RIEMANN-CESÀRO METHODS OF SUMMABILITY III

HIROSHI HIROKAWA

(Received October 15, 1957; in revised form, May 23. 1958 and December 7, 1958)

1. Introduction. Let s_n^α be the Cesàro sum of a series $\sum_{n=0}^{\infty} a_n$ with $a_0 = 0$, that is, A_n^α being Andersen's notation,

$$s_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^\alpha a_\nu,$$

and let σ_n^α be the Cesàro mean of the series $\sum_{n=0}^{\infty} a_n$, that is, $\sigma_n^\alpha = s_n^\alpha / A_n^\alpha$. The series $\sum_{n=0}^{\infty} a_n$ is said to be evaluable (C, α), $\alpha > -1$, to s if $\sigma_n^\alpha \to s$ as $n \to \infty$ and to be evaluable $|C, \alpha|$, $\alpha > -1$, to s if the series $\sum_{n=0}^{\infty} |\sigma_n^\alpha - \sigma_{n+1}^\alpha|$ is convergent and if $\sigma_n^\alpha \to s$ as $n \to \infty$. In the following, let p be a positive integer and let α be a real number such that $\alpha \geqq -1$. The series $\sum_{n=0}^{\infty} a_n$ is said to be evaluable to s by Riemann-Cesàro method of order p and index α, or briefly, to be evaluable (R, p, α) to s, if the series

$$C_{p,\alpha}^{-1} \cdot t^{\alpha+1} \sum_{n=1}^{\infty} s_n^\alpha \left(\frac{\sin nt}{nt} \right)^p, \tag{1. 1}$$

where

$$C_{p,\alpha} = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^\infty u^{\alpha-p} (\sin u)^p \, du, & -1 < \alpha < p-1 \text{ or } \alpha = 0,\ p = 1, \\ 1, & \alpha = -1, \end{cases} \tag{1. 2}$$

converges in some interval $0 < t < t_0$ and its sum tends to s as $t \to 0+$. Under this definition, the summabilities $(R, p, -1)$ and $(R, p, 0)$ are reduced to the well-known summabilities (R, p) and (R_p), respectively. In our ealier papers [3, 4] we have investigated some properties on this summability. The purpose of this paper is to study further properties on this. One of our problems is to establish the inclusion relation between the methods with same order and distinguished indices. Concerning this problem, Marcinkiewicz [6] proved the following theorems.

THEOREM A. *A series may be evaluable* (R_2) *without being evaluable* $(R, 2)$.

THEOREM B. *A series may be evaluable* $(R, 2)$ *without being evaluable* (R_2).

These theorems were also proved by Hardy and Rogosinski [2] and Kuttner [5]. We shall prove, by the method used by Marcinkiewicz, the following theorems 1 and 2, in the sections 2 and 3, respectively.

THEOREM 1. *There exists a series evaluable* $(R, 2, \alpha+1)$, $-1 \leqq \alpha < 0$, *but not evaluable* $(R, 2, \alpha)$.

THEOREM 2. *There exists a series evaluable* $(R, 2, \alpha)$, $-1 \leqq \alpha < 0$, *but not evaluable* $(R, 2, \alpha+1)$.

The present author [3] proved the following theorem.

THEOREM C. *Suppose that the series* $\sum_{n=0}^{\infty} a_n$ *is evaluable* $(C, r+1)$, $r > -1$, *to* s *and*

$$\sum_{\nu=1}^{n} |s_\nu^r| = O(n^{r+1}). \tag{1. 3}$$

Then the series $\sum_{n=0}^{\infty} a_n$ *is evaluable* (R, p, α) *to* s *when* $-1 \leqq \alpha < r < p-1$.

Recently, Rajagopal [8] proved the following

THEOREM D. *Suppose that the series* $\sum_{n=0}^{\infty} a_n$ *is Abel evaluable to* s *and that the condition* (1. 3) *holds. Then the series* $\sum_{n=0}^{\infty} a_n$ *is evaluable* (R, p, r) *to* s *when* $-1 \leqq r < p-1$.

Concerning these theorems, in the section 4, we shall prove the following

THEOREM 3. *Under the asumption of Theorem D, the series* $\sum_{n=0}^{\infty} a_n$ *is evaluable* (R, p, α) *to* s *when* $-1 \leqq \alpha < p-1$ *and* $-1 < r < p-1$.

Further the author [3] proved the following

THEOREM E. *Let* α *be an integer such that* $0 \leqq \alpha + 1 < p$. *If the series* $\sum_{n=0}^{\infty} a_n$ *evaluable* $|C, p|$ *to* s, *then it is evaluable* (R, p, α) *to* s. *Further, if the series* $\sum_{n=1}^{\infty} a_n$ *is evaluable* $|C, 1|$ *to* s, *then it is also evaluable* $(R, p, 0)$ *to* s.

An improvement of this theorem is the following, which is proved in the last section 5.

THEOREM 4. *Let α be a real number such that* $0 \quad \alpha+1<p$. *If the series* $\sum_{n=0}^{\infty} a_n$ *is evaluable* $|C,p|$ *to* s, *then it is evauable* (R, p, α) *to* s.

I take this opportunity of expressing my heartfelt thanks to Professor G. Sunouchi for his kind encouragement and valuable suggestions during the preparation of this paper.

2. Proof of Theorem 1. We shall first prove that there exists an even function $f(t) \sim \sum b_\nu \cos s\nu t$ satisfying the conditions:

(2. 1) $$\sum |b_\nu| \qquad \text{converges,}$$

(2. 2) $$\lim_{t\to 0+} t^\alpha (f(t) - f(0)) = 0,$$

and

(2. 3) $$\limsup_{t\to 0+} t^{\alpha+1} \left| \frac{\tilde{f}(2t)}{t} - \frac{\tilde{F}(2t) - \tilde{F}(0)}{t^2} \right| = +\infty,$$

where $\tilde{f}(t)$ is the function conjugate to $f(t)$ and where $\tilde{F}(t)$ is an integral of $\tilde{f}(t)$.

We may take two decreasing positive sequences $\{\lambda_i\}$ and $\{\varepsilon_i\}$ such that

(2. 4) $$0 < \lambda_i < \frac{1}{2}, \ 0 < \lambda_{i+1} < \lambda_i - 2^\alpha \lambda_i \quad (i = 1, 2, 3, \ldots\ldots),$$

(2. 5) $$\sum_{i=1}^{\infty} \varepsilon_i \qquad \text{converges}$$

and

(2. 6) $$\lim_{i\to\infty} \varepsilon_i \log \lambda_i = -\infty.$$

Now, we shall define the function $f(t)$ as follows.

