SEMI-GROUPS OF OPERATORS IN FRÉCHET SPACE AND APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction. This paper is concerned with semi-groups of operators in Fréchet space and its application to the Cauchy problem for some linear partial differential equations with constant coefficients.

A topological vector space is called a Fréchet space if it is locally convex, complete and metrizable.

We shall deal with a semi-group of operators $\{T(\xi); 0 \leqq \xi < \infty\}$ satisfying the following conditions:

(1) For each $\xi \geqq 0$, $T(\xi)$ is a continuous linear operator from a Fréchet space X into itself and

$$T(\xi + \eta) = T(\xi)T(\eta) \qquad \text{for } \xi, \eta \geqq 0,$$
$$T(0) = I \text{ (the identity)}.$$

(2) There exists a non-negative number σ such that

$$\{e^{-\sigma\xi}T(\xi)x ; \xi \geqq 0\}$$

is bounded in X for each $x \in X$.

(3) $$\lim_{\xi \downarrow 0} T(\xi)x = x \qquad \text{for each } x \in X.$$

Since a Banach space is obviously a Fréchet space, our semi-groups are an extention of semi-groups of class (C_0) in Banach space. (For semi-groups in Banach space see the book of E. Hille and R. S. Phillips [3].)

We first remark that the conditions (1) and (3) imply the condition (2) if X is a Banach space. For $M \equiv \sup_{0 \leqq \xi \leqq 1} \|T(\xi)\| < \infty$ by the uniform boundedness theorem, and hence $\|T(\xi)\| \leqq M \cdot \exp(\xi \log M)$ for each $\xi \geqq 0$. But this is not true in general if X is a Fréchet space.

EXAMPLE. We consider real valued functions of one real variable. C^∞ denotes the space of ∞ times continuously differentiable functions. It is well known that the space C^∞ becomes a Fréchet space under the family of semi-norms $\{p_{m,k}(\cdot) ; m, k = 0, 1, 2, \cdots\cdots\}$, where

$$p_{m,k}(x) = \sup_{|t| \leqq k} |x^{(m)}(t)| \qquad \text{for each } x \in C^\infty. \tag{1.1}$$

We define

(1.2) $$[T(\xi)x](t) = x(\xi + t) \qquad \text{for each } \xi \geqq 0,\ x \in C^\infty.$$

Then $\{T(\xi); 0 \leqq \xi < \infty\}$ is obviously a semi-group of operators satisfying the conditions (1) and (3). But this semi-group does not imply the condition (2). In fact, for $x_0(t) = e^{t^2} \in C^\infty$,

$$p_{0,k}(e^{-\sigma\xi}T(\xi)x_0) = \sup_{|t| \leqq k} e^{-\sigma\xi}e^{(\xi+t)^2} \geqq e^{\xi(\xi-2k-\sigma)}.$$

Hence

$$\overline{\lim_{\xi\to\infty}}\, p_{0,k}(e^{-\sigma\xi}T(\xi)x_0) = \infty \qquad \text{for each } \sigma \geqq 0,$$

so that $\{e^{-\sigma\xi}T(\xi)x_0; \xi \geqq 0\}$ is not bounded for each $\sigma \geqq 0$.

§ 2 — § 5 are devoted to investigations of such semi-groups and we can obtain results similarly as for semi-groups of class (C_0) in Banach space. In § 6, these results are applied to the Cauchy problem for the parabolic equation and the wave equation.

2. Preliminaries.

We first prove the following

THEOREM 2. 1. *If $\{T_\alpha\}$ is a family of continuous linear operators from a Fréchet space X_1 into a Fréchet space X_2 such that the set $\{T_\alpha x\}$ is bounded for each $x \in X_1$, then for each neighborhood $N_2 \in \Sigma_2$ there exists a neighborhood $N_1 \in \Sigma_1$ such that $T_\alpha(N_1) \subset N_2$ for all α, where $\Sigma_i (i=1, 2)$ is a complete system of convex neighborhoods of the origin in X_i.*

PROOF. Since X_i is locally convex and metrizable, its topology is also determined by a family of denumerable semi-norms $\{p_{i1}, p_{i2}, p_{i3}, \cdots\cdots\}$. Let us put

(2. 1) $$\|x\|_i = \sum_{n=1}^{\infty} \frac{p_{in}(x)}{2^n(1 + p_{in}(x))} \qquad \text{for } x \in X_i.$$

Then X_i is a quasi-normed space under the quasi-norm (2. 1) and $\|\cdot\|_i$-topology is equivalent to the original topology in X_i. Thus X_i becomes a complete quasi-normed space, so that each T_α is a continuous linear operator from a complete quasi-normed space X_1 into a complete quasi-normed space X_2 and the set $\{T_\alpha x\}$ is bounded in the complete quasi-normed space X_2 for each $x \in X_1$. Hence, by the Mazur-Orlicz theorem [5], for any $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that $\|T_\alpha x\|_2 \leqq \varepsilon$ for all α and $\|x\|_1 \leqq \delta$. Then the theorem is proved from the equivalence of the quasi-normed topology (2. 1) and the original topology in X_i.

COROLLARY 2.1. *Let $\{T_\alpha\}$ be a family of operators satisfying the assumptions in Theorem 2. 1. If the limit $\lim_{\alpha\to\infty} T_\alpha x$ exists on a dense subspace D in X_1, then the limit $\lim_{\alpha\to\infty} T_\alpha x$ exists on the whole space X_1 and $T = \lim_{\alpha\to\infty}$*

T_α is a continuous linear operator from X_1 into X_2.

PROOF. For any $N_2 \in \Sigma_2$ there exists a $N_2' \in \Sigma_2$ such that $N_2' + N_2' + N_2' \subset N_2$, and, by Theorem 2.1, there exists a symmetric neighborhood $N_1 \in \Sigma_1$ such that

$$T_\alpha(N_1) \subset N_2' \qquad \text{for all } \alpha. \tag{2.2}$$

Let x be any fixed element in X_1 and let x_0 be an element in D snch that $\pm(x_0 - x) \in N_1$. Then, by assumption, there exists a number $\alpha_0 < 0$ such that $T_\alpha x_0 - T_{\alpha'} x_0 \in N_2'$ for $\alpha, \alpha' > \alpha_0$. Hence

$$\begin{aligned} T_\alpha x - T_{\alpha'} x &= T_\alpha(x - x_0) + (T_\alpha x_0 - T_{\alpha'} x_0) + T_{\alpha'}(x_0 - x) \\ &\in N_2' + N_2' + N_2' \subset N_2 \end{aligned}$$

for $\alpha, \alpha' > \alpha_0$. The second part follows from (2.2).

