ON THE PROJECTION OF NORM ONE IN W*-ALGEBRAS II

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In this paper, we shall study the projection of norm one in W^*-algebras following [7]. Firstly, we obtain the general decomposition theorem of a projection of norm one π from a W^*-algebra $\mathbf{M}$ to its C^*-subalgebra $\mathbf{N}$ showing that $\mathbf{N}$ is decomposed into the maximal W^*-representable direct summand and the rest. Restricting ourself to the case of $\mathbf{N}$ being a W^*-representable $*$-subalgebra, we prove that π is decomposed into three parts by three orthogonal central projections z_1, z_2, z_3 of $\mathbf{N}$. The first component is a normal projection of norm one from $\mathbf{M}$ to $\mathbf{N}z_1$, the second singular one to $\mathbf{N}z_2$ and z_1, z_2 are maximal central projections having these properties. In the last section we discuss on the σ-weak continuity property of π and the relation to the other continuity. We can prove that π is σ-weakly continuous if and only if the kernel of π is σ-weakly closed.

1. Preliminaries. Consider a W^*-algebra $\mathbf{M}$, its conjugate space $\mathbf{M}^*$ and the space $\mathbf{M}_*$ of all σ-weakly continuous linear functionals on $\mathbf{M}$. We define the operators R_a and L_a on $\mathbf{M}^*$ for each $a \in \mathbf{M}$ such that

$$< x, R_a\varphi > = < xa, \varphi > \text{ and } < x, L_a\varphi > = < ax, \varphi >$$

for all $a \in \mathbf{M}$, $\varphi \in \mathbf{M}^*$. The following properties are easily verified: $R_{(\lambda a+\mu b)} = \lambda R_a + \mu R_b$, $L_{(\lambda a+\mu b)} = \lambda L_a + \mu L_b$, $R_{ab} = R_a R_b, L_{ab} = L_b L_a$, where a and b are arbitrary elements of $\mathbf{M}$ and λ, μ complex numbers.

A subspace of $\mathbf{M}^*$ which is invariant both for every R_a and every L_a is called an invariant subspace. It can be shown that there exists a one-to-one correspondence between the σ-weakly closed ideal m of $\mathbf{M}$ and the closed invariant subspace V of $\mathbf{M}_*$ such that $m = V^0$ and $V = m^0$ where V^0 and m^0 denote the polar of V and m in $\mathbf{M}$ and $\mathbf{M}_*$ respectively.

A positive linear functional φ is called singular if there exists no non-zero positive normal linear functional ψ such as $\psi \leqq \varphi$; we denote the closed subspace generated by all singular linear functionals on $\mathbf{M}$ by $\mathbf{M}_*^\perp$. $\mathbf{M}_*^\perp$ is an invariant subspace of $\mathbf{M}^*$. It can be shown that any closed invariant subspace V is decomposed such as

$$V = (V \cap \mathbf{M}_*) \underset{l^1}{\oplus} (V \cap \mathbf{M}_*^\perp), \text{ in particular } \mathbf{M}^* = \mathbf{M}_* \underset{l^1}{\oplus} \mathbf{M}_*^\perp,$$

the sum being l^1-direct sum.

A uniformly continuous linear homomorphism π from a W^*-algebra $\mathbf{M}$ to a W^*-algebra $\mathbf{N}$ is called singular if ${}^t\pi(\mathbf{N}_*) \subset \mathbf{M}_*^\perp$, where ${}^t\pi$ denote the transpose of π. We can prove that a positive singular mapping π from $\mathbf{M}$ to $\mathbf{N}$ has the property that there exists no non-zero normal linear homomorphism π' such that $\pi'(a) \leqq \pi(a)$ for all positive element $a \in \mathbf{M}$. Corresponding to the decomposition of a linear functional on a W^*-algebra it can be shown that any uniformly continuous linear homomorphism from a W^*-algebra to the another is decomposed into the σ-weakly continuous part and the singular part. Some of these proofs are found in [6].

A C^*-algebra is called W^*-representable if it is faithfully representable as a W^*-algebra on some Hilbert space $\mathbf{H}$.

Through our discussions we assume that a C^*-algebra has always a unit.

2. Decomposition of the projection of norm one. Let $\mathbf{N}$ be a C^*-algebra in which each upper bounded increasing directed set of self-adjoint elements has a supremum in $\mathbf{N}$. We define a "normal" linear functional on $\mathbf{N}$ as usually.