(2. 7) $$f(t) = \sum_{i=1}^{\infty} f_i(t),$$

where

$$\begin{aligned} f_i(t) &= 0, && t = \lambda_i - 2^\alpha\lambda_i \text{ and } t = \lambda_i, \\ &= \varepsilon_i \lambda_i^{-\alpha}{}_i && t = \lambda_i - \lambda_i^{1-\alpha}, \\ &= \text{linear}, && (\lambda_i - 2^\alpha \lambda_i,\ \lambda_i - \lambda_i^{1-\alpha}) \text{ and } (\lambda_i - \lambda_i^{1-\alpha},\ \lambda_i), \\ &= 0 && \text{elsewhere}, \\ &= f_i(-t). \end{aligned}$$

Then, as is shown in the following, we see that the function $f(t)$ satisfies the three conditions (2. 1), (2. 2), (2. 3). It is obvious that

$$f'(t) = \frac{\varepsilon_i \lambda_i^{-\alpha}}{2^\alpha \lambda_i - \lambda_i^{1-\alpha}} \qquad (\lambda_i - 2^\alpha \lambda_i < t < \lambda_i - \lambda_i^{1-\alpha})$$
$$= -\frac{\varepsilon_i \lambda_i^{-\alpha}}{\lambda_i^{1-\alpha}} = -\frac{\varepsilon_i}{\lambda_i} \qquad (\lambda_i - \lambda_i^{1-\alpha} < t < \lambda_i).$$

We have then

$$\int_{-\pi}^{\pi} |f'(t)|^{1-\alpha} dt = 2\int_0^{\lambda_1} |f'(t)|^{1-\alpha} dt$$
$$= 2\left\{\sum_{i=1}^{\infty}\int_{\lambda_i-2^\alpha\lambda_i}^{\lambda_i-\lambda_i^{1-\alpha}} + \sum_{i=1}^{\infty}\int_{\lambda_i-\lambda_i^{1-\alpha}}^{\lambda_i}\right\}$$
$$= 2\left\{\sum_{i=1}^{\infty}\left(\frac{\varepsilon_i \lambda_i^{-\alpha}}{2^\alpha\lambda_i - \lambda_i^{1-\alpha}}\right)^{1-\alpha}\cdot(2^\alpha\lambda_i - \lambda_i^{1-\alpha}) + \sum_{i=1}^{\infty}\left(\frac{\varepsilon_i}{\lambda_i}\right)^{1-\alpha}\cdot\lambda_i^{1-\alpha}\right\}$$
$$= 2\left\{\sum_{i=1}^{\infty}\varepsilon_i^{1-\alpha}\lambda_i^{-\alpha(1-\alpha)}(2^\alpha\lambda_i - \lambda_i^{1-\alpha})^\alpha + \sum_{i=1}^{\infty}\varepsilon_i^{1-\alpha}\right\}$$
$$= 2\left\{\sum_{i=1}^{\infty}\varepsilon_i^{1-\alpha}\lambda_i^{\alpha^2}(2^\alpha - \lambda_i^{-\alpha})^\alpha + \sum_{i=1}^{\infty}\varepsilon_i^{1-\alpha}\right\}$$
$$\leqq 2\,\{(2^\alpha - \lambda_1^{-\alpha})^\alpha + 1\}\sum_{i=1}^{\infty}\varepsilon_i^{1-\alpha} < +\infty,$$

since the series $\sum_{i=1}^{\infty}\varepsilon_i^{1-\alpha}$ converges by (2. 5). Thus, the derivative $f'(t)$ belongs to the class $L^{1-\alpha}$, so that, using Tonelli's theorem [10 ; p. 138], (2. 1) follows because $f(t)$ is absolutely continuous. We shall next show that the function $f(t)$ is satisfying (2. 2). Let $\lambda_i - 2^\alpha \lambda_i \leqq t \leqq \lambda_i - \lambda_i^{1-\alpha}$. Then

$$\frac{f(t)}{\varepsilon_i \lambda_i^{-\alpha}} \leqq \frac{t}{\lambda_i - \lambda_i^{1-\alpha}} \leqq \left(\frac{t}{\lambda_i - \lambda_i^{1-\alpha}}\right)^{-\alpha};$$

hence

$$t^\alpha f(t) \leqq \varepsilon_i \lambda_i^{-\alpha}(\lambda_i - \lambda_i^{1-\alpha})^\alpha.$$

If $\lambda_i - \lambda_i^{1-\alpha} \leqq t \leqq \lambda_i$, then

$$\frac{f(t)}{\varepsilon_i \lambda_i^{-\alpha}} \leqq 1 \leqq \left(\frac{t}{\lambda_i - \lambda_i^{1-\alpha}}\right)^{-\alpha};$$

hence

$$t^\alpha f(t) \leqq \varepsilon_i \lambda_i^{-\alpha}(\lambda_i - \lambda_i^{1-\alpha})^\alpha.$$

Therefore, we get, remenbering that $f(0) = 0$,

$$0 \leqq \lim_{t\to 0+} t^{\alpha}(f(t)-f(0)) \leqq \lim_{i\to\infty} \varepsilon_i \lambda_i^{-\alpha}(\lambda_i - \lambda_i^{1-\alpha})^{\alpha} = \lim_{i\to\infty} \varepsilon_i (1-\lambda_i^{-\alpha})^{\alpha} = 0,$$

and, then, we see that $\lim_{t\to 0+} t^{\alpha}(f(t)-f(0)) = 0$, which is the required. We shall now prove that the function $f(t)$ satisfies (2. 3). Let $S_n = \sum_{i=1}^{n} f_i$, $R_n = \sum_{=n+1}^{\infty} f_i$ and, for any function g, let

$$\Delta(g,\ u) = u^{\alpha+1}\left|\frac{g(2u)}{u} - \frac{G(2u)-G(0)}{u^2}\right|,$$

where $G(u)$ is an integral of g. Let us denote by $\widetilde{S}_n$ and $\widetilde{R}_n$ the function conjugate to S_n and R_n, respectively. We have then, by the well-known formula,

$$\widetilde{S}_{n-1}(u) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{S_{n-1}(x)\,dx}{2\tan\frac{1}{2}(x-u)}$$

$$= -\frac{1}{\pi}\left(\int_{-\lambda_1}^{-\lambda_{n-1}+2^{\alpha}\lambda_{n-1}} + \int_{\lambda_{n-1}-2^{\alpha}\lambda_{n-1}}^{\lambda_1}\right)$$