COROLLARY 2.2. *If $\{T(\xi); 0 \leqq \xi < \infty\}$ is a semi-group of operators satisfying the conditions (1), (2) and (3), then $\{e^{-\sigma\xi}T(\xi)x;\ 0 \leqq \xi < \infty, x \in B\}$ is bounded for each bounded set $B \subset X$. Especially, for any fixed $\omega > 0$, $\{T(\xi)x; 0 \leqq \xi \leqq \omega, x \in B\}$ is bounded.*

PROOF. Theorem 2.1 shows that for each $N \in \Sigma$ there exists an $N' \in \Sigma$ such that $e^{-\sigma\xi}T(\xi)(N') \subset N$ for all $\xi \geqq 0$, where Σ denotes a complete system of convex neighborhoods of the origin in X. Since B is a bounded set, there exists a positive number α_B such that $\alpha_B B \subset N'$. Hence $\alpha_B \cdot e^{-\sigma\xi} T(\xi)(B) \subset N$ for all $\xi \geqq 0$.

3. Infinitesimal generator and resolvent. Let $\{T(\xi); 0 \leqq \xi < \infty\}$ be a semi-group of operators satisfying the conditions (1), (2) and (3). It is clear that, for each $x \in X$, $T(\xi)x$ is a continuous function of $\xi \in [0, \infty)$.

The infinitesimal generator is defined as the limit

$$\lim_{h \downarrow 0} \frac{T(h) - I}{h} x = Ax \tag{3.1}$$

whenever this limit exists, the domain $D(A)$ of A being the set of elements for which the limit exists. For $x \in D(A)$ we have

$$\frac{dT(\xi)x}{d\xi} = AT(\xi)x = T(\xi)Ax \qquad \text{for } \xi > 0. \tag{3.2}$$

THEOREM 3.1. *The infinitesimal generator A is a closed linear operator and $D(A)$ is dense in X.*

PROOF. Let x be any fixed element in X. $T(\xi)x$ is a continuous function on $[0, \infty)$ with values in X, so that we can define the Riemann integral

$$\frac{1}{\eta}\int_0^\eta T(\xi)x\, d\xi \ (\equiv y_\eta)$$

for each $\eta > 0$. It is clear that $y_\eta \to x$ as $\eta \downarrow 0$ and that $y_\eta \in D(A)$. Hence $D(A)$ is dense in X. By (3. 2), we have

$$\frac{1}{\eta}(T(\eta)x - x) = \frac{1}{\eta}\int_0^\eta T(\xi)Ax\,d\xi \qquad \text{for } x \in D(A).$$

Suppose that $\{x_n\}$ is a sequence of elements in $D(A)$ and that $x_n \to x_0$, $Ax_n \to z_0$. The above formula holds for $x = x_n$, so that

$$\frac{1}{\eta}(T(\eta)x_n - x_n) = \frac{1}{\eta}\int_0^\eta T(\xi)Ax_n d\xi.$$

Theorem 2. 1 shows that for any closed convex neighborhood N there exists a number $n_0 > 0$ such that $T(\xi)(Ax_n - z_0) \in N$ for $n > n_0$, $0 \leqq \xi \leqq \eta$. Hence $\eta^{-1}\int_0^\eta T(\xi)(Ax_n - z_0)d\xi \in N$ for $n > n_0$, that is,

$$\frac{1}{\eta}\int_0^\eta T(\xi)Ax_n d\xi \to \frac{1}{\eta}\int_0^\eta T(\xi)z_0 d\xi$$

as $n \to \infty$. Thus we have for each $\eta > 0$

$$\frac{1}{\eta}\int_0^\eta T(\xi)z_0\,d\xi = \frac{1}{\eta}[T(\eta)x_0 - x_0].$$

When $\eta \to 0$ the left hand side tends to z_0, so that $x_0 \in D(A)$ and $Ax_0 = z_0$. This completes the proof.

Let x be any fixed element in X and let us put

$$R_w(\lambda\,; A)x = \int_0^w e^{-\lambda\xi}T(\xi)x\,d\xi$$

for each $w > 0$ and $\lambda > 0$. (We can define the integral of Riemann type since $e^{-\lambda\xi}T(\xi)x$ is a continuous function on $[0, \infty)$ with values in X.) P denotes a family of denumerable semi-norms determining the topology of X. Then we have for any semi-norm $p \in P$

$$p(R_w(\lambda; A)x - R_{w'}(\lambda\,; A)x) \leqq \int_{w'}^w e^{-\lambda\xi}p(T(\xi)x)d\xi.$$

By the assumption (2) there exists a constant $M_p > 0$ such that $p(T(\xi)x) \leqq e^{\sigma\xi}M_p$ for all $\xi \geqq 0$. Hence if $\lambda > \sigma$, then

$$p(R_w(\lambda\,; A)x - R_{w'}(\lambda\,; A)x) \leqq M_p\int_{w'}^w e^{-(\lambda-\sigma)\xi}d\xi \to 0$$

as $w, w' \to \infty$. Thus the limit $\lim_{w\to\infty} R_w(\lambda\,; A)x$ exists.

We shall define $R(\lambda\,; A)$ for each $\lambda > \sigma$ by

$$R(\lambda\,; A)x = \lim_{w\to\infty} R_w(\lambda\,; A)x = \int_0^\infty e^{-\lambda\xi}T(\xi)x\,d\xi. \tag{3. 3}$$

THEOREM 3.2. *For each* $\lambda > \sigma$, $R(\lambda; A)$ *is a continuous linear operator from* X *into itself, and*

$$(\lambda - A)R(\lambda; A)x = x \qquad \text{for all } x \in X,$$
$$R(\lambda; A)(\lambda - A)x = x \qquad \text{for all } x \in D(A).$$

PROOF. It is clear that $R(\lambda; A)$ is a linear operator from X into itself. Let $x_n \to 0$, $x_n \in X$. Then the sequence $\{x_n\}$ is bounded, so that $\{e^{-\sigma\xi}T(\xi)x_n; \xi \geqq 0, n = 1, 2, 3, \ldots\}$ is a bounded set by the Corollary 2.2. For each semi-norm $p \in P$ there exists a positive constant M_p such that $p(T(\xi)x_n) \leqq e^{\sigma\xi} M_p$ for each $\xi \geqq 0$ and $n \geqq 1$. From the definition of $R(\lambda; A)$ we have

$$p(R(\lambda; A)x_n) \leqq \int_0^\infty e^{-\lambda\xi} p(T(\xi)x_n)\, d\xi.$$

Since $\lim_{n\to\infty} p(T(\xi)x_n) = 0$ for all $\xi \geqq 0$ and $e^{-\lambda\xi}p(T(\xi)x_n) \leqq M_p e^{-(\lambda-\sigma)} \in L^1$ for all n, the convergence theorem shows that $p(R(\lambda; A)x_n) \to 0$ as $n \to \infty$. Hence $R(\lambda; A)x_n \to 0$ as $n \to \infty$, that is, $R(\lambda; A)(\lambda > \sigma)$ is a continuous linear operator. The second part can be proved similarly as in the case of Banach space.

