i = 1, 2, \ldots, m^2)$. Accordingly $\rho(K) = 0$ by (1). This contradicts the above consequence $\rho(K)=2\pi$. Hence there exists a fixed point.

2. **Case $n \geqq 3$.** We denote a point in a real Euclidian n space by $X=(x_1,\dots,x_n)$ and define $\|X\|=\sqrt{x_1^2+\dots+x_n^2}$. The sum of any two points $X=(x_i)$ and $Y=(y_i)$ is defined as $X+Y=(x_1+y_1,\dots, x_n+y_n)$.

We assume f maps continuously a sphere $\|X\| \leqq 1$ into itself.

It is easily seen that if a certain extension $\bar{f}$ of f which is defined by the following formulas has a fixed point, this point is also fixed under the original f, and vice versa:

$$\bar{f}(X)=\begin{cases} f(X) & \text{if } \|X\| \leqq 1 \\ f(X/\|X\|) & \text{if } \|X\| \geqq 1. \end{cases}$$

To see the existence of a fixed point for $\bar{f}$, we consider the regularisation $\bar{f}_\delta$ of $\bar{f}$ defined by

$$\bar{f}_\delta(X)=\int_{\|Y\|\leqq\delta}\bar{f}(X+Y)dV\Big/\int_{\|Y\|\leqq\delta}1dV,\ dV=dy_1\dots dy_n,\ 0<\delta<1.$$

$\bar{f}_\delta(X)$ tends uniformly to $\bar{f}(X)$ when δ tends to 0. Furthermore we have $\|\bar{f}_\delta(X)\| \leqq 1$ for any allowable δ.

Now assume that $\bar{f}_\delta$ has a fixed point $X(\delta)$ for every δ, then, the compactness of the unit sphere gives rise to the existence of a positive decreasing sequence $\{\delta_n\}$ such that $\lim_{n\to\infty} X(\delta_n)$ exists. Next, in view of the uniform convergence of $\{\bar{f}_{\delta_n}(X)\}$ together with the continuity of $\bar{f}(X)$, it follows that $X_0=\lim X(\delta_n)$ is a fixed point of $\bar{f}$.

Hence, the problem is reduced to show that $\bar{f}_\delta$ has a fixed point. Noting that every coordinate of $\bar{f}_\delta$, the regularisation of $\bar{f}$, has partial derivatives of the n-th order and replacing $\bar{f}_\delta$ by f for simplicity of notation, we may assume, without loss of generality, that

f is a continuous mapping from an n space R^n into the unit sphere $\|X\| \leqq 1$, and every coordinate y_i of $f(X)$ has partial derivatives of the n-th order, consequently $\dfrac{\partial^2 y_i}{\partial x_j \partial x_k}=\dfrac{\partial^2 y_i}{\partial x_k \partial x_j}$ holds.

We proceed to the next step of our proof. For every X such that $X \neq \varepsilon f(X)$ where $|\varepsilon|<1.5$, we define $f(X|\varepsilon)$ by

$$f(X|\varepsilon)=\frac{X-\varepsilon f(X)}{\|X-\varepsilon f(X)\|}. \tag{1}$$

The function $f(X|\varepsilon)$ has evidently derivatives of the n-th order and is continuous with respect to (X,ε) whenever it is defined. Moreover it is a regular function of ε.

Since $X-\varepsilon f(X)$ does not vanish on the surface of the cube K: $|x_1| \leqq 2, \dots, |x_n| \leqq 2$, $f(X|\varepsilon)$ is continuous there. Take a point $X_i=(x_1,\dots,x_n)$ and consecutive $n-1$ points

$$X_j = (x_1, \dots, x_j + dx_j, \dots, x_n) \quad (j = 1, \dots, i-1,\ i+1, \dots, n)$$

lying on the surface S_i of K defined by $x_i = 2$. We calculate the limit of the ratio of the volume of a tetrahedron with vertices $f(X_1|\varepsilon), \dots, f(X_n|\varepsilon)$ and 0, to that of another tetrahedron with vertices $X_1, \dots, X_n$ and 0. If we put $f(X|\varepsilon) = (y_1, \dots, y_n)$ and $D(y_j/x_k) = \partial y_j/\partial x_k$, this limit is given by

