ON THE DEFINITION OF CESÀRO-PERRON INTEGRALS

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1. Introduction. The Cesàro-Perron integral was defined by J. C. Burkill [1] using the Cesàro-continuous upper and lower functions.

G. Sunouchi and M. Utagawa [3] proved that the Cesàro-Perron scale of integration can be defined without assuming the Cesàro-continuity of upper and lower functions and that the indefinite integral is Cesàro-continuous.

We denote by $CP_0$ and $CP$ the Burkill's Cesàro-Perron integral and the generalized Cesàro-Perron integral defined by G. Sunouchi and M. Utagawa respectively. It is clear that $CP$-integral includes $CP_0$-integral. But, in this paper, we will prove the equivalence of these integrals by using the Cesàro-Denjoy integral introduced by W. L. C. Sargent [2].

I must express my best thanks to Dr. G. Sunouchi for his suggestions and criticisms.

2. $CP_0$-integral and $CP$-integral.

DEFINITION 2.1. We put

$$C(f, a, b) = \frac{1}{b - a}\int_a^b f(t) \, dt,$$

where the integral is taken in the restricted Denjoy sense.

If $\lim_{h \to 0} C(f, x_0, x_0 + h) = f(x_0)$, then $f(x)$ is termed Cesàro-continuous at $x_0$.

If $\overline{CD} f(x_0) = \underline{CD} f(x_0)$, where

$$\lim_{h \to 0} \left\{ C(f, x_0, x_0 + h) - f(x_0) \right\} \frac{1}{2} h = \overline{CD} f(x_0)$$

and

$$\lim_{h \to 0} \left\{ C(f, x_0, x_0 + h) - f(x_0) \right\} \frac{1}{2} h = \underline{CD} f(x_0),$$

then $f(x)$ is called Cesàro differentiable at $x_0$ and we denote the common value by $CD f(x_0)$.

DEFINITION 2.2. $U(x)$ [$L(x)$] is termed upper [lower] function of a measurable $f(x)$ in $[a, b]$, provided that

*) Numbers in brackets refer to the bibliography at the end.
(i) $U(a) = 0$ [L(a) = 0],
(ii) $U(x)$ [L(x)] is Cesàro-continuous on $[a, b],
(iii) \overline{CD} U(x) > -\infty$ [$\overline{CD} L(x) < +\infty$] at each point $x$,
(iv) $\overline{CD} U(x) \geq f(x)$ [$\overline{CD} L(x) \leq f(x)$] at each point $x$.

**DEFINITION 2.3.** If $f(x)$ has upper and lower functions in $[a, b]$ and l. u. b. $U(b) = g. l. b. L(b)$, then $f(x)$ is termed *integrable in Cesàro-Perron sense* or $CP_0$-integrable. The common value of the two bounds is called the definite $CP_0$-integral of $f(x)$ and denote by $(CP_0) \int_a^b f(t) \, dt$.

**DEFINITION 2.4.** If in the definition 2.2, the condition (ii) is omitted, then the Perron-scale of integration constructed by the Definition 2.3 is called *CP-integral* and its definite on $[a, b]$ is denoted by $(CP) \int_a^b f(t) \, dt$.

The CP-integral has the following properties, cf. [3].

**THEOREM 2.1.** The function $U(x) - L(x)$ is increasing and non-negative.

**THEOREM 2.2.** If $f(x)$ is CP-integrable in $[a, b]$, then $f(x)$ is so also in any subinterval.

**THEOREM 2.3.** The indefinite integral $F(x) = (CP) \int_a^x f(t) \, dt$ is Cesàro-continuous.

**THEOREM 2.4.** The function $F(x)$ is Cesàro differentiable almost everywhere and $CD F(x) = f(x)$, a.e.

### 3. Cesàro-Denjoy integral.

**DEFINITION 3.1.** The function $f(x)$ is said to be $AC^*$ on a set $E$ if it is Denjoy-integrable in the restricted sense in an interval containing $E$, and if to each positive number $\varepsilon$, there corresponds a number $\delta$ such that

$$\sum_{r=1}^n \sup_{x \in (a_r, b_r)} | C(f, a_r, x) - f(a_r) | < \varepsilon, \quad (1)$$

$$\sum_{r=1}^n \sup_{x \in (a_r, b_r)} | C(f, b_r, x) - f(b_r) | < \varepsilon, \quad (2)$$

for all finite non-overlapping sequence of intervals $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ with end points on $E$ and such that

$$\sum_{r=1}^n (b_r - a_r) < \delta. \quad (3)$$

If the inequalities (1) and (2) are replaced by the following conditions
respectively
\[ \sum_{r=1}^{n} \inf_{x \in (a_r, b_r)} |C(f,a_r,x) - f(a_r)| > -\varepsilon, \] (4)
\[ \sum_{\gamma=1}^{n} \inf_{y \in (b_{\gamma}, b_{\gamma+1})} |f(b_{\gamma}) - C(f,b_{\gamma},x)| > -\varepsilon, \] (5)
then \( f(x) \) is called \( AC^* \) below on \( E \). There is a corresponding definition of \( AC^* \) above on \( E \). If the set \( E \) is the sum of a countable number of sets \( E_n \) on each of which \( f \) is \( AC^* \) and if \( f \) is Cesàro-continuous on \( E \), then \( f \) is termed \( ACG^* \) on \( E \), cf. [2].