From this theorem we get the resolvent equation

$$R(\lambda; A) - R(\mu; A) = -(\lambda - \mu)R(\lambda; A)R(\mu; A) \tag{3.4}$$

for each $\lambda, \mu > \sigma$.

THEOREM 3.3. *For each* $x \in X$ *and* $k = 1, 2, 3, \cdots$

$$\frac{d^k}{d\lambda^k} R(\lambda; A)x = (-1)^k k!\, [R(\lambda; A)]^{k+1}x \qquad (\lambda > \sigma). \tag{3.5}$$

PROOF. From (3.4)

$$\frac{1}{h}[R(\lambda + h; A)x - R(\lambda; A)x] - (-1)[R(\lambda; A)]^2 x$$
$$= hR(\lambda + h; A)[R(\lambda; A)]^2 x \quad \text{for all } x \in X.$$

Then for any semi-norm $p \in P$ we have

$$p\left(\frac{1}{h}[R(\lambda + h; A)x - R(\lambda; A)x] - (-1)[R(\lambda; A)]^2 x\right)$$
$$= |h| p(R(\lambda + h; A)[R(\lambda; A)]^2 x) \leqq |h| \int_0^\infty e^{-(\lambda+h)\xi} p(T(\xi)[R(\lambda; A)]^2 x) d\xi$$
$$\leqq |h| M_p \frac{1}{\lambda + h - \sigma} \to 0$$

as $|h| \to 0$, where M_p is a constant such that $e^{-\sigma\xi}p(T(\xi)[R(\lambda; A)]^2 x) \leqq M_p$ for all $\xi \geqq 0$. This asserts that

$$\frac{d}{d\lambda}R(\lambda\,;A)x = (-1)[R(\lambda\,;A)]^2x \qquad \text{for all } x \in X.$$

Using the induction we see that (3.5) holds for each $k \geqq 1$.

THEOREM 3.4. *For each bounded set B, the set*

$$\{[(\lambda-\sigma)R(\lambda\,;A)]^n x\,;x \in B, \lambda > \sigma,\ n = 1,2,3,\ldots\ldots\}$$

is bounded.

PROOF. From the definition of $R(\lambda\,;A)$

$$\frac{d^k}{d\lambda^k}R(\lambda\,;A)\,x = (-1)^k\int_0^\infty \xi^k e^{-\lambda\xi}T(\xi)x\,d\xi,$$

so that by (3.5)

$$[(\lambda-\sigma)R(\lambda\,;A)]^{k+1}\,x = \frac{(\lambda-\sigma)^{k+1}}{k\,!}\int_0^\infty \xi^k e^{-\lambda\xi}T(\xi)x\,d\xi.$$

Thus for each semi-norm $p \in P$ we have

$$p([(\lambda-\sigma)R(\lambda\,;A)]^{k+1}x) \leqq \frac{(\lambda-\sigma)^{k+1}}{k\,!}\int_0^\infty \xi^k e^{-\lambda\xi}p(T(\xi)x)\,d\xi.$$

Corollary 2.2 shows that there exists a constant $M_p > 0$ such that $p(T(\xi)x) \leqq M_p e^{\sigma\xi}$ for all $\xi \geqq 0$ and $x \in B$. Hence

$$p([(\lambda-\sigma)R(\lambda\,;A)]^{k+1}x) \leqq M_p$$

for all $x \in B$, $\lambda > \sigma$ and $k = 0, 1, 2,\cdots\cdots$.

THEOREM 3.5. *For each $x \in X$*

$$\lim_{\lambda\to\infty} \lambda R(\lambda\,;A)x = x.$$

PROOF. By Theorem 3.2

$$\lambda R(\lambda\,;A)x - x = R(\lambda\,;A)Ax \qquad \text{for } x \in D(A),$$

and Theorem 3.4 asserts that for each semi-norm $p \in P$ there exists a constant $M_p > 0$ such that $p(R(\lambda\,;A)Ax) \leqq M_p(\lambda-\sigma)^{-1}$ for all $\lambda > \sigma$. Hence

$$p(\lambda R(\lambda\,;A)x - x) = p(R(\lambda\,;A)Ax) \leqq M_p(\lambda-\sigma)^{-1} \to 0$$

as $\lambda \to \infty$, so that $\lim_{\lambda\to\infty} \lambda R(\lambda\,;A)x = x$ for $x \in D(A)$. Since $D(A)$ is dense in X and since $\{\lambda R(\lambda\,;A)x\,;\lambda >> \sigma\}$ is bounded for each $x \in X$, the theorem follows from Corollary 2.1.

4. Representation theorem.

We now define

$$T_\lambda^n(\xi)x = \exp(-\lambda\xi)\sum_{k=0}^{n}\frac{(\lambda\xi)^k}{k\,!}[\lambda R(\lambda\,;A)]^k x$$

for each $x \in X$. For each fixed $\lambda > \sigma$, $\xi \geqq 0$ and $x \in X$, the sequence $\{T_\lambda^n(\xi)x;$ $n = 0, 1, 2,\ldots\ldots\}$ is a Cauchy sequence. Indeed, for any semi-norm $p \in P$,

$$p(T_\lambda^n(\xi)x - T_\lambda^m(\xi)x) \leqq \exp(-\lambda\xi)\sum_{m+1}^{n}\frac{(\lambda\xi)^k}{k!}p([\lambda R(\lambda;A)]^k x)$$

and there exists, by Theorem 3.4, a constant $M_p > 0$ such that $(\lambda-\sigma)^k p([R(\lambda;A)]^k x) \leqq M_p$ for all k, and so that

$$p(T_\lambda^n(\xi)x - T_\lambda^m(\xi)x) \leqq M_p \exp(-\lambda\xi)\sum_{m+1}^{n}\frac{(\lambda\xi)^k}{k!}\frac{\lambda^k}{(\lambda-\sigma)^k} \to 0$$

as $n, m \to \infty$.