Then we se easily that $\widetilde{S}'_{n-1}(0)$ exists and $\widetilde{S}_{n-1}(u)$ is continuous in an interval containing origin. Therefore we have, $\tilde{\sigma}(u)$ denoting an integral of $\widetilde{S}_{n-1}(u)$,

$$\lim_{h\to 0}\frac{\tilde{\sigma}(2h)-\tilde{\sigma}(0)}{h^2} = \lim_{h\to 0}\frac{2\,\widetilde{S}_{n-1}(2h)}{2h} = 2\,\widetilde{S}'_{n-1}(0),$$

remembering that $\widetilde{S}_{n-1}(0) = 0$. Thus, for a fixed n, $\Delta(\widetilde{S}_{n-1},\ u) \to 0$ as $u \to 0$. Hence, if the sequence $\{\lambda_i\}$ decreases rapidly enough,

$$\Delta(\widetilde{S}_{n-1},\ \lambda_n) < 1 \qquad (n = 1,\ 2,\ 3, \ldots\ldots),$$

and then we may suppose that this condition is satisfied in addition to the conditions (2. 4), (2. 5) and (2. 6). Hence

$$\Delta(\widetilde{f},\ \lambda_n) \geqq \Delta(\widetilde{R}_{n-1},\ \lambda_n) - \Delta(\widetilde{S}_{n-1},\ \lambda_n) \geqq \Delta(\widetilde{R}_{n-1},\ \lambda_n) - 1.$$

Now

$$|\widetilde{R}_{n-1}(\lambda_n)| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} R_{n-1}(t)\frac{dt}{2\tan\frac{1}{2}(\lambda_n - t)}\right|$$

$$= \frac{1}{\pi}\int_{-\lambda_n}^{\lambda_n} R_{n-1}(t)\frac{dt}{2\tan\frac{1}{2}(\lambda_n - t)}$$

$$\geqq \frac{1}{\pi} \int_{\lambda_n - 2^\alpha \lambda_n}^{\lambda_n - \lambda_n^{1-\alpha}} R_{n-1}(t) \frac{dt}{2 \tan \frac{1}{2}(\lambda_n - t)}$$

$$\geqq \frac{1}{4\pi} \int_{\lambda_n - 2^{\alpha-1}\lambda_n - \frac{1}{2}\lambda_n^{1-\alpha}}^{\lambda_n - \lambda_n^{1-\alpha}} \varepsilon_n \lambda_n^{-\alpha} \frac{dt}{\lambda_n - t}$$

$$= \frac{\varepsilon_n \lambda_n^{-\alpha}}{4\pi} \left[\log \left\{ \lambda_n \left(2^{\alpha-1} + \frac{1}{2} \lambda_n^{-\alpha} \right) \right\} - \log \lambda_n^{1-\alpha} \right]$$

$$= \frac{\varepsilon_n \lambda_n^{-\alpha}}{4\pi} \left\{ \log \lambda_n + \log \left(2^{\alpha-1} + \frac{1}{2} \lambda_n^{-\alpha} \right) - (1-\alpha) \log \lambda_n \right\}$$

$$> \frac{\varepsilon_n \lambda_n^{-\alpha}}{4\pi} \{ \alpha \log \lambda_n + \log 2^{\alpha-1} \}$$

and hence, using (2. 6),

$$\lambda_n^\alpha |\widetilde{R}_{n-1}(\lambda_n)| > \frac{\alpha}{4\pi} \varepsilon_n \log \lambda_n + \frac{1}{4\pi} \varepsilon_n \log 2^{\alpha-1} \to +\infty \qquad \text{as } n \to \infty.$$

On the other hand, using the Riesz theorem [10; p. 148],

$$\left| \int_0^{\lambda_n} \widetilde{R}_{n-1}(t)\, dt \right| \leqq \lambda_n^{1/2} \left(\int_0^{\lambda_n} \widetilde{R}_{n-1}^2(t)\, dt \right)^{1/2}$$

$$\leqq \lambda_n^{1/2} \left(\int_{-\pi}^{\pi} \widetilde{R}_{n-1}^2(t)\, dt \right)^{1/2}$$

$$\leqq K \lambda_n^{1/2} \left(\int_{-\pi}^{\pi} R_{n-1}^2(t)\, dt \right)^{1/2}$$

$$\leqq K \lambda_n^{1/2} \left(\int_{-\lambda_n}^{\lambda_n} R_{n-1}^2(t)\, dt \right)^{1/2}$$

$$\leqq 2K \lambda_n^{1/2} (\varepsilon_n^2 \lambda_n^{-2\alpha+1})^{1/2} = 2K \varepsilon_n \lambda_n^{1-\alpha} = o(\lambda_n^{1-\alpha}),$$

where K is a constant. Therefore

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{1-\alpha}} \int^{\lambda_n} \widetilde{R}_{n-1}(t)\, dt = 0,$$

and then

$$\lim_{n\to\infty} \Delta(\widetilde{f}, \lambda_n) = +\infty,$$

which proves (2. 3). Thus the function $f(t)$ satisfies the conditions (2. 1), (2. 2) and (2. 3). Using this function, we shall now define a series $\sum_{n=1}^\infty a_n$ evaluable $(R, 2, \alpha+1)$ but not evaluable $(R, 2, \alpha)$. Let the series $\sum_{n=1}^\infty a_n$ be defined by the relation $s_n^{\alpha+1} = n^2 b_n$, $s_n^{\alpha+1}$ being the $(C, \alpha+1)$ sum of the series Σa_n. Then, by a lemma due to Marcinkiewicz [6; Lemma 1],

$$(2.8)\qquad \begin{aligned} t^{\alpha+1}\sum_{n=1}^{\infty} s_n^{\alpha}\left(\frac{\sin nt}{nt}\right)^2 &= -t^{\alpha+1}\left(\frac{\tilde{f}(2t)}{t} - \frac{\tilde{F}(2t)-\tilde{F}(0)}{t^2}\right) + \rho(t), \\ t^{\alpha+2}\sum_{a=1}^{\infty} s_n^{\alpha+1}\left(\frac{\sin nt}{nt}\right)^2 &= -\frac{1}{2}\, t^{\alpha}(f(2t)-f(0)), \end{aligned}$$

$\rho(t)$ denoting a continuous function for all t. Thus (2. 2) and (2. 3) show that the series $\sum_{n=1}^{\infty} a_n$ is evaluable $(R, 2, \alpha+1)$ but not evaluable $(R, 2, \alpha)$ and the proof of Theorem is complete.