LEMMA 1. *The set V of all finite linear combinations of normal linear functionals on $\mathbf{N}$ is a closed invariant subspace of $\mathbf{N}^*$.*

PROOF. Let $\{a_\alpha\}$ be an upper-bounded increasing directed set of self-adjoint elements in $\mathbf{N}$ with $a = \sup_\alpha a_\alpha$. It is clear that we have $b^*ab = \sup_\alpha b^*a_\alpha b$ for any invertible element b.

Now, for any element $c \in \mathbf{N}$ there exists a positive number $\lambda > 0$ such that $\lambda 1 + c$ is invertible. Hence

$$(\lambda 1 + c)^* a(\lambda 1 + c) = \sup_\alpha (\lambda 1 + c)^* a_\alpha (\lambda 1 + c),$$

and we get

$$\sup_\alpha < (\lambda 1 + c)^* a_\alpha (\lambda 1 + c), \varphi > \, = \, < (\lambda 1 + c)^* a (\lambda 1 + c), \varphi >$$

for every positive element $\varphi \in V$.

On the other hand,

$$\begin{aligned} < (\lambda 1 + c)^* a_\alpha (\lambda 1 + c), \varphi > &= \lambda^2 < a_\alpha, \varphi > + \lambda < c^* a_\alpha, \varphi > \\ &\quad + \lambda < a_\alpha c, \varphi > + < c^* a_\alpha c, \varphi >, \\ < (\lambda 1 + c)^* a (\lambda 1 + c), \varphi > &= \lambda^2 < a, \varphi > + \lambda < c^* a, \varphi > \\ &\quad + \lambda < ac, \varphi > + < c^* ac, \varphi > . \end{aligned}$$

Then a usual computation applying Schwarz's inequality shows that $< c^* a_\alpha, \varphi >$ and $< a_\alpha c, \varphi >$ converge to their corresponding terms. Therefore, we get that $< c^* a_\alpha c, \varphi >$ converges to $< c^* ac, \varphi >$ for every positive $\varphi \in V$ and for any $c \in \mathbf{N}$, i.e. $\sup_\alpha < c^* a_\alpha c, \varphi > = < c^* ac, \varphi >$. Therefore, we have $L_c{}^* R_c V \subset V$ for all $c \in \mathbf{N}$. Then an equality

$$\begin{aligned} 4L_a{}^* R_b &= L_{(a+b)}{}^* R_{(a+b)} - L_{(a-b)}{}^* R_{(a-b)} \\ &\quad + iL_{(a-ib)}{}^* R_{(a-ib)} - iL_{(a+ib)}{}^* R_{(a+ib)} \end{aligned}$$

proves that $L_a{}^* R_b V \subset V$ for all $a, b \in \mathbf{N}$. That is, V is an invariant subspace of $\mathbf{N}^*$.

Denote by $\bar{V}$ the norm-closure of V and suppose $a = \sup_\alpha a_\alpha$ for an upper-bounded increasing directed set $\{a_\alpha\}$ of self-adjoint elements in $\mathbf{N}$. Since

$< a_\alpha, \varphi >$ converges to $< a, \varphi >$ for every $\varphi \in V$ and since $\{a_\alpha\}$ are uniformly norm-bounded, we see that $< a_\alpha, \psi >$ converges to $< a, \psi >$ for all $\psi \in \bar{V}$; especially for any positive element $\psi \in \bar{V}$ we have $\sup_\alpha < a_\alpha, \psi > = < a, \psi >$, that is, ψ is normal. Thus every positive element $\psi \in \bar{V}$ belongs to V. By [4: Theorem 1] $\bar{V}$ is algebraically spanned by positive elements in $\bar{V}$, so that $\bar{V}$ is contained in V and V is closed.

LEMMA 2. *Suppose* **N** *is a* C^**-algebra above-mentioned. Then* **N** *is a direct sum of a* W^**-representable algebra and a* C^**-algebra on which no normal linear functional exists.*

PROOF. Consider an invariant subspace V in Lemma 1. The polar V^0 of V in **N** is a closed two-sided ideal of **N**. Let $\{e_\alpha\}$ be a maximal family of orthogonal projections of V^0 with $e = \sup_\alpha e_\alpha$. We have $e \in V^0$.

It is clear $\mathbf{N}e \subset V^0$. Suppose there exists an element a in V^0 such that $b = a - ae \neq 0$, then we can get an element c as cb^*b becomes a non-zero projection of V^0. Moreover $cb^*be = 0$. This contradicts the maximality of $\{e_\alpha\}$. Hence $V^0 \subset \mathbf{N}e$, which implies $V^0 = \mathbf{N}e$. As V^0 is self-adjoint, $V^0 = \mathbf{N}e = e\mathbf{N}$.