Then, for each $\lambda > \sigma$, $\xi \geqq 0$ and $x \in X$, the limit

$$T_\lambda(\xi)x = \lim_{n\to\infty} T_\lambda^n(\xi)x = \exp(-\lambda\xi)\sum_{k=0}^{\infty}\frac{(\lambda\xi)^k}{k!}[\lambda R(\lambda;A)]^k x \tag{4.1}$$

exists. Since $T_\lambda^n(\xi)$ is a continuous linear operator from X into itself, it follows from Corollary 2.1 that $T_\lambda(\xi)$ is a continuous linear operator from X into itself.

THEOREM 4.1. *For each fixed $\lambda > \sigma$ and $x \in X$, $T_\lambda(\xi)x$ is a continuous function on $[0, \infty)$ with values in X. Furthermore the set*

$$\left\{\exp\left(\frac{-\sigma\xi}{1-\sigma/\lambda}\right)T_\lambda(\xi)x\,;\, x \in B, \xi \geqq 0 \text{ and } \lambda > \sigma\right\}$$

is bounded if B is a bounded set.

PROOF. $T_\lambda^n(\xi)x$ is a continuous function of $\xi \in [0, \infty)$ and (4.1) holds uniformly with respect to ξ in any finite interval of ξ, so that $T_\lambda(\xi)x$ is a continuous function of $\xi \in [0, \infty)$. By Theorem 3.4 we have for each seminorm $p \in P$

$$\begin{aligned} p(T_\lambda(\xi)x) &\leqq \exp(-\lambda\xi)\sum_{k=0}^{\infty}\frac{(\lambda\xi)^k}{k!}p([\lambda R(\lambda;A)]^k x) \\ &\leqq M_p \exp(-\lambda\xi)\sum_{k=0}^{\infty}\frac{(\lambda\xi)^k}{k!}\left(\frac{\lambda}{\lambda-\sigma}\right)^k = M_p \exp\frac{\xi\sigma}{1-\sigma/\lambda} \end{aligned}$$

for all $\xi \geqq 0$, $\lambda > \sigma$ and $x \in B$, where $M_p > 0$ is a constant.

THEOREM 4.2. *For each fixed $\lambda, \mu > \sigma$ and $x \in X$*

$$\frac{d}{d\eta}T_\lambda(\xi-\eta)T_\mu(\eta)x = T_\lambda(\xi-\eta)T_\mu(\eta)(\mu AR(\mu;A)x - \lambda AR(\lambda;A)x) \tag{4.2}$$

$$(0 \leqq \eta \leqq \xi).$$

PROOF. An elementary calculus shows that for each semi-norm $p \in P$

$$\sum_{k=k_0}^{\infty} p\left(\frac{d}{d\xi}\left\{\exp(-\lambda\xi)\frac{(\lambda\xi)^k[\lambda R(\lambda;A)]^k}{k!}x\right\}\right)$$

$$\leqq M_p\lambda\Big(\sum_{k=k_0}^{\infty}\frac{(\lambda\omega)^k}{k\,!}\Big(\frac{\lambda}{\lambda-\sigma}\Big)^k+\frac{\lambda}{\lambda-\sigma}\sum_{k_0-1}^{\infty}\frac{(\lambda\omega)^k}{k\,!}\Big(\frac{\lambda}{\lambda-\sigma}\Big)^k\Big)$$

$(0\leqq\xi\leqq\omega)$, where M_p is a constant such that $p([(\lambda-\sigma)R(\lambda\,;A)]^k x)\leqq M_p$ for all $\lambda>\sigma$ and $k=0,1,2,\cdots\cdots$. The right hand side tends to zero as $k_0\to\infty$ and hence the series

$$\sum_{k=0}^{\infty}\frac{d}{d\xi}\Big\{\exp(-\lambda\xi)\frac{(\lambda\xi)^k[\lambda R(\lambda\,;A)]^k}{k\,!}x\Big\}$$

converges uniformly with respect to ξ in any finite interval of ξ. Therefore $\sum_{k=0}^{\infty}\frac{d}{d\xi}\Big\{\exp(-\lambda\xi)\frac{(\lambda\xi)^k[\lambda R(\lambda\,;A)]^k}{k\,!}x\Big\}$ is a continuous function of $\xi\in[0,\infty)$ and then

$$\int_0^{\xi}\sum_{k=0}^{\infty}\frac{d}{d\eta}\Big\{\exp(-\lambda\eta)\frac{(\lambda\eta)^k[\lambda R(\lambda\,;A)]^k}{k\,!}x\Big\}d\eta$$

$$=\sum_{k=0}^{\infty}\exp(-\lambda\xi)\frac{(\lambda\xi)^k[\lambda R(\lambda;A)]^k}{k\,!}x-x$$

for all $\xi>0$, and so that

$$\frac{d}{d\xi}\sum_{k=0}^{\infty}\exp(-\lambda\xi)\frac{(\lambda\xi)^k[\lambda R(\lambda;A)]^k}{k\,!}x=\sum_{k=0}^{\infty}\frac{d}{d\xi}\Big(\exp(-\lambda\xi)\frac{(\lambda\xi)^k[\lambda R(\lambda;A)]^k}{k\,!}x\Big)$$

$$=\exp(-\lambda\xi)\sum_{k=0}^{\infty}\frac{(\lambda\xi)^k[\lambda R(\lambda;A)]^k}{k\,!}(\lambda^2R(\lambda\,;A)x-\lambda x).$$

Since $\lambda^2R(\lambda\,;A)x-\lambda x=\lambda AR(\lambda\,;A)x$ by Theorem 3. 2, we have

$$\frac{d}{d\xi}T_\lambda(\xi)x=T_\lambda(\xi)\lambda AR(\lambda\,;A)x\qquad\text{for }\xi\geqq0.\tag{4. 3}$$

Then the formula (4. 2) follows from (4. 3), Therems 4. 1, and 2. 1, and the property $T_\mu(\eta)AR(\lambda\,;A)=AR(\lambda\,;A)T_\mu(\eta)$.