3. Proof of Theorem 2. We take the two sequences $\{\lambda_i\}$ and $\{\varepsilon_i\}$ in the former section, that is, these sequences satisfy the conditions (2. 4), (2. 5) and (2. 6). Further let the sequence $\{\lambda_i\}$ satisfy

$$(3.1)\qquad 2\lambda_{i+1} < \lambda_i.$$

Then we define the function $\tilde{g}(t)$ in the same way as we defined $f(t)$ in (2, 7) except that now we suppose $\tilde{g}(t)$ to be an odd function. Let us denote by $g(t)$ the function conjugate to $\tilde{g}(t)$. Let

$$g(t) \sim \sum b_n \cos nt.$$

Then the convergency of the series $\sum |b_n|$ is seen, by the method analogous to one which we proved (2. 1). Furthermore, we have easily

$$(3.2)\qquad \lim_{t\to 0+} \{t^{\alpha}\tilde{g}(2t) - t^{\alpha-1}(\tilde{G}(2t) - \tilde{G}(0))\} = 0.$$

Now we shall prove that

$$(3.3)\qquad \limsup_{t\to 0+} t^{\alpha}|g(t) - g(0)| = +\infty.$$

Let us write

$$\tilde{S}_n = \sum_{i=1}^{n} \tilde{g}_i \qquad \text{and} \qquad \tilde{R}_n = \sum_{i=n+1}^{\infty} g_i,$$

and denote by S_n and R_n the function conjugate to $\tilde{S}_n$ and $\tilde{R}_n$, respectively. Then we see easily that $S_{n-1}(t)$ is even and have the derivative at origin. Hence $S'_{n-1}(0) = 0$. Therefore, if the sequence $\{\lambda_i\}$ is chosen suitably,

$$(3.4)\qquad \left|\frac{S_{n-1}(\lambda_n) - S_{n-1}(0)}{\lambda_n}\right| < 1 \qquad (n = 2, 3, 4, \ldots\ldots\ldots\ldots),$$

and we may suppose that this condition is satisfied in addition to the conditions (2. 4), (2. 5) and (2. 6). Let us write

$$\lambda_n^\alpha |g(\lambda_n) - g(0)| \quad \lambda_n^\alpha |g_n(\lambda_n) - g_n(0)| - \lambda_n^\alpha |S_{n-1}(\lambda_n) - S_{n-1}(0)|$$
$$- \lambda_n^\alpha |R_n(\lambda_n) - R_n(0)|$$
$$= U_n - V_n - W_n.$$

Then, by (3. 4),

$$V_n = \lambda_n^{1+\alpha} \left| \frac{S_{n-1}(\lambda_n) - S_{n-1}(0)}{\lambda_n} \right| < 1 \qquad (n = 2, 3, 4, \ldots\ldots\ldots\ldots), \tag{3. 5}$$

and, using (3. 1),

$$W_n \leqq \frac{\lambda_n^\alpha}{\pi} \left| \int_{-\lambda_{n+1}}^{\lambda_{n+1}} \tilde{R}_n(t) \frac{\sin \frac{1}{2}\lambda_n}{2 \sin \frac{1}{2}(t - \lambda_n) \sin \frac{1}{2} t} dt \right|$$

$$\leqq \frac{\lambda_n^{\alpha+1}}{\lambda_n - \lambda_{n+1}} \int_{-\lambda_{n+1}}^{\lambda_{n+1}} \frac{\tilde{R}_n(t)}{2 \sin \frac{1}{2} t} dt$$

$$= \frac{\lambda_n^{\alpha+1}}{\lambda_n - \lambda_{n+1}} \sum_{i=n+1}^{\infty} \int_{\lambda_i - 2^\alpha \lambda_i}^{\lambda_i} \frac{\tilde{g}_i(t)}{\sin \frac{1}{2} t} dt \tag{3. 6}$$

$$\leqq \frac{2\lambda_n^{\alpha+1}}{\lambda_n - \lambda_{n+1}} \sum_{i=n+1}^{\infty} \frac{\varepsilon_i \lambda_i^{-\alpha}}{\lambda_i - 2^\alpha \lambda_i} \cdot 2^\alpha \lambda_i$$

$$\leqq \frac{2^{\alpha+1} \lambda_n}{\lambda_n - \lambda_{n+1}} \cdot \frac{1}{1 - 2^\alpha} \sum_{i=n+1}^{\infty} \varepsilon_i = o\left(\frac{\lambda_n}{\lambda_n - \lambda_{n+1}} \right) = o(1).$$

Since

$$\lambda_n^\alpha |g_n(0)| = \frac{\lambda_n^\alpha}{\pi} \int_{-\pi}^{\pi} \tilde{g}_n(t) \frac{dt}{2 \tan \frac{t}{2}}$$

$$= \frac{2\lambda_n^\alpha}{\pi} \int_0^{\lambda_n} \tilde{g}_n(t) \frac{dt}{2\tan \frac{t}{2}}$$

$$= \frac{2\lambda_n^\alpha}{\pi} \int_{\lambda_n - 2^\alpha \lambda_n}^{\lambda_n} \tilde{g}_n(t) \frac{dt}{2 \tan \frac{t}{2}}$$

$$\leqq \frac{2\lambda_n^\alpha}{\pi(\lambda_n - 2^\alpha \lambda_n)} \cdot \varepsilon_n \lambda_n^{-\alpha} \cdot 2^\alpha \lambda_n = \frac{2^{\alpha+1} \varepsilon_n}{\pi(1 - 2^\alpha)} = o(1),$$

we have, arguing as in the former section,

$$U_n = \frac{\lambda_n^\alpha}{\pi} \int_{-\pi}^{\pi} \tilde{g}_n(t) \left\{ \frac{1}{2\tan\frac{1}{2}(t-\lambda_n)} - \frac{1}{2\tan\frac{t}{2}} \right\} dt$$

$$(3.\ 7) \qquad \geqq \frac{\lambda_n^\alpha}{\pi} \left| \int_{-\lambda_n}^{\lambda_n} \tilde{g}_n(t) \frac{dt}{2\tan\frac{1}{2}(t-\lambda_n)} \right| - \frac{\lambda_n^\alpha}{\pi} \left| \int_{-\lambda_n}^{\lambda_n} \tilde{g}_n(t) \frac{dt}{2\tan\frac{t}{2}} \right|$$

$$\geqq \frac{\varepsilon_n}{2\pi} \{\alpha \log \lambda_n + \log 2^{\alpha-1}\} + o(1) \to +\infty, \qquad \text{as } n \to \infty.$$

