ON THE RIESZ SUMMABILITY OF FOURIER SERIES

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Let $f(x)$ be an integrable and periodic function with period 2π, and let

$$(1) \qquad f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

F.T. Wang [4] proved the following theorem:
If $1 < \alpha < 2$, and the series

$$\sum_{n=2}^{\infty}(a_n^2 + b_n^2)(\log n)^{\alpha-1}$$

converges, then the Fourier series (1) is summable $(R, \exp(\log n)^\alpha, \delta)$ almost everywhere, for any positve δ.

In this note we shall give some better results than the above theorem.

THEOREM 1. *If $1 < \alpha < \infty$, and the series*

$$\sum_{n=2}^{\infty}(a_n^2 + b_n^2)\{\log(\log n)\}$$

converges, then Fourier series (1) *is summable* $(R, \exp(\log n)^\alpha, \delta)$ *almost everywhere for any positive* δ.

THEOREM 2. *If $0 < \alpha \leqq 1$, and the series*

$$\sum_{n=2}^{\infty}(a_n^2 + b_n^2)(\log n)^\alpha$$

converges, then the Fourier series (1) *is summable* $(R, \exp\{\exp(\log n)^\alpha\}, \delta)$, *almost everywhere for any positive* δ.

In Theorem 2, if we put $\alpha = 1$, then the convergency of $\sum_{n=2}^{\infty}(a_n^2 + b_n^2)\log n$ implies the (R, e^n, δ) summability of (1) almost everywhere. Since (R, e^n, δ) summation is equivalent to convergence, this case is nothing but the theorem of Kolmogoroff-Seliverstoff-Plessner. Thus our theorems link the theorem of Kolmogoroff-Seliverstoff-Plessner and the theorem of Fejér-Lebesgue. Improvement of our results may be difficult.

Our theorems are easy consequences of the following two propositions.

PROPOSITION 1. *The Lebesgue constant of* $(R, \Lambda_n, 1)$ *summation of the*

Fourier series (1) *is* $\log(n\lambda_n/\Lambda_n)$ *where* $\lambda_n=\Lambda_n-\Lambda_{n-1}\geqq 0$, *provided that*
(2) λ_n *is non-decreasing* *and*
(3) λ_n/Λ_n *is non-increasing.*

PROPOSITION 2. *Let the Lebesgue constant of the* $(R, \Lambda_n, 1)$ *summation of the orthogonal development*

$$f(x)\sim\sum_{n=1}^{\infty} c_n\varphi_n(x) \tag{4}$$

be $w(n)$, *then*

$$\sum_{n=2}^{\infty} a_n^2 w(n) < \infty \tag{5}$$

implies $(R,\Lambda_n, 1)$*-summability of the series* (4) *almost everywhere.*

The Lebesgue constant of (R, Λ_n, δ)-summability was given by K.Matsumoto [3]. But Proposition 1, the special case of his result, is very simple. For the sake of completeness we give the proof of Proposition 1. The Cesàro summability case of Prop. 2 was given by S. Kaczmarz [1]. The method of proof of this proposition is the line of Plessner.

PROOF OF PROPOSITION 1. If we put

$$\sum_{k=0}^{n}\lambda_k = \Lambda_n, \qquad \lambda_k\geqq 0$$

then $\Lambda_n\to\infty$ as $n\to\infty$, since λ_n is non-decreasing. The kernel of $(R, \Lambda_n, 1)$-summability of Fourier series (1) is

$$K_n(t)=\frac{1}{2\Lambda_n\sin(t/2)}\sum_{k=0}^{n}\lambda_k\sin\left(k+\frac{1}{2}\right)t$$

and the Lebesgue constant is

$$\begin{aligned} w(n) &= \int_0^{\pi}|K_n(t)|\,dt = \int_0^{\lambda_n/\Lambda_n}+\int_{\lambda_n/\Lambda_n}^{\pi} \\ &= P_n+Q_n, \end{aligned}$$