The same agument shows that

$$\frac{d}{d\eta}T_\lambda(\xi-\eta)T(\eta)x=T_\lambda(\xi-\eta)T(\eta)(A-\lambda R(\lambda\,;A)A)x\tag{4. 4}$$

for $x\in D(A),\lambda>\sigma$ and $0\leqq\eta\leqq\xi$.

THEOREM 4. 3. *For each* $\xi\geqq0$ *and* $x\in X$

$$T(\xi)x=\lim_{\lambda\to\infty}\exp(-\lambda\xi)\sum_{k=0}^{\infty}\frac{(\lambda\xi)^k[\lambda R(\lambda;A)]^k}{k\,!}x.\tag{4. 5}$$

PROOF. By (4. 4) we have

$$T(\xi)x-T_\lambda(\xi)x=\int_0^{\xi}\frac{d}{d\eta}T_\lambda(\xi-\eta)T(\eta)xd\eta$$

$$= \int_0^{\xi} T_\lambda(\xi - \eta) T(\eta) (Ax - \lambda R(\lambda ; A)Ax) \, d\eta$$

for $x \in D(A)$. It follows from Theorems 4. 1 and 2. 1 that for any closed convex neighborhood N of the origin there exists a neighborhood $N' \in \Sigma$ such that $T_\lambda(\xi - \eta)N' \subset N$ for all $\lambda > 2\sigma$ and $\eta \in [0, \xi]$. Further there exists by Theorem 2. 1 a neighborhood $N'' \in \Sigma$ such that $T(\eta)N'' \subset N'$ for all $\eta \in [0, \xi]$, and Theorem 3. 5 asserts that there exists a number $\lambda_0 > 0$ such that $Ax - \lambda R(\lambda ; A)Ax \in N''$ for $\lambda > \lambda_0$. Hence if $\lambda > \max(2\sigma, \lambda_0)$, then $T_\lambda(\xi - \eta)T(\eta)(Ax - \lambda R(\lambda ; A)Ax) \in N$ for all $\eta \in [0, \xi]$. Hence we get

$$\frac{1}{\xi}[T(\xi)x - T_\lambda(\xi)x] \in N$$

for $\lambda > \max(\lambda_0, 2\sigma)$, that is, $\lim_{\lambda \to \infty} T_\lambda(\xi)x = T(\xi)x$ for each $\xi > 0$ and $x \in D(A)$. We have by Corollary 2. 1 that the limit $T'(\xi)x = \lim_{\lambda \to \infty} T_\lambda(\xi)x$ exists for all $x \in X$ and that $T'(\xi)$ is a continuous linear operator. Since $T'(\xi)x = T(\xi)x$ for $x \in D(A)$ and since the operators $T'(\xi)$ and $T(\xi)$ are continuous, we have $T(\xi)x = T'(\xi)x$ for all $x \in X$ and $\xi > 0$. If $\xi = 0$, then $T(\xi)x = x = T_\lambda(\xi)x$ for all $\lambda > \sigma$ and $x \in X$. Therefore (4. 5) holds for all $x \in X$ and $\xi \geqq 0$.

5. Generation of semi-groups. Collecting the previous results we get the following

THEOREM 5. 1. *If* $\{T(\xi) ; 0 \leqq \xi < \infty\}$ *is a semi-group of operators satisfying the conditions* (1), (2) *and* (3), *then*

(1′) *the infinitesimal generator* A *is a closed linear operator and its domain* $D(A)$ *is dense in* X,

(2′) *for each* $\lambda > \sigma$ *there exists a continuous linear operator* $R(\lambda ; A)$ *from* X *into itself such that*

$$(\lambda - A)R(\lambda ; A)x = x \qquad \textit{for } x \in X,$$
$$R(\lambda ; A)(\lambda - A)x = x \qquad \textit{for } x \in D(A),$$

(3′) *for each* $x \in X$ *the set*

$$\{[(\lambda - \sigma)R(\lambda ; A)]^n x ; \lambda > \sigma, \ n = 0, 1, 2, \cdots\cdots\}$$

is bounded.

Then we have

$$T(\xi)x = \lim_{\lambda \to \infty} \exp(-\lambda\xi) \sum_{k=0}^{\infty} \frac{(\lambda\xi)^k [\lambda R(\lambda ; A)]^k}{k\,!} x.$$

We now consider the converse problem for the theory of semi-groups, namely, what properties should an operator A possess in order that it be the infinitesimal generator of a semi-group of operators satisfying the conditions (1), (2) and (3)?

Let A be a linear operator satisfying the following conditions:

(1′) A is a closed linear operator from the domain $D(A)$ into X and $D(A)$ is dense in X.

(2′) For each $\lambda > \sigma$, where σ is some non-negative constant, there exists a continuous linear operator $R(\lambda ; A)$ from X into itself such that

$$(\lambda - A)R(\lambda ; A)x = x \qquad \text{for } x \in X,$$
$$R(\lambda ; A)(\lambda - A)x = x \qquad \text{for } x \in D(A).$$

(3′) For each $x \in X$ the set

$$\{[(\lambda - \sigma)R(\lambda ; A)]^n x ; \lambda > \sigma, n = 0, 1, 2, \cdots\cdots\}$$

is bounded.

Under these assumptions it follows from the previous arguments that

$$T_\lambda(\xi)x = \exp(-\lambda\xi) \sum_{k=0}^{\infty} \frac{(\lambda\xi)^k [\lambda R(\lambda ; A)]^k}{k!} x \tag{5.1}$$

is well defined for each $\xi \geqq 0$, $\lambda > \sigma$ and $x \in X$, and that Theorems 3.5, 4.1 and 4.2 hold.