Summing up (3. 5), (3. 6) and (3. 7), we get (3. 3). We shall now define a series evaluable $(R, 2, \alpha)$ but not evaluable $(R, 2, \alpha+1)$. Let the series $\sum_{n=1}^{\infty} a_n$ be defined by the relation $s_n^{\alpha+1} = n^2 b_n$, $s_n^{\alpha+1}$ being the $(C, \alpha+1)$ sum of the series $\sum_{n=1}^{\infty} a_n$. Then, by (2. 8), (3. 2) and (3. 3), we see that the series $\sum_{n=1}^{\infty} a_n$ is evaluable $(R, 2, \alpha)$ but not evaluable $(R, 2, \alpha+1)$. Thus the theorem is completely proved.

4. Proof of Theorem 3. For the proof, we need the following Lemmas.

LEMMA 1. *Let $r > -1$. Suppose that* (1. 3) *holds and the series* $\sum_{n=0}^{\infty} a_n$ *is Abel evaluable to s. Then*

$$(4.\ 1) \qquad \sigma_n^{r+1} = O(1)$$

and

$$(4.\ 2) \qquad \sigma_n^{r+1+\varepsilon} \to s \qquad \text{as } n \to \infty.$$

where ε is an arbitrary positive number.

PROOF. Rajagopal [8 ; series-analogue of Lemma 6] proved (4. 1) and

$$(4.\ 3) \qquad \sigma_n^{r+2} \to s \qquad \text{as } n \to \infty.$$

Then, by the well-known theorem [1 ; Theorem 70], (4. 2) follows.

LEMMA 2. *Let $r > -1$. Then the condition* (1. 3) *implies*

$$(4.\ 4) \qquad \sum_{n=0}^{m} |\sigma_n^r| = O(m)$$

and, for $\varepsilon > 0$,

$$\sum_{n=0}^{m} |\sigma_n^{r+\epsilon}| = O(m),$$

and conversely (4. 4) *implies* (1. 3).

Since $\Gamma(r+1)A_n^r \sim n^r$ $(r \neq -1, -2, \ldots\ldots)$, Lemma is obtained, in obvious way, using Abel transformation.

We shall now prove Theorem. The three cases are considered:

$$\text{(I) } \alpha < r, \text{ (II) } \alpha = r, \text{ (III) } \alpha > r.$$

We shall first prove that Theorem in the case (I) is reduced to Theorem C. Since $r + 1 < p$, we take ε such that $r + 1 + \varepsilon < p$. Then, by Lemma 1, the series $\sum_{n=0}^{\infty} a_n$ is evaluable $(C, r + 1 + \varepsilon)$ and, by Lemma 2, (1. 3) implies

$$\sum_{\nu=0}^{n} |s_\nu^{r+\epsilon}| = O(n^{r+1+\epsilon}).$$

Hence, by Theorem C, the series $\sum_{n=0}^{\infty} a_n$ evaluable (R, p, α) to s. Theorem in the case (II) is Theorem D. The case (III) is reduced to the case (II), reasoning that when the conditions of Theorem are satisfied for r, they are also satisfied for α, by Lemma 2. Thus Theorem is proved.

5. **Proof of Theorem 4.** For the proof we need the following

LEMMA 3. *Let the series* $\sum_{\nu=1}^{\infty} b_\nu$ be converge and let $c_n = \sum_{\nu=n}^{\infty} b_\nu$. *Then*

$$\sum_{\nu=n}^{m} a_\nu b_\nu = a_n c_n - a_m c_{m+1} - \sum_{\nu=n}^{m-1} c_{\nu+1} \Delta a_\nu,$$

where $\Delta a_\nu = a_\nu - a_{\nu+1}$. *In particular, if* $a_m c_{m+1} \to 0$ *as* $m \to \infty$,

$$\sum_{\nu=n}^{\infty} a_\nu b_\nu = a_n c_n - \sum_{\nu=n}^{\infty} c_{\nu+1} \Delta a_\nu.$$

Proof of Lemma is obvious, so that we omit it. We shall now prove Theorem. Since Theorem in which α is an integer is Theorem E, we suppose that α is not an integer and write the greatest integer less than α by β. Then $p - 2 \quad \beta$, by $\alpha + 1 < p$. We may suppose, without loss of generality, that $s = 0$. Let $\varphi(t) = (t^{-1} \sin t)^p$ and let $\Delta^m \varphi(nt)$ denote the m-th difference of $\varphi(nt)$ with respect to n. Then Obreschkoff [7] showed

$$\Delta^m \varphi(nt) = O(t^m) \tag{5. 1}$$

and

(5. 2) $$\Delta^m \varphi(nt) = O(t^{m-p} n^{-p}),$$

for $m, n = 1, 2, 3, \ldots\ldots$ and $t > 0$. By the repeated use of Abel's transformation,

$$t^{\alpha+1} \sum_{n=1}^{m} s_n^\alpha \varphi(nt) = t^{\alpha+1} \sum_{n=1}^{m-p+\beta+1} s_n^{\alpha+p-\beta-1} \Delta^{p-\beta-1} \varphi(nt)$$

$$+ t^{\alpha+1} \sum_{i=1}^{p-\beta-1} s_{m-i+1}^{\alpha+i} \Delta^{i-1} \varphi((m-i+1)t).$$

Since the summability $|C, p|$ implies the summability (C, p), when $0 \quad i \quad p$,