$$\frac{1}{n!}\begin{vmatrix} y_1 + D(y_1/x_1)dx_1 & \vdots & \overset{i}{\smile}\,y_1 & \vdots & y_1 + D(y_1/x_n)\,dx_n \\ \vdots & & \vdots & & \vdots \\ y_n + D(y_n/x_1)dx_1 & \vdots & y_n & \vdots & y_n + D(y_n/x_n)\,dx_n \end{vmatrix} : \frac{1}{n!}\begin{vmatrix} x_1 + dx_1 & \vdots & \overset{i}{\smile}\,x_1 & \vdots & x_1 \\ \vdots & & \vdots & & \vdots \\ x_n & \vdots & x_n & \vdots & x_n + dx_n \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} D(y_1/x_1) & \vdots & \overset{i}{\smile}\,y_1 & \vdots & D(y_1/x_n) \\ \vdots & & \vdots & & \vdots \\ D(y_n/x_1) & \vdots & y_n & \vdots & D(y_n/x_n) \end{vmatrix} \quad (x_i = 2).$$

We define for every X such that $X \neq \varepsilon f(X)$

$$F^i(X|\varepsilon) = \begin{vmatrix} D(y_1/x_1) & \vdots & \overset{i}{\smile}\,y_1 & \vdots & D(y_1/x_n) \\ \vdots & & \vdots & & \vdots \\ D(y_n/x_1) & \vdots & y_n & \vdots & D(y_n/x_n) \end{vmatrix}. \tag{2}$$

As the height of the latter tetrahedron is 2, $F^i(X|\varepsilon)_{x_i=2}$ is just the ratio by which the area element at X_i on S_i is magnified under the mapping $f(X|\varepsilon)$, $f(X|\varepsilon)$ being regarded as a mapping from S_i on the surface of the unit sphere. In a similar manner, $-F^i(X|\varepsilon)_{x_i=-2}$ is the corresponding magnifing ratio for area elements under the mapping $f(X|\varepsilon)$ from S_i' defined by $x_i = -2$ on the surface of the unit sphere.

Since $f(X|0) = X/\|X\|$ maps homeomorphically the surface of K on that of unit sphere and, as is easily shown, $F^i(X|0)_{x_i=2} \geqq 2/(2\sqrt{n})^n$ and $-F^i(X|0)_{x_i=-2} \geqq 2/(2\sqrt{n})^n$ hold, $F^i(X|\varepsilon)_{x_i=2}$ and $-F^i(X|\varepsilon)_{x_i=-2}$ are also positive for ε sufficiently near 0, and consequently for such a small ε, $f(X|\varepsilon)$ is a homeomorphism from the surface of K on that of the unit sphere. For every image point is an inner point and the set of all image points must be closed as the image of a compact set, and therefore the image of the surface of K is the surface of the unit sphere. Therefore we have for small ε

$$\sum_{i=1}^{n}\int \cdots \int \{F^i(X|\varepsilon)_{x_i=2} - F^i(X|\varepsilon)_{x_i=-2}\}dx_1 \dots dx_{i-1}\,dx_{i+1} \dots dx_n = \text{const.} \tag{3}$$

where the constant equals to the area of the surface of the unit sphere.

If we use a complex number ξ instead of real number ε, $f(X|\xi)$ can be defined by the same formula as (1) whenever $X \neq \xi f(X)$, where $\|X - \xi f(X)\|$

denotes a complex number $\left\{\sum (x_i - \xi f_i(X))^2\right\}^{1/2}$. Though we must determine, in a precise consideration, which value $\left\{\sum (x_i - \xi f_i(X))^2\right\}^{1/2}$ represents, we define the value only for X and ξ such that $\|X\| > 1.8$ and $|\xi| < 1.7$, because we use X near the surface of K. For any X such that $\|X\| > 1.8$, we define $\left\{\sum (x_i - \xi f_i(X))^2\right\}^{1/2}$ in such a way that it represents a regular function which takes a positive number at $\xi = 0$. Thus if $\|X\| > 1.8$ holds, coordinates $y_j = y_j(X|\xi)$ of $f(X|\xi)$ are, as is easily seen, regular in a circle $|\xi| < 1.7$ and continuous with respect to (X, ξ) in this region. Moreover $y_j(X|\xi)$ has, as is easily shown by our assumption, partial derivatives $D(y_j/x_k|\xi) = \dfrac{\partial y_j(X|\xi)}{\partial x_k}$ continuous with respect to (X, ξ) in the same region.