We now prove that for each fixed $x \in X$ the limit $\lim_{\lambda \to \infty} T_\lambda(\xi)x$ exists uniformly with respect to ξ in any finite interval of ξ. In fact, by Theorem 4.2, we have

$$T_\mu(\xi)x - T_\lambda(\xi)x = \int_0^\xi \frac{d}{d\eta} T_\lambda(\xi - \eta) T_\mu(\eta) x \, d\eta$$
$$= \int_0^\xi T_\lambda(\xi - \eta) T_\mu(\eta)[\mu R(\mu ; A) - \lambda R(\lambda ; A)] A x d\eta$$

for $x \in D(A)$. By Theorems 4.1 and 4.2 for any closed neighborhood $N \in \Sigma$ there exists a neighborhood $N' \in \Sigma$ such that $T_\lambda(\xi - \eta)T_\mu(\eta)N' \subset N$ for all μ, $\lambda > 2\sigma$ and $0 \leqq \eta \leqq \xi \leqq \omega$, where ω is any fixed number, and Theorem 3.5 shows that there exists a number $\lambda_0 > 0$ such that $[\mu R(\mu; A) - \lambda R (\lambda ; A)]Ax \in N'$ for all λ, $\mu > \lambda_0$. Then we have for each $0 \leqq \xi \leqq \omega$

$$\xi^{-1}(T_\mu(\xi)x - T_\lambda(\xi)x) \in N$$

if λ, $\mu > \max(\lambda_0, 2\sigma)$, so that $T_\mu(\xi)x - T_\lambda(\xi)x \in \omega N$ for all $\xi \in [0, \omega]$ if λ, $\mu > \max(\lambda_0, 2\sigma)$. Hence for each fixed $x \in D(A)$ the limit $\lim_{\mu \to \infty} T_\mu(\xi)x$ exists uniformly with respect to ξ in any finite interval. Corollary 2.1 concludes that for each fixed $x \in X$ the limit $\lim_{\mu \to \infty} T_\mu(\xi)x$ exists uniformly with respect to ξ in any finite interval and this limit is a continuous linear operator from X into itself.

We define

$$T(\xi)x = \lim_{\lambda\to\infty} T_\lambda(\xi)x \qquad \text{for each } \xi \geqq 0 \text{ and } x \in X. \tag{5.2}$$

Since $T_\lambda(\xi)x$ is a continuous function of $\xi \in [0,\infty)$, $T(\xi)x$ is continuous with respect to $\xi \in [0, \infty)$ for each $x \in X$.

An elementary argument shows that for $\lambda > \sigma$

$$T_\lambda(\xi + \eta) = T_\lambda(\xi)T_\lambda(\eta) \text{ and } T_\lambda(0) = I,$$

and hence we have by (5.2) and Theorem 2.1 the semi-group property

$$T(\xi + \eta) = T(\xi)T(\eta) \text{ and } T(0) = I.$$

Finally, from Theorem 4.1, we have that the set $\{e^{-\sigma\xi} T(\xi)x;\ 0 \leqq \xi < \infty\}$ is bounded for each $x \in X$. Thus we obtain the following

THEOREM 5.2. *If A is an operator satisfying the assumptions* (1′), (2′) *and* (3′), *then A is the infinitesimal generator of a semi-group of operators* $\{T(\xi);\ 0 \leqq \xi < \infty\}$ *satisfying the conditions* (1), (2) *and* (3). *Further*

$$T(\xi)x = \lim_{\lambda\to\infty} \exp(-\lambda\xi) \sum_{k=0}^{\infty} \frac{(\lambda\xi)^k[\lambda R(\lambda;A)]^k}{k!} x$$

for all $x \in X$ and $\xi \geqq 0$.

PROOF. It has already been observed that the family of operators which is defined by (5.2) satisfies the conditions (1), (2) and (3). We shall now show that A is the infinitesimal generator of $\{T(\xi); 0 \leqq \xi < \infty\}$.
By (4.3)

$$\frac{1}{\xi}(T_\lambda(\xi)x - x) = \frac{1}{\xi}\int_0^\xi T_\lambda(\eta)\lambda R(\lambda;A)\,Ax\,d\eta \tag{5.3}$$

for each $x \in D(A)$. Let $N \in \Sigma$ be any neighborhood and let N' be a closed convex neighborhood of origin such that $N' + N' \subset N$. By Theorems 4.1 and 2.1 there exists a neighborhood $N'' \in \Sigma$ such that $T_\lambda(\eta)N'' \subset N'$ for all $\lambda > 2\sigma$ and $\eta \in [0, \xi]$, and by Theorem 3.5 there exists a number $\lambda_0' > 0$ such that $\lambda R(\lambda;A)Ax - Ax \in N''$ for all $\lambda > \lambda_0'$. Since (5.2) holds uniformly in any finite interval of ξ, there exists a number $\lambda_0 > 0$ such that $(T_\lambda(\eta) - T(\eta))Ax \in N'$ for all $\lambda > \lambda_0$ and $\eta \in [0, \xi]$. Thus if $\lambda > \max(\lambda_0, \lambda_0', 2\sigma)$, then

$$\begin{aligned}
&\frac{1}{\xi}\left(\int_0^\xi T_\lambda(\eta)\lambda R(\lambda;A)\,Ax\,d\eta - \int_0^\xi T(\eta)Ax\,d\eta\right)\\
&= \frac{1}{\xi}\int_0^\xi T_\lambda(\eta)(\lambda R(\lambda;A)Ax - Ax)d\eta + \frac{1}{\xi}\int_0^\xi (T_\lambda(\eta) - T(\eta))Ax\,d\eta\\
&\quad \in N' + N' \subset N.
\end{aligned}$$

Then passing to the limit with λ in (5.3) we have

$$\frac{1}{\xi}(T(\xi)x - x) = \frac{1}{\xi}\int_0^{\xi} T(\eta)Axd\eta$$

for $x \in D(A)$, so that $\lim_{\xi \downarrow 0} \frac{1}{\xi}(T(\xi)x - x) = Ax$ for $x \in D(A)$.

Let A' be the infinitesimal generator of $\{T(\xi); 0 \leqq \xi < \infty\}$ and $D(A')$ be its domain. Since $D(A') \supset D(A)$ and $A'x = Ax$ for $x \in D(A)$, it follows from the assumption (2′) and Theorem 5. 1 (2′) that $R(\lambda; A')(\lambda - A)x = x = R(\lambda; A)(\lambda - A)x$ for $x \in D(A)$. Hence we have $R(\lambda; A) = R(\lambda; A')$ according to $(\lambda - A)[D(A)] = X$, so that $A = A'$. This concludes the proof of Theorem 5. 2.

6. Applications to partial differential equations. The theory of semi-groups of operators in Banach space has been applied to the Cauchy problem for linear partial differential equations by E. Hille [2], P. D. Lax & A. N. Milgram [4] and K. Yosida [7], [8].

In this section we shall apply to the Cauchy problem our semi-group theory.