(5. 3) $$s_n^i = o(n^p).$$

Then, using (5. 2), for a fixed $t > 0$,

$$t^{\alpha+1} \sum_{i=1}^{p-\beta-1} s_{m-i+1}^{\alpha+i} \Delta^{i-1} \varphi((m-i+1)t)$$

$$= o\left(\sum_{i=1}^{p-\beta-1} (m-i+1)^p \cdot (m-i+1)^{-p}\right) = o(1),$$

when $m \to \infty$. Therefore,by the identity

$$s_n^{\alpha+p-\beta-1} = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-\beta-1} s_\nu^{p-1},$$

$$t^{\alpha+1} \sum_{n=1}^{\infty} s_n^\alpha \varphi(nt) = t^{\alpha+1} \sum_{n=1}^{\infty} s_n^{\alpha+p-\beta-1} \Delta^{p-\beta-1} \varphi(nt)$$

$$= t^{\alpha+1} \sum_{n=1}^{\infty} \Delta^{p-\beta-1} \varphi(nt) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-\beta-1} s_\nu^{p-1}$$

(5. 4) $$= t^{\alpha+1} \sum_{\nu=1}^{\infty} s_\nu^{p-1} \sum_{n=\nu}^{\infty} A_{n-\nu}^{\alpha-\beta-1} \Delta^{p-\beta-1} \varphi(nt)^{1)}$$

$$= t^{\alpha+1} \sum_{\nu=1}^{\infty} s_\nu^p \sum_{n=\nu}^{\infty} A_{n-\nu}^{\alpha-\beta-1} \Delta^{-\beta} \varphi(nt)$$

$$= \sum_{\nu=1}^{\infty} \sigma_\nu^p (t^{\alpha+1} A_\nu^p \sum_{n=\nu}^{\infty} A_{n-\nu}^{\alpha-\beta-1} \Delta^{p-\beta} \varphi(nt))$$

$$= \sum_{\nu=1}^{\infty} (\sigma_\nu^p - \sigma_{\nu+1}^p) U_\nu(t),$$

where

$$U_m(t) = t^{\alpha+1} \sum_{\nu=1}^{m} A_\nu^p \sum_{n=\nu}^{\infty} A_{n-\nu}^{\alpha-\beta-1} \Delta^{p-\beta} \varphi(nt),$$

provided that the inversion in (5. 4) is legitimate and that, for $0 < t < 1$ and

---

1) Here, we use $\sum_{n=\nu}^{\infty} A_{n-\nu}^{\alpha-\beta-1} \Delta^{p-\beta-1} \varphi(nt) = O(t^{-\alpha-1} \nu^{-p})$ in (5. 11) below and $s_\nu^p = o(\nu^p)$.

$m = 1, 2, 3, \ldots\ldots,$

(5. 5) $$U_m(t) = O(1).$$

Therefore, for the proof, it is sufficient to prove that the series

(5. 6) $$\sum_{\nu=1}^{\infty} (\sigma_\nu^p - \sigma_{\nu+1}^p) U_\nu(t)$$

converges in $0 < t < t_0$ and its sum tends to zero as $t \to 0+$. To justify that the inversion in (5. 4) is legitimate, we must show that,for a fixed $t > 0$,

(5. 7) $$\lim_{N\to\infty} \sum_{\nu=1}^{N} s_\nu^{p-1} \sum_{n=N+1}^{\infty} A_{n-\nu}^{\alpha-\beta-1} \Delta^{p-\beta-1} \varphi(nt) = 0.$$

Since $\beta \leqq p-2$ and

$$\sum_{\nu=n}^{\infty} \Delta^{p-\beta-1} \varphi(\nu t) = \Delta^{p-\beta-2} \varphi(nt),$$

we have, by Lemma 3, for ν such that $0 \quad \nu \leqq N$,

(5. 8) $$\begin{aligned}\sum_{n=N+1}^{\infty} A_{n-\nu}^{\alpha-\beta-1} \Delta^{p-\beta-1} \varphi(nt) &= A_{N-\nu+1}^{\alpha-\beta-1} \Delta^{p-\beta-2} \varphi((N+1)t) \\ &\quad + \sum_{n=N+1}^{\infty} A_{n-\nu}^{\alpha-\beta-2} \Delta^{p-\beta-2} \varphi((n+1)t) \\ &= O((N-\nu+1)^{\alpha-\beta-1} N^{-p}).\end{aligned}$$

Then, putting $[N/2] = \mu$,

$$\sum_{\nu=1}^{N} s_\nu^{p-1} \sum_{n=N+1}^{\infty} A_{n-\nu}^{\alpha-\beta-1} \Delta^{p-\beta-1} \varphi(nt) = \Big(\sum_{\nu=1}^{\mu-1} + \sum_{\nu=\mu}^{N}\Big) = I_1 + I_2,$$

say. Now, by the series-analogue of Rajagopal's lemma [8 ; Lemma 10],

(5. 9) $$\sum_{\nu=1}^{n} |s_\nu^{p-1}| = O(n^p),$$

from our assumption. Hence, by (5. 8),

$$\begin{aligned}I_1 &= O\Big(\sum_{\nu=1}^{\mu-1} |s_\nu^{p-1}| (N-\nu+1)^{\alpha-\beta-1} N^{-p}\Big) \\ &= O\Big(N^{-p} N^{\alpha-\beta-1} \sum_{\nu=1}^{N} |s_\nu^{p-1}|\Big) \\ &= O(N^{\alpha-\beta-1}) = o(1).\end{aligned}$$

Concerning I_2, we take an arbitrarily fixed L large enough. Then, using (5. 9) and (5. 3) for $i = p-1$,

$$\begin{aligned}I_2 &= O\Big(N^{-p} \sum_{\nu=\mu}^{N} |s_\nu^{p-1}| (N+1-\nu)^{\alpha-\beta-1}\Big) \\ &= O\Big(N^{-p} \sum_{\nu=1}^{N+1-\mu} |s_{N+1-\nu}^{p-1}| \nu^{\alpha-\beta-1}\Big)\end{aligned}$$

$$= O\Big(N^{-p}\sum_{\nu=n}^{L-1}\Big) + O\Big(N^{-p}\sum_{\nu=L}^{N+1-}\Big)$$

$$= o(L^{\alpha-\beta}) + O\Big(N^{-p}\sum_{\nu=1}^{L}|s_{N+1-\nu}^{p-1}|\cdot L^{\alpha-\beta-1}\Big)$$

$$+ O\Big(N^{-p}\sum_{=1}^{N+1-\mu}|s_{N+1-\nu}^{p-1}|(N+1-\mu)^{\alpha-\beta-1}\Big) + O\Big(N^{-p}\sum_{\nu=L}^{N-\mu}\Big(\sum_{n=1}^{\nu}|s_{n+N+1}^{p-1}|\Big)\nu^{\alpha-\beta-2}\Big)$$