The function \( f(x) \) is \( AC^* \) on \( E \) if and only if \( f(x) \) is both \( AC^* \) below and \( AC^* \) above on \( E \).

**Definition 3.2.** The function \( f(x) \) defined on \([a, b]\) is called integrable in the Cesàro-Denjoy sense or \( CD \)-integrable provided that there exists a function \( F(x) \) \( ACG^* \) on \([a, b]\) and such that
\[ \text{CD} F(x) = f(x), \text{ a.e.} \]

We call the function \( F(x) \) the indefinite \( CD \)-integral and define the definite \( CD \)-integral as \( F(b) - F(a) \), cf. [2].

The following results have been proved by Sargent, cf. [2].

**Theorem 3.1.** If \( \text{CD} f(x) > -\infty \) at each point of \( E \), then \( E \) is the sum of a countable number of sets on each of which \( f(x) \) is \( AC^* \) below.

**Theorem 3.2.** The \( CD \)-integral is a descriptive definition of the \( CP_0^* \)-integral.

**4. Theorem**

**Theorem.** The \( CP \)-integral is equivalent to the \( CP_0^* \)-integral.

**Proof.** Since the \( CD \)-integral is equivalent to the \( CP_0^* \)-integral, it is sufficient to prove that the \( CD \)-integral includes the \( CP \)-integral and that the following equality holds,
\[ (\text{CD}) \int_{a}^{b} f(t) \, dt = (\text{CP}) \int_{a}^{b} f(t) \, dt. \] (6)
Let \( F(x) = (\text{CP}) \int_{a}^{x} f(t) \, dt \). Then, by Theorems 2.3 and 2.4, the function \( F(x) \) is Cesàro-continuous on \([a, b]\) and \( \text{CD} F(x) = f(x) \) a.e.

We shall prove that \( F(x) \) is \( ACG^* \) on \([a, b]\).

For a given \( \varepsilon > 0 \), we can select the upper and lower functions \( U(x), L(x) \) such that
\[ U(b) - L(b) \leq \frac{1}{2} \ \varepsilon \]  
(7)

and

\[ CD \ U(x) > - \infty \ (a \leq x \leq b). \]  
(8)

It follows from (8) and Theorem 3.1 that \([a, b]\) is the sum of a countable number of sets \(E_n\) on each of which \(U(x)\) is \(AC^*\) below. Consequently, for any finite non-overlapping intervals \((a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\) with end point on \(E_n\) and such that

\[ \sum_{r=1}^m (b_r - a_r) < \delta_n, \]
we have

\[ \sum_{r=1}^m \inf \{C(U, a_r, x)\} > - \frac{\varepsilon}{2} \]  
(9)

and

\[ \sum_{r=1}^m \inf \{|U(b_r) - C(U, b_r, x)|\} > - \frac{\varepsilon}{2} \]  
(10)

Suppose that \(a_r < x < b_r\). Then it follows that

\[ C(F, a_r, x) - F(a_r) = C(U, a_r, x) - U(a_r) - \frac{1}{x - a_r} \int_{a_r}^x [U(t) - F(t)] \ dt \]
\[ + |U(a_r) - F(a_r)| \]
\[ \geq C(U, a_r, x) - U(a_r) - |U(b_r) - F(b_r)| \]
\[ + |U(a_r) - F(a_r)|, \]

since \(U(x) - F(x)\) is increasing and non-negative by Theorem 2.1. Therefore, we obtain from (7) and (9)

\[ \sum_{r=1}^m \inf \{|C(F, a_r, x) - F(a_r)|\} \geq \]
\[ \sum_{r=1}^m \inf \{|C(U, a_r, x) - U(a_r)| - |U(b) - F(b)|\} > - \varepsilon. \]

Similarly, we have from (7) and (10)

\[ \sum_{r=1}^m \inf \{|F(b_r) - C(F, b_r, x)|\} > - \varepsilon. \]

Hence the function \(F(x)\) is \(AC^*\) below on \(E_n\).

Since \(-f(x)\) is \(CP\)-integrable and its indefinite integral is \(-F\), the interval \([a, b]\) is the sum of a countable number of sets \(E'_m\) on each of which \(-F\) is \(AC^*\) below. Therefore \(F\) is \(AC\) above on \(E'_m\) and is \(AC^*\) on \(E_n \cap E'_m\).

Since \(F\) is Cesàro-continuous on \([a, b]\) and \([a, b] = \sum \sum E_n \cap E'_m\), the
function $F(x)$ is $ACG^*$ on $[a, b]$. Thus, $f$ is $CD$-integrable on $[a, b]$ and

$$(CD)\int_a^b f(t) \, dt = F(b) - F(a) = (CP)\int_a^b f(t) \, dt.$$  

REFERENCES


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