6. 1. Preliminaries. Let H be the space of real-valued C^∞-functions (infinite times continuously differentiable functions) defined on m-dimensional euclidean space E^m such that its partial derivatives of all orders belong to the space L^2. It is clear that the space H becomes a pre-Hilbert space under the inner product

$$(6.1) \qquad (\varphi, \psi)_n = \sum_{|k| \leqq n} \int_{E^m} D^{(k)} \varphi(t) D^{(k)} \psi(t) dt,$$

where $D^{(k)} = \dfrac{\partial^{k_1+k_2+\cdots+k_m}}{\partial t_1^{k_1} \partial t_2^{k_2} \cdots \partial t_m^{k_m}}$, $|k| = \sum_{i=1}^{m} k_i$ and $dt = dt_1\, dt_2 \cdots\cdots dt_m$.

Let H_n be the completion of H with respect to the norm

$$(6.2) \qquad \|\varphi\|_n = (\varphi, \varphi)_n^{1/2}.$$

The following theorem is due to P. D. Lax & A. N. Milgram [4].

THEOREM 6. 1. *Let a bilinear functional $B(\varphi, \psi)$ defined on the Hilbert space H_n satisfying the followings;*

$$|B(\varphi, \psi)| \leqq \gamma \|\varphi\|_n \|\psi\|_n \qquad \text{for all } \varphi, \psi \in H_n,$$
$$B(\varphi, \varphi) \geqq \delta \|\varphi\|_n^2 \qquad \text{for all } \varphi \in H_n,$$

where γ, δ are some positive constants. Then there exists a bounded linear operator S from H_n onto itself such $\|S\| \leqq \delta^{-1}$ and $(\varphi, \psi)_n = B(\varphi, S\psi)$ for all $\varphi, \psi \in H_n$.

We shall introduce a topology into the vector space H. Let D be a partial differential operator of the form $\partial^{k_1+k_2+\cdots+k_m}/\partial t_1^{k_1} \partial t_2^{k_2} \cdots \partial t_m^{k_m}$ and let us put

$$p_D(\varphi) = \left(\int_{E^m} |(D\varphi)(\boldsymbol{t})|^2 d\boldsymbol{t}\right)^{1/2} \quad \text{for all } \varphi \in H. \tag{6.3}$$

It is a semi-norm of the vector space H and the totality of those semi-norms corresponding to all partial differential operators defines a topology of H. We shall again denote by H the topological vector space H provided with this topology.

LEMMA 6. 1. *If* $\lim_{\alpha\to 0} f_\alpha = f$ *in* H, *then for each partial differential operator* D *)

$$\lim_{\alpha\to 0}(Df_\alpha)(\boldsymbol{t}) = (Df)(\boldsymbol{t})$$

holds uniformly with respect to $\boldsymbol{t}$ *in any compact set in* E^m.

PROOF. We now prove the lemma for $m = 2$. We may assume $f = 0$ without loss of generality. By the assumption and the Schwarz inequality we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\partial^{\delta_1+1}}{\partial t_1^{\delta_1}\partial t_2} e^{-(t_1^2+t_2^2)} f_\alpha(t_1, t_2)\right| dt_1\, dt_2 \to 0$$

as $\alpha \to 0$, that is, for arbitrary $\varepsilon > 0$ there exists a number $\alpha_0 = \alpha_0(\varepsilon) > 0$ such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\partial^{\delta_1+1}}{\partial t_1^{\delta_1}\partial t_2} e^{-(t_1^2+t_2^2)} f_\alpha(t_1, t_2)\right| dt_1\, dt_2 < \varepsilon \tag{6.4}$$

for $|\alpha| \leqq \alpha_0$, where $\delta_1 = 0$ or 1. Then

$$\left\{\begin{aligned} &\int_{-\infty}^{\infty}\left|\frac{\partial^{\delta_1}}{\partial t_1^{\delta_1}} e^{-(t_1^2+s_2^2)} f_\alpha(t_1, s_2) - \frac{\partial^{\delta_1}}{\partial t_1^{\delta_1}} e^{-(t_1^2+\varepsilon_2^2)} f_\alpha(t_1, \varepsilon_2)\right| dt_1 \\ &\leqq \int_{-\infty}^{\infty} dt_1 \int_{\varepsilon_2}^{s_2}\left|\frac{\partial^{\delta_1+1}}{\partial t_1^{\delta_1}\partial t_2} e^{-(t_1^2+t_2^2)} f_\alpha(t_1, t_2)\right| dt_2 \\ &\leqq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\partial^{\delta_1+1}}{\partial t_1^{\delta_1}\partial t_2} e^{-(t_1^2+t_2^2)} f_\alpha(t_1, t_2)\right| dt_1\, dt_2 < \varepsilon \end{aligned}\right. \tag{6.5}$$

for all $|\alpha| \leqq \alpha_0$, s_2 and ε_2. Now

$$\int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty}\left|\frac{\partial^{\delta_1}}{\partial t_1^{\delta_1}} e^{-(t_1^2+t_2^2)} f_\alpha(t_1, t_2)\right| dt_1 < \infty.$$

Hence we see that

$$\lim \int_{-\infty}^{\infty}\left|\frac{\partial^{\delta_1}}{\partial t_1^{\delta_1}} e^{-(t_1^2+t_2^2)} f_\alpha(t_1, t_2)\right| dt_1 = 0$$

*) From now the symbol D (or D_t) denotes a partial differential operator of the form $\partial^{k_1+\cdots+k_m}/\partial t_1^{k_1}\cdots\partial t_m^{k_m}$.

when t_2 tends to $-\infty$ (or $+\infty$) without taking the values of t_2 which form a set of finite measure. Therefore for each α there exists a sequence $\{\varepsilon_2^k\} = \{\varepsilon_2^k(\alpha)\}$ such that

$$\lim_{\varepsilon_2^k\to-\infty}\int_{-\infty}^{\infty}\left|\frac{\partial^{\delta_1}}{\partial t_1^{\delta_1}}e^{-(t_1^2+(\varepsilon_2^k)^2)}f_\alpha(t_1,\varepsilon_2^k)\right|dt_1=0.$$