say. The inner sum of P_n is

$$\begin{aligned} S_n &= \sum_{k=0}^{n}\lambda_k\sin\left(k+\frac{1}{2}\right)t=\sum_{k=0}^{n}\lambda_k\sin kt+\sum_{n=0}^{k}\lambda_k\left\{\sin\left(k+\frac{1}{2}\right)t-\sin kt\right\} \\ &= \sum_{k=0}^{n}\lambda_k\sin kt+\sum_{k=0}^{n}\lambda_k\cos\left(2k+\frac{1}{2}\right)t\sin\frac{t}{2} \end{aligned}$$

and

$$|S_n|\leqq\left|\sum_{k=0}^{n}\lambda_k\sin kt\right|+\Lambda_n\sin\frac{t}{2}$$

$$\leqq \sum_{k=0}^{n-1} \Lambda_k \sin\frac{t}{2}\left|\cos(k+1)t\right| + \Lambda_n \sin nt + \Lambda_n \sin\frac{t}{2}$$

$$\leqq \sin\frac{t}{2}\sum_{k=0}^{n-1}\Lambda_k + \Lambda_n \sin nt + \Lambda_n \sin\frac{t}{2}.$$

Since λ_n/Λ_n is non-increasing,

$$\sum_{k=0}^{n}\Lambda_k \leqq \frac{\Lambda_n}{\lambda_n}\sum_{k=0}^{n-1}\frac{\lambda_k}{\Lambda_k}\Lambda_k = \frac{\Lambda_n^2}{\lambda_n}.$$

Thus we have

$$P_n = \int_0^{\lambda_n/\Lambda_n}\frac{1}{\Lambda_n}\Big(\sum_{k=0}^{n-1}\Lambda_k\Big)dt + \int_0^{\lambda_n/\Lambda_n}\frac{1}{\Lambda_n}\left|\frac{\sin nt}{\sin(t/2)}\right|dt + \int_0^{\lambda_n/\Lambda_n} dt$$

$$\leqq \frac{1}{\Lambda_n}\frac{\Lambda_n^2}{\lambda_n}\frac{\lambda_n}{\Lambda_n} + \frac{1}{\Lambda_n}\int_0^{\lambda_n/\Lambda_n}\left|\frac{\sin nt}{\sin(t/2)}\right|dt + \frac{\lambda_n}{\Lambda_n}$$

$$\sim \log\Big(\frac{n\lambda_n}{\Lambda_n}\Big).$$

On the other hand, by the partial summation of Abel, we get

$$\sum_{k=0}^{n}\lambda_k \sin\Big(k+\frac{1}{2}\Big)t = O\Big(\frac{\lambda_n}{t}\Big),$$

since λ_n is non-decreasing.

$$Q_n = \int_{\lambda_n/\Lambda_n}^{\pi}\frac{1}{2\Lambda_n\sin(t/2)}\Big\{\sum_{k=0}^{n}\lambda_k\sin\Big(k+\frac{1}{2}\Big)t\Big\}dt$$

$$= \int_{\lambda_n/\Lambda_n}^{\pi}\frac{\lambda_n}{\Lambda_n\sin(t/2)}O\Big(\frac{1}{t}\Big)dt = O\Big(\frac{\lambda_n}{\Lambda_n}\int_{\lambda_n/\Lambda_n}^{\pi}\frac{dt}{t^2}\Big) = O(1).$$