Now, for a self-adjoint element a, $ea \in e\mathbf{N} = \mathbf{N}e$, so that $eae = ea = ae$. Hence e is a central projection of **N**.

By [3], $\mathbf{N}(1-e)$ is W^*-representable and the decomposition $\mathbf{N} = \mathbf{N}(1-e) + \mathbf{N}e$ is a desired one.

With these preparations we show the following

THEOREM 1. *If* π *is a projection of norm one from a* W^**-algebra* **M** *to its* C^**-subalgebra* **N**, π *is decomposed into* π_1 *and* π_2 *by a central projection* z *of* **N** *where* π_1 *is a projection of norm one from* **M** *to a* W^**-representable algebra* $\mathbf{N}z$ *and* π_2 *a projection of norm one from* **M** *to* $\mathbf{N}(1-z)$ *on which no normal linear functional exists.*

PROOF. As in the proof of Theorem 4 in [7], **N** is a C^*-algebra in which each upper-bounded increasing directed set of self-adioint elements has a supremum in **N**. Hence there exists, by Lemma 2, a central projection z of **N** such that $\mathbf{N}z$ is a W^*-representable algebra and $\mathbf{N}(1-z)$ a C^*-algebra on which any non-zero normal linear functional does not exist. If we put

$$\pi_1(a) = \pi(a)z \qquad \pi_2(a) = \pi(a)(1-z) \text{ for all } a \in \mathbf{M},$$

then we have the decomposition $\pi = \pi_1 + \pi_2$ as stated above.

In the following lines we restrict ourself to the case $\pi = \pi_1$. Let π be a projection of norm one from a W^*-algebra **M** to a W^*-representable *-subalgebra **N**. **N** is the conjugate space of a Banach space $\mathbf{N}_*$, the space of all σ-weakly continuous linear functionals when we represent **N** on a Hilbert space **H** so as **N** to be a W^*-algebra on **H**. We can consider $\sigma(\mathbf{N}, \mathbf{N}_*)$-topology on **N**. If π is continuous in $\sigma(\mathbf{M}, \mathbf{M}_*)$ and $\sigma(\mathbf{N}, \mathbf{N}_*)$-topologies we say also that π is σ-weakly continuous. Thus, π is σ-weakly continuous if and only if π is normal [2]. Moreover, we denote also the space of all singular linear functionals on a represented W^*-algebra of $\mathbf{N}$ by $\mathbf{N}_*^+$. If π satisfies the condition ${}^t\pi(\mathbf{N}_*) \subset \mathbf{M}_*^+$, π is also called singular. With these considerations we have

THEOREM 2. *Let π be a projection of norm one from a W^*-algebra $\mathbf{M}$ to its W^*-representable $*$-subalgebra $\mathbf{N}$, then there exist three central projections z_1, z_2, z_3 in $\mathbf{N}$ with $z_1 + z_2 + z_3 = 1$ such that if we put $\pi_i(a) = \pi(a)z_i (i = 1, 2, 3)$ for $a \in \mathbf{M}$, π_1 is a normal projection of norm one from $\mathbf{M}$ to $\mathbf{N}z_1$ and π_2 a singular one from $\mathbf{M}$ to $\mathbf{N}z_2$; z_1 and z_2 are maximal central projections having these properties.*

PROOF. Put $V = {}^t\pi^{-1}(\mathbf{M}_*) \cap \mathbf{N}_*$, then V is an invariant subspace of $\mathbf{N}_*$. In fact, by [7; Theorem 1] $\pi(axb) = a\pi(x)b$ for every $a, b \in \mathbf{N}$ so that

$$< x, {}^t\pi(L_aR_bV) > = < a\pi(x)b, V > = < \pi(axb), V > = < x, L_aR_b{}^t\pi(V) >$$

for every $a, b \in \mathbf{N}$ and $x \in \mathbf{M}$.