$$= o(1) + O(L^{\alpha-\beta-1}) + O(N^{\alpha-\beta-1}) + O\Big(\sum_{\nu=L}^{\infty}\nu^{\alpha-\beta-2}\Big)$$

$$= o(1) + O(L^{\alpha-\beta-1}).$$

Since L is arbitrary, by $\alpha-\beta-1<0$, $I_2=o(1)$ which proves, together with $I_1=o(1)$, (5. 7). We shall next prove that (5. 5) holds. Now, by Lemma putting $\rho=[t^{-1}]$,

$$\sum_{n=\nu}^{\infty}A_{n-\nu}^{\alpha-\beta-1}\Delta^{p-\beta}\varphi(nt) = \Big(\sum_{n=\nu}^{\nu+\rho-1} + \sum_{n=\nu+\rho}^{\infty}\Big)$$

$$= O\Big(t^{-\beta}\nu^{-p}\sum_{n=\nu}^{\nu+\rho-1}A_{n-\nu}^{\alpha-\beta-1}\Big) + A_{\rho}^{-\beta-1}\Delta^{p-\beta-1}\varphi((\nu+\rho)t)$$

$$- \sum_{n=\nu+\rho}^{\infty}A_{n-\nu+1}^{\alpha-\beta-2}\Delta^{p-\beta-1}\varphi((n+1)t)$$

$$= O(t^{-\beta}\nu^{-p}A_{\rho-1}^{\alpha-\beta}) + O(t^{-\beta-1}\nu^{-p}A_{\rho}^{\alpha-\beta-1})$$

$$+ O\Big(t^{-\beta-1}(\nu+\rho)^{-p}\sum_{n=\rho}^{\infty}n^{\alpha-\beta-2}\Big)$$

$$= O(t^{-\alpha}\nu^{-p}) + O(t^{-\alpha}\nu^{-p}) + O(t^{-\alpha}\nu^{-p})$$

$$= O(t^{-\alpha}\nu^{-p}).$$

Hence, if $mt \quad 1$, then

$$U_m(t) = O\Big(t\sum_{\nu=1}^{m}\nu^{p}\cdot\nu^{-p}\Big) = O(mt) = O(1). \tag{5. 10}$$

Thus,we have to prove for the case in which $mt>1$. For $p\geqq 2$,

$$\sum_{n=\nu}^{\infty}A_{n-\nu}^{\alpha-\beta-1}\Delta^{p-\beta-2}\varphi(nt) = O\Big(t^{-\beta-2}\sum_{n=\nu}^{2\nu-1}A_{n-\nu}^{\alpha-\beta-1}n^{-p}\Big) + O\Big(t^{-\beta-2}\sum_{n=2\nu}^{\infty}(n-\nu)^{-\beta-1}n^{-p}\Big)$$

$$= O\Big(t^{-\beta-2}\nu^{-p}\sum_{n=\nu}^{2\nu}A_{n-\nu}^{\alpha-\beta-1}\Big) + O\Big(t^{-\beta-2}\nu^{\alpha-\beta-1}\sum_{n=2\nu}^{\infty}n^{-p}\Big)$$

$$= O(t^{-\beta-2}\nu^{-p+\alpha-\beta}) + O(t^{-\beta-2}\nu^{-p+\alpha-\beta})$$

$$= O(t^{-\beta-2}\nu^{-p+\alpha-\beta}).$$

For $p = 1$, then $-1 < \alpha < 0$ so that $\beta = -1$. Since

$$\sum_{n=\nu}^{\infty} \frac{\sin nt}{n} = O(t^{-1}\nu^{-1}),$$

if $\nu t > 1$, then

$$\sum_{n=\nu}^{\infty} A_{n-\nu}^{\alpha-\beta-1} \Delta^{p-\beta-2} \varphi(nt) = \sum_{n=\nu}^{\infty} A_{n-\nu}^{\alpha} \frac{\sin nt}{nt} = \left(\sum_{n=\nu}^{2\nu-1} + \sum_{n=2\nu}^{\infty} \right)$$

$$= O\left(t^{-1}\nu^{-1} \sum_{n=\nu}^{2\nu} A_{n-\nu}^{\alpha}\right) + A_{\nu}^{\alpha} \cdot O(t^{-2}\nu^{-1}) + O\left(\sum_{n=2\nu}^{\infty} \nu^{-1}t^{-1}(n-\nu)^{\alpha}\right)$$

$$= O(t^{-1}\nu^{\alpha}) + O(t^{-1}\nu^{\alpha}) + \left(t^{-1} \sum_{n=2\nu}^{\infty} (n-\nu)^{\alpha-1}\right)$$

$$= O(t^{-1}\nu^{\alpha}) = O(t^{-\beta-2}\nu^{-p+\alpha-\beta}).$$

Similarly, for p 1, putting $\rho = [t^{-1}]$,

$$\sum_{n=\nu}^{\infty} A_{n-\nu}^{\alpha-\beta-1} \Delta^{p-\beta-1} \varphi(nt) = \left(\sum_{n=\nu}^{\nu+\rho-1} + \sum_{n=\nu+\rho}^{\infty} \right) = O(t^{-\alpha-1}\nu^{-p}). \tag{5.11}$$

Now

$$U_m(t) = t^{\alpha+1} \sum_{\nu=1}^{m} A_{\nu}^{p} \sum_{n=\nu}^{\infty} A_{n-\nu}^{\alpha-\beta-1} \Delta^{p-\beta}\varphi(nt) = t^{\alpha+1} \left(\sum_{\nu=1}^{-1} + \sum_{\nu=\rho}^{m} \right) = U_{m1} + U_{m2},$$

say, where $U_{m1} = O(1)$ by (5. 10). Since

$$\sum_{\nu=i}^{\infty} \sum_{n=\nu}^{\infty} A_{n-\nu}^{\alpha-\beta-1} \Delta^{p-\beta-j} \varphi(nt) = \sum_{n=i}^{\infty} A_{n-i}^{\alpha-\beta-1} \Delta^{p-\beta-j-1} \varphi(nt), \qquad (j = 0,\ 1),$$