Thus we get

$$w(n) = \log\Big(\frac{n\lambda_n}{\Lambda_n}\Big).$$

PROOF OF PROPOSITION 2. Let us put

$$f(x) \sim \sum_{n=1}^{\infty} a_n\varphi_n(x)$$

and its n-th $(R, \Lambda_n, 1)$ mean is

$$\sigma_n(x) = \int_a^b f(t)\, K_n(x,t)dt$$

and put

$$v_n(x) = \sup_{1\leqq k\leqq n}\frac{\sigma_k(x)}{\sqrt{w(k)}} = \frac{\sigma_p(x)}{\sqrt{w(p)}}$$

where $p=p(x)$. Then

$$v_n(x)=\int_a^b f(t)\frac{K_p(x,t)}{\sqrt{w(p)}}dt$$

and put

$$\begin{aligned}I_n &= \int_a^b v_n(x)dx \\ &= \int_a^b\int_a^b \frac{f(t)}{\sqrt{w(p)}} K_p(x,t)\,dxdt \\ &= \int_a^b f(t)dt\int_a^b\frac{K_p(x,\,t)}{\sqrt{w(p)}}dx.\end{aligned}$$

Applying Hölder's inequality we have

$$\begin{aligned}I_n^2 &\leqq \int_a^b |f(t)|^2\,dt\int_a^b\left\{\int_a^b\frac{K_p(x,t)}{\sqrt{w(p)}}\,dx\right\}^2 dt \\ &\leqq \|f\|^2\int_a^b\int_a^b\int_a^b\frac{K_{p_1}(x_1,\,t)}{\sqrt{w(p_1)}}\cdot\frac{K_{p_2}(x_2,\,t)}{\sqrt{w(p_2)}}dx_1dx_2dt\end{aligned}$$

where $p_1 = p(x_1)$ and $p_2 = p(x_2)$. Since

$$K_p(x,\,t)=\sum_{n=0}^{p}\left(1-\frac{\Lambda_n}{\Lambda_p}\right)\varphi_p(x)\varphi_p(t),$$

we have

$$\begin{aligned}&\int_a^b K_{p_1}(x_1,\,t)K_{p_2}(x_2,\,t)\,dt \\ &=\sum_{n=1}^{r}\left(1-\frac{\Lambda_n}{\Lambda_{p_1}}\right)\left(1-\frac{\Lambda_n}{\Lambda_{p_2}}\right)\varphi_n(x_1)\varphi_n(x_2),\end{aligned}$$

where $r = r(x_1,\,x_2) = \min.(p(x_1),\,p(x_2))$. Applying the partial summation successively, the above sum equals to

$$\begin{aligned}&\sum_{n=1}^{r}\left\{-\left(\frac{1}{\Lambda_{p_1}}+\frac{1}{\Lambda_{p_2}}\right)+\frac{2\Lambda_n}{\Lambda_{p_1}\Lambda_{p_2}}\right\}\lambda_n D_n\,(x_1,x_2) \\ &=\sum_{n=1}^{r}\frac{2\lambda_n}{\Lambda_{p_1}\Lambda_{p_2}}\Lambda_n K_n(x_1,\,x_2) \\ &\qquad+\left\{-\left(\frac{1}{\Lambda_{p_1}}+\frac{1}{\Lambda_{p_2}}\right)+\frac{2\Lambda_r}{\Lambda_{p_1}\Lambda_{p_2}}\right\}\Lambda_r K_r(x_1,\,x_2).\end{aligned}$$

Thus we have

$$\begin{aligned}I_n^2\leqq \mathrm{A}\int_a^b\int_a^b&\frac{1}{\sqrt{w(p_1)}\sqrt{w(p_2)}}\Bigg[\sum_{n=1}^{r}\frac{2\lambda_n}{\Lambda_{p_1}\Lambda_{p_2}}\Lambda_n K_n(x_1,x_2) \\ &+\left\{-\left(\frac{1}{\Lambda_{p_1}}+\frac{1}{\Lambda_{p_2}}\right)+\frac{2\Lambda_r}{\Lambda_{p_1}\Lambda_{p_2}}\right\}\Lambda_r K_r(x_1,\,x_2)\Bigg]dx_1dx_2\end{aligned}$$

$$= A\int\int_{M_1} + A\int\int_{M_2} = AJ_1 + AJ_2$$

say, where A is a constant and

$$M_1 = \{(x_1, x_2) \mid p(x_1) \geqq p(x_2)\} \text{ and } M_2 = \{(x_1, x_2) \mid p(x_1) < p(x_2)\}.$$