The polar V^0 of V in $\mathbf{N}$ is a $\sigma(\mathbf{N}, \mathbf{N}_*)$-closed two-sided ideal. Hence there exists a central projection z_1 of $\mathbf{N}$ such that $V^0 = \mathbf{N}(1 - z_1)$. Put $\pi_1(a) = \pi(a)z_1$ for all $a \in \mathbf{M}$, then $< a, {}^t\pi_1(\mathbf{N}_*) > = < \pi(a)z, \mathbf{N}_* > = < \pi(a), R_{z_1}\mathbf{N}_* > = < \pi(a), V > = < a, {}^t\pi(V) >$ for all $a \in \mathbf{M}$, because $V = V^{00}$(bipolar in $\mathbf{N}_*$) $= (\mathbf{N}(1 - z_1))^0 = R_{z_1}\mathbf{N}_*$. Therefore ${}^t\pi_1(\mathbf{N}_*) \subset \mathbf{M}_*$ i.e. π_1 is normal. Moreover one can easily verify that π_1 is a projection of norm one from $\mathbf{M}$ to $\mathbf{N}z_1$.

Next, suppose that there exists a central projection h such that if we put $\pi_1'(a) = \pi(a)h$, π_1' is a normal projection of norm one. Since ${}^t\pi_1'(\mathbf{N}_*) \subset \mathbf{M}_*$, an equality

$$< a, {}^t\pi_1'(\mathbf{N}_*) > = < \pi(a)h, \mathbf{N}_* > = < \pi(a), \quad R_h\mathbf{N}_* > = < a, {}^t\pi(R_h\mathbf{N}_*) >$$

implies $R_h\mathbf{N}_* \subset V = R_{z_1}\mathbf{N}_*$. Hence $h \leqq z_1$.

As for ${}^t\pi^{-1}(\mathbf{M}_*^+) \cap \mathbf{N}_*$ we proceed the same computation and get $\pi_2(a) = \pi(a)z_2$ where $({}^t\pi^{-1}(\mathbf{M}_*^+) \cap \mathbf{N}_*)^0 = \mathbf{N}(1 - z_2)$

Now it is clear that z_1 and z_2 are orthogonal. We set $z_3 = (1 - z_1)(1 - z_2)$ and define $\pi_3(a) = \pi(a)z_3$ for all $a \in \mathbf{M}$. This yields the decomposition of π described in our theorem.

REMARK. If $\pi_3 \neq 0$ we can further decompose this projection into normal part and singular part and both are non-zero $\mathbf{N}$-modul linear homomorphisms but these are no more projections of norm one from $\mathbf{M}$ to certain direct summands of $\mathbf{N}$. Moreover the normal part is an onto-mapping. But we omit all these proofs here.

3. The continuity of the projection of norm one. We begin with the following

LEMMA 3. *Let $\mathbf{M}$, $\mathbf{N}$, and π be the same as in above discussions, then we have ${}^t\pi^{-1}(\mathbf{M}_*) \subset \mathbf{N}_*$.*

PROOF. ${}^t\pi^{-1}(\mathbf{M}_*)$ is a closed invariant subspace of $\mathbf{N}^*$ as shown in the proof of Theorem 2, hence

$$ {}^t\pi^{-1}(\mathbf{M}_*) = {}^t\pi^{-1}(\mathbf{M}_*) \cap \mathbf{N}_* \underset{n}{\oplus} {}^t\pi^{-1}(\mathbf{M}_*) \cap \mathbf{N}_*^\perp. $$

Take a positive element φ of ${}^t\pi^{-1}(\mathbf{M}_*) \cap \mathbf{N}_*^\perp$. then ${}^t\pi(\varphi)$ is a normal linear functional of $\mathbf{M}$. Therefore if $\{a_\alpha\}$ is a bounded increasing directed set of self-adjoint elements of $\mathbf{N}$ and $a_0 = \sup_\alpha a_\alpha$ in $\mathbf{M}$ we have $\sup_\alpha < a_\alpha, \varphi >$ $= \sup_\alpha < \pi(a_\alpha), \varphi > = \sup_\alpha < a_\alpha, {}^t\pi(\varphi) > = < a_0, {}^t\pi(\varphi) > = <\pi(a_0), \varphi >$. We get, however, $\pi(a_0) = \sup_\alpha a_\alpha$ in $\mathbf{N}$, so that φ is normal on $\mathbf{N}$ and this implies $\varphi = 0$.

Hence we get ${}^t\pi^{-1}(\mathbf{M}_*) \cap \mathbf{N}_*^\perp = 0$ by [4: Theorem 1], which leads to ${}^t\pi^{-1}(\mathbf{M}_*) \subset \mathbf{N}_*$.