and

$$A_{\nu}^{p} - A_{\nu+1}^{p} = -A_{\nu+1}^{p-1},$$

we have, by Lemma 3,

$$U_{m2} = t^{\alpha+1} \Big[A_{\rho}^{p} \sum_{n=\rho}^{\infty} A_{n-\rho}^{\alpha-\beta-1} \Delta^{p-\beta-1} \varphi(nt) - A_{m}^{p} \sum_{n=m+1}^{\infty} A_{n-m-1}^{\alpha-\beta-1} \Delta^{p-\beta-1} \varphi(nt)$$

$$+ \sum_{\nu=\rho}^{m-1} A_{\nu+1}^{p-1} \sum_{n=\nu+1}^{\infty} A_{n-\nu-1}^{\alpha-\beta-1} \Delta^{p-\beta-1} \varphi(nt) \Big]$$

$$= t^{\alpha+1} \Big[A_{\rho}^{p} \sum_{n=\rho}^{\infty} A_{n-\rho}^{\alpha-\beta-1} \Delta^{p-\beta-1} \varphi(nt) - A_{m}^{p} \sum_{n=m+1}^{\infty} A_{n-m-1}^{\alpha-\beta-1} \Delta^{p-\beta-1} \varphi(nt)$$

$$+ A_{\rho+1}^{p-1} \sum_{n=\rho+1}^{\infty} A_{n-\rho-1}^{\alpha-\beta-1} \Delta^{p-\beta-2} \varphi(nt) - A_{m}^{p-1} \sum_{n=m+1}^{\infty} A_{n-m-1}^{\alpha-\beta-1} \Delta^{p-\beta-2}\varphi(nt)$$

$$+ \sum_{\nu=\rho}^{m-2} A_{\nu+2}^{p-2} \sum_{n=\nu+2}^{\infty} A_{n-\nu-2}^{\alpha-\beta-1} \Delta^{p-3-2} \varphi(nt) \Big]$$

$$= O(t^{\alpha+1}\cdot\rho^{p}\cdot t^{-\alpha-1}\rho^{-p}) + O(t^{\alpha+1}m^{p}\cdot t^{-\alpha-1}m^{-p})$$
$$+ O(t^{\alpha+1}\rho^{p-1}\cdot t^{-\beta-2}\rho^{-p+\alpha-\beta}) + O(t^{\alpha+1}m^{p-1}\cdot t^{-\beta-2}m^{-p+\alpha-\beta})$$
$$+ O\left(t^{\alpha+1}\sum_{\nu-\rho}^{\infty}\nu^{p-2}\cdot t^{-\beta-2}\nu^{-p+\alpha-\beta}\right)$$
$$= O(1) + O((\rho t)^{\tau-\beta-1}) + O((mt)^{\alpha-\beta-1}) + O((\rho t)^{\alpha-\beta-1})$$
$$= O(1).$$

This proves (5. 5) for $mt > 1$. We shall now consider the series (5. 6). In virture of (5. 5), by the assumption that

$$\sum_{n=1}^{\infty} \sigma_n^p - \sigma_{n+1}^p| < +\infty,$$

the series (5. 6) is converges (absolutely) in $0 < t < 1$. Further, for an arbitrary positive number ε, there exists an $N = N(\varepsilon)$ such that

$$\left|\sum_{\nu=N}^{\infty}(\sigma_\nu^p - \sigma_{\nu+1}^p)\,U_\nu(t)\right| < \varepsilon.$$

On the other hand, for a fixed $\nu > 0$, obviously

$$t^{\alpha+1}\sum_{n=\nu}^{\infty} A_{n-\nu}^{\alpha-\beta-1}\,\Delta^{p-\beta}\varphi(nt) = O\left(t^{\alpha+1}\sum_{n=\nu}^{\infty}(n-\nu+1)^{\alpha-\beta-1}t^{-\beta}n^{-p}\right)$$
$$= O\left(t^{\alpha-\beta+1}\sum_{n=\nu}^{\infty}(n-\nu+1)^{\alpha-\beta-p-1}\right)$$
$$= O\,(t^{\alpha-\beta+1}).$$

Then, for m N,

$$U_m(t) = t^{\alpha+1}\sum_{\nu=1}^{m}A_\nu^p\sum_{n=\nu}^{\infty}A_{n-\nu}^{\alpha-\beta-1}\,\Delta^{p-\beta}\varphi(nt)$$
$$= O\left(t^{\alpha-\beta+1}\sum_{\nu=1}^{m}\nu^p\right) = O(t^{\alpha-\beta-1}N^{p+1}).$$

Hence

$$\lim_{t\to 0+}\sum_{\nu=1}^{N}(\sigma_\nu^p - \sigma_{\nu+1}^p)\,U_\nu(\mathrm{t}) = 0.$$

Therefore we have

$$\limsup_{t\to 0+}\left|\sum_{\nu=1}^{\infty}(\sigma_\nu^p - \sigma_{\nu+1}^p)\,U_\nu(t)\right| \leqq \varepsilon$$

Since ε is arbitrary, we get

$$\lim_{t\to 0+}\sum_{\nu=1}^{\infty}(\sigma_\nu^p - \sigma_{\nu+1}^p)\,U_\nu(t) = 0,$$

and Theorem is completely proved.

REFERENCES

[1] G. H. HARDY, Divergent series, 1949, Oxford.

[2] G. H. HARDY and W. W. ROGOSINSKI, Note on Fourier series (IV), summability (R_2), Proc. Cambridge Phil. Soc. 43 (1947), 10-25.

[3] H. HIROKAWA, Riemann-Cesàro methods of summability, Tôhoku Math. Journ. 7 (1955), 279-295.

[4] H. HIROKAWA, Riemann-Cesàro methods of summability II, Tôhoku Math. Journ. 9 (1957), 13-26.

[5] B. KUTTNER, Some relations between diffrent kinds of Riemann summability, Proc. London Math. Soc. 40 (1936), 524-540.

[6] J. MARCINKIEWICZ, On Riemann's two methods of summation, Journ. Londn Math. Soc. 10 (1936), 268-272.

[7] N. OBRESCHKOFF, Über das Riemannsche Summierungsverhahren, Math. Zeits. 48 (1942), 441-454.

[8] C. T. RAJAGOPAL, On the Riemann- Cesàro summability of series and integrals, Tôhoku Math. Journ. 9 (1957), 247-263.

[9] M. RIESZ, Sur un théorème de la moyenne et ses applications, Acta Szeged, 1 (1922), 114-126.

[10] A. ZYGMUND, Trigonometrical series, 1935, Warszawa.

CHIBA INSTITUTE OF TECHNOLOGY, CHIBA, JAPAN.