$$\begin{aligned} J_2 &\leqq \int_a^b \int_a^b \frac{1}{w(p_1)} \Big[\sum_{n=1}^{p_1} \frac{2}{\Lambda^2_{p_1}} \lambda_n \Lambda_n \mid K_n(x_1, x_2) \mid \\ &\qquad + \Big\{ \frac{2}{\Lambda_{p_1}} + \frac{2\Lambda_{p_1}}{\Lambda^2_{p_1}} \Big\} \Lambda_{p_1} \mid K_{p_1}(x_1, x_2) \mid \Big] dx_1\, dx_2 \\ &\leqq \int_a^b \frac{1}{w(p_1)} \Big[\sum_{n=1}^{p_1} \frac{2\lambda_n \Lambda_n}{\Lambda^2_{p_1}} w(n) + \frac{4\Lambda_{p_1}}{\Lambda_{p_1}} w(p_1) \Big] dx_1 \\ &\leqq \int_a^b \frac{1}{w(p_1)} \Big[\frac{w(p_1)}{\Lambda^2_{p_1}} \Lambda^2_{p_1} + w(p_1) \Big] dx_1 \\ &\leqq 2(b-a). \end{aligned}$$

The other term J_1 is identical. Thus we get

$$\int_a^b \left| \sup_{1 \leqq n} \frac{\sigma_n(x)}{\sqrt{w(n)}} \right| dx \leqq \sqrt{8(b-a)}\, \|f\|.$$

From this,

$$\lim_{n\to\infty} \frac{\sigma_n(x)}{\sqrt{w(n)}}$$

exist and are finite, almost everywhere. Since $w(n)$ is Lebesgue constant and $\sum a_n^2 w(n) < \infty$, then $s_{n_k}(x)$ converge almost everywhere, for

$$k \leqq w(n_k) < k+1,$$

by Rademacher's argument. The convergence of $\sigma_n(x)$ is routine argument (see, Kaczmarz-Steinhaus [2], p.193).

PROOF OF THEOREM 1. If we put

$$\Lambda_n = \exp\{(\log n)^\alpha\}, \ (1 < \alpha < \infty),$$

then $\lambda_n \sim n^{-1}(\log n)^{\alpha-1} \exp\{(\log n)^\alpha\}$

and the hypotheses of Propositions 1 and 2 are satisfied. So the Lebesgue constant of $(R, \exp\{(\log n)^\alpha\}, 1)$ summation is

$$\begin{aligned} w(n) &= \log \frac{n \cdot n^{-1}(\log n)^{\alpha-1} \exp\{(\log n)^\alpha\}}{\exp\{(\log n)^\alpha\}} = \log(\log n)^{\alpha-1} \\ &\sim \log(\log n). \end{aligned}$$

Thus the Fourier series is $(R, \exp\{(\log n)^\alpha\}, 1)$ summable almost everywhere provided that

$$\sum_{n=2}^{\infty} (a_n^2 + b_n^2) \log(\log n)$$

converges. We can conclude that $(R, \exp\{\log n)^{\alpha}\}, 1)$ summability implies $(R, \exp\{(\log n)^{\alpha}\}, \delta)$ summability almost everywhere if $f(x) \in L^2(-\pi, \pi)$, following the argument of Wang [4]. Thus we get Theorem 1.

PROOF OF THEOREM 2. This is analogous to the above proof. Since

$$\Lambda_n = \exp\{\exp(\log n)^{\alpha}\},$$

we have

$$\lambda_n = \exp\{\exp(\log n)^{\alpha}\}\cdot\exp(\log n)^{\alpha}(\log n)^{\alpha-1}\cdot n^{-1}$$

and

$$w(n) \sim \log\{\exp(\log n)^{\alpha}(\log n)^{\alpha-1}\}$$
$$\sim (\log n)^{\alpha}.$$

The remaining part of proof is immediate.

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