THEOREM 3. *Let $\mathbf{M}$ be a W^*-algebra, $\mathbf{N}$ a W^*-representable $*$-subalgebra, π a projection of norm one from $\mathbf{M}$ to $\mathbf{N}$ and $\pi = \pi_1 + \pi_2 + \pi_3$ is the decomposition mentioned above. Then*

1^0 π is normal if and only if $\pi^{-1}(0)$ is σ-weakly closed;

2^0 $\pi_1 = 0$ if and only if $\pi^{-1}(0)$ is σ-weakly dense in $\mathbf{M}$.

PROOF. 1^0 It suffices to prove the sufficiency. Put $V = {}^t\pi^{-1}(\mathbf{M}_*)$ and let $\pi^{-1}(0)^0$ be the polar of $\pi^{-1}(0)$ in $\mathbf{M}^*$. Then we have, by the classical theorem of Banach space, $\pi^{-1}(0)^0 = {}^t\pi(\mathbf{N}^*)$. Therefore $\pi^{-1}(0)^0 \cap \mathbf{M}_* = {}^t\pi(\mathbf{N}_*) \cap \mathbf{M}_* = {}^t\pi(V)$.

Now, from the hypothesis, $(\pi^{-1}(0)^0 \cap \mathbf{M}_*)^* = \mathbf{M}/\pi^{-1}(0)$ (the factor space of $\mathbf{M}$ by $\pi^{-1}(0)$) because $\pi^{-1}(0)^0 \cap \mathbf{M}_*$ is the polar of $\pi^{-1}(0)$ in $\mathbf{M}_*$. We represent an element of $\mathbf{M}/\pi^{-1}(0)$ by $\bar{a}$ for $a \in \mathbf{M}$.

If we assume, for some $a \in \mathbf{N}$, $< a, \varphi > = 0$ for all $\varphi \in V$ then $< a, \varphi >$ $= < \pi(a), \varphi > = < a, {}^t\pi(\varphi) > = < \bar{a}, {}^t\pi(\varphi) >$ for all $\varphi \in V$. Hence $\bar{a} = 0$ i.e. $a \in \pi^{-1}(0)$ which implies $a = 0$. Therefore V is $\sigma(\mathbf{N}_*, \mathbf{N})$-dence in $\mathbf{N}_*$ by Lemma 3. On the other hand V is a closed subspace of $\mathbf{N}_*$, whence $V = \mathbf{N}_*$. This completes the proof.

2^0. If $\pi_1 = 0$ then z_1 in Theorem 2 is zero, so that $(V \cap \mathbf{N}_*)^0 = V^0 = \mathbf{N}$. Hence $V = 0$ and this implies ${}^t\pi(V) = \pi^{-1}(0)^0 \cap \mathbf{M}_* = 0$. Therefore $\overline{\pi^{-1}(0)}$, the σ-weak closure of $\pi^{-1}(0)$, is $\mathbf{M}$ for $\overline{\pi^{-1}(0)} = (\pi^{-1}(0)^0 \cap \mathbf{M}_*)^0 = \mathbf{M}$.

The above argument is invertible so we get the sufficiency of 2^0.

At the last, we summarize the conditions for the continuity of a projection of norm one. For the convenience we assume that $\mathbf{N}$ is a W^*-subalgebra of $\mathbf{M}$.

THEOREM 4. *Let π be a projection of norm one from a W^*-algebra $\mathbf{M}$ to its W^*-subalgebra $\mathbf{N}$. Then the next six conditions are equivalent;*

1^0 π is σ-weakly continuous;

2^0 π is strongest continuous;

3^0(resp. 4^0) π is σ-weakly (resp. weakly) continuous on the unit sphere of $\mathbf{M}$;

5^0(resp. 6^0) π is strongestly (resp. strongly) continuous on the unit sphere of $\mathbf{M}$.

PROOF. 1^0 implies 2^0 by 3^0 of Theorem 1 in [7]. $1^0 \rightleftarrows 3^0 \rightleftarrows 4^0, 2^0 \rightleftarrows 5^0 \rightleftarrows 6^0$ are trivial ones.

Take a σ-weakly continuous linear functional φ of $\mathbf{N}$, then φ is strongly continuous on the unit sphere of $\mathbf{N}$, so that if π is strongly continuous on the unit sphere of $\mathbf{M}$ $'\pi(\varphi)$ is strongly continuous on the unit sphere of $\mathbf{M}$. Therefore $'\pi(\varphi)$ is σ-weakly continuous on $\mathbf{M}$, that is, π is σ-weakly continuous. Hence 6^0 implies 1^0. Thus all proofs are completed.

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