Now, we will show that $D(y_j/x_k|\xi)$ is analytic. Denoting $X = (x_1, \ldots, x_n)$ and $Y = (x_1, \ldots, x_k + \Delta x_k, \ldots x_n)$, the Cauchy's integral formula gives us, for every ξ such that $|\xi| < 1.6$,

$$\frac{y_j(Y|\xi) - y_j(X|\xi)}{\Delta x_k} = \frac{1}{2\pi i}\int_C \frac{1}{\zeta - \xi}\,\frac{y_j(Y|\zeta) - y_j(X|\zeta)}{\Delta x_k}\,d\zeta$$

where C denotes a circle with a center 0 and radius 1.6. When Δx_k tends to zero, the integrand of the right side tends to $\dfrac{D(y_j/x_k|\zeta)}{\zeta - \xi}$ uniformly with respect to ζ on C. Therefore we have

$$D(y_j/x_k|\xi) = \frac{1}{2\pi i}\int_C \frac{D(y_j/x_k|\zeta)}{\zeta - \xi}\,d\zeta.$$

Hence $D(y_j/x_k|\xi)$ is a regular function of $\xi(|\xi| < 1.6)$, as was to be shown.

Thus $D(y_j/x_k|\xi)$ is continuous with respect to (X, ξ) and regular with respect to ξ. Replacing $D(y_j/x_k)$ by $D(y_j/x_k|\xi)$ in (2), we define $F^i(X|\xi)$ in the same way. Then $F^i(X|\xi)$ is continuous with respect to (X, ξ) on the region defined by $\|X\| > 1.8$ and $|\xi| < 1.6$, and regular in $|\xi| < 1.6$. Therefore $F^i(X|\xi)_{x_i=2}$ is also regular in $|\xi| < 1.6$ and continuous with respect to (X, ξ) when X ranges on the surface S_i of K. By an analogous method as above which depends only on the Cauchy's integral formula and on the theory of uniform convergence, we can show that

$$\int \cdot\cdot \int F^i(X|\xi)_{x_i=2}\,dx_1 \ldots dx_{i-1}\,dx_{i+1} \ldots dx_n$$

is regular in $|\xi| < 1.5$, integral domain being $-2 \leq x_j \leq 2$, $j = 1, 2, \ldots, i-1, i+1, \ldots, n$. By the similar consideration, we know that

$$\sum_{i=1}^{n} \int \cdot\cdot \int \{F^i(X|\xi)_{x_i=2} - F^i(X|\xi)_{x_i=-2}\}\,dx_1 \ldots dx_{i-1}\,dx_{i+1} \ldots dx_n \tag{4}$$

is regular in a circle $|\xi| < 1.5$.

Since (3) holds for every real ε sufficiently small, the representation (4) equals to a constant in $|\xi| < 1.5$ by a well-known property of analytic functions. Putting $\xi = 1$ in (4), we have

$$\sum_{i=1}^{n} \int \cdot\cdot \int \{F^i(X|1)_{x_i=2} - F^i(X|1)_{x_i=-2}\} dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n \neq 0 . \qquad (5)$$

It will be shown that, if f has no fixed point, the left side of (5) is zero. In fact, if f has no fixed point, then $F^i(X|1)$ is defined and has continuous derivatives everywhere by our assumption. Therefore the left side of the representation (5) equals to

$$\sum_{i=1}^{n} \int \cdot\cdot \int_K \frac{\partial F^i(X|1)}{\partial x_i} dx_1 \ldots dx_n = \int \cdot\cdot \int_K \sum_{i=1}^{n} \frac{\partial F^i(X|1)}{\partial x_i} dx_1 \ldots dx_n .$$

Denote Δ_j^i the determinant which is obtained by differentiating the j-th column of $F^i(X|1)$ with respect to x_i, then we have

$$\sum_{i=1}^{n} \frac{\partial F^i(X|1)}{\partial x_i} = \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta_j^i = \sum_{i=1}^{n} \Delta_i^i ,$$

because $i \neq j$ implies $\Delta_j^i = -\Delta_i^j$ as is easily shown. Since $y_1^2 + \ldots + y_n^2 = 1$ implies $\Delta_i^i = 0$, we have

$$\sum_{i=1}^{n} \frac{\partial F^i(X|1)}{\partial x_i} = \sum_{i=1}^{n} \Delta_i^i = 0 .$$

Hence if f has no fixed point, the left side of the representation (5) vanishes. This contradicts (5). Therefore f has a fixed point.

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