Combining this and (6.5) we have

$$\int_{-\infty}^{\infty}\left|\frac{\partial^{\delta_1}}{\partial t_1^{\delta_1}}e^{-(t_1^2+s_2^2)}f_\alpha(t_1,s_2)\right|dt_1<\varepsilon \tag{6.6}$$

for all $|\alpha|\leqq\alpha_0$ and s_2. Let us take $\delta_1=1$, then

$$\begin{cases} |e^{-(s_1^2+s_2^2)}f_\alpha(s_1,s_2)-e^{-(\varepsilon_1^2+s_2^2)}f_\alpha(\varepsilon_1,s_2)| \\ \leqq\displaystyle\int_{\varepsilon_1}^{s_1}\left|\frac{\partial}{\partial t_1}e^{-(t_1^2+s_2^2)}f_\alpha(t_1,s_2)\right|dt_1\leqq\int_{-\infty}^{\infty}\left|\frac{\partial}{\partial t_1}e^{-(t_1^2+s_2^2)}f_\alpha(t_1,s_2)\right|dt_1<\varepsilon\end{cases} \tag{6.7}$$

for all s_1,ε_1,s_2 and $|\alpha|\leqq\alpha_0$. On the other hand if we put $\delta_1=0$ in (6.6), then

$$\int_{-\infty}^{\infty}\left|e^{-(t_1^2+s_2^2)}f_\alpha(t_1,s_2)\right|dt_1<\varepsilon$$

for all s_2 and $|\alpha|\leqq\alpha_0$. Hence, as in the preceding case, we see that for each fixed s_2 and $|\alpha|\leqq\alpha_0$ there exists a sequence $\{\varepsilon_1^k\}=\{\varepsilon_1^k(\alpha,s_2)\}$ such that

$$\lim_{\varepsilon_1^k\to-\infty}e^{-((\varepsilon_1^k)^2+s_2^2)}f_\alpha(\varepsilon_1^k,s_2)=0.$$

Combining this and (6.7) we have $|e^{-(s_1^2+s_2^2)}f_\alpha(s_1,s_2)|<\varepsilon$ for all s_1 and s_2 if $|\alpha|\leqq\alpha_0$. Thus $f_\alpha(s_1,s_2)\to0$ uniformly with respect to (s_1,s_2) in any compact set.

Finally if $\lim_{\alpha\to0}f_\alpha=0$ in H then it is obvious that $\lim_{\alpha\to0}Df_\alpha=0$ in H for each partial differential operator D, so that $(Df_\alpha)(\boldsymbol{t})\to0$ uniformly with respect to $\boldsymbol{t}$ in any compact set.

Using the same method we can also prove the lemma for $m\geqq3$.

The following lemma is easily proved.

LEMMA 6.2. *The space H is a Fréchet space.*

6.2. Parabolic equation. Let A be a partial differential operator of the $2n$-th order in m-dimensional euclidean space E^m;

$$A=-(-1)^n\sum_{|\rho|,|\nu|=0}^{n}a^{\rho;\nu}D^{(\rho)}D^{(\nu)}, \tag{6.8}$$

where $D^{(\rho)} = \dfrac{\partial^{\rho_1+\cdots+\rho_m}}{\partial t_1^{\rho_1}\cdots\partial t_m^{\rho_m}}$, $|\rho| = \sum_{i=1}^{m}\rho_i$ and the coefficients $a^{\rho;\nu} = a^{\rho_1,\dots,\rho_m;\nu_1,\dots,\nu_m}$ are real constants.

In this section we consider the Cauchy problem for the parabolic equation in m-dimensional euclidean space E^m;

$$\begin{cases} \dfrac{\partial u(\xi, t)}{\partial \xi} = Au(\xi, t), & \xi \geqq 0, \\ u(0, t) = f(t). \end{cases}$$

We assume that

(6. 9) $$a^{\rho;\nu} = a^{\nu;\rho} \qquad \text{for } |\rho| = |\nu| = n,$$

and there exists a constant $\varepsilon_0 > 0$ such that

(6. 10) $$\sum_{|\rho|=|\nu|=n} a^{\rho;\nu} t_1^{\rho_1}\cdots\cdots t_m^{\rho_m}\, t_1^{\nu_1}\cdots\cdots t_m^{\nu_m} \geqq \varepsilon_0\Big(\sum_{j=1}^{m} t_j^2\Big)^n$$

for each $(t_1,\dots\dots, t_m) \in E^m$. Therefore A is an elliptic differential operator. We define the adjoint operator A^* by

(6. 11) $$A^* = -(-1)^n \sum_{|\rho|,|\nu|=0}^{n} (-1)^{|\rho|+|\nu|}\, a^{\rho;\nu} D^{(\nu)} D^{(\rho)}.$$

We can easily prove

(6. 12) $$(Af, g)_0 = (f, A^* g)_0 \qquad \text{for all } f, g \in H,$$

where $(f, g)_0 = \int_{E^m} f(t) g(t)\, dt$.

LEMMA 6. 3. (*Gårding's inequalities*) *There exist positive constants* δ_0, $\lambda_0(=\lambda_0(\delta_0))$ *and* K *such that if* $\lambda \geqq \lambda_0$ *then*

(6. 13) $$((\lambda - A)f, f)_0 = (f, (\lambda - A^*)f)_0 \geqq \delta_0 \|f\|_n^2 \qquad \textit{for all } f \in H,$$

(6. 14) $$|(Af, g)_0 - (f, Ag)_0| \leqq K\|f\|_n \|g\|_{n-1} \qquad \textit{for all } f, g \in H.$$

Further for each $\lambda > 0$ *there exists a constant* M_λ *such that*

(6. 15) $$|((\lambda - A)f, g)_0| = |(f, (\lambda - A^*)g)_0| \leqq M_\lambda \|f\|_n \|g\|_n$$

for all $f, g \in H$.

For the proof see Gårding's paper [1]. In our case the coefficients are constants, so that we see that the inequalities hold in the space H.

LEMMA 6. 4. *Let* λ *be any fixed number such that* $\lambda > \lambda_0$. *Then, for any function* $f \in L^2 \cap C^\infty$, *the equation*

$$\lambda u - Au = f$$

has a solution $u_f \in H_n \cap C^\infty$.

PROOF. Let us define a bilinear functional

$$\bar{B}_\lambda(u, v) = (\lambda u - A^* u, v)_0$$

for all $u, v \in H$. From Gårding's inequalities we have

$$|\bar{B}_\lambda(u, v)| \leqq M_\lambda \|u\|_n \|v\|_n, \ \bar{B}_\lambda(u, u) \geqq \delta_0 \|u\|_n^2.$$

Since H_n is the completion of H with respect to the norm $\|\cdot\|_n$, $\bar{B}_\lambda(u, v)$ may be extended to the bilinear functional $B_\lambda(u, v)$ defined on H_n satisfying

(6. 16) $$|B_\lambda(u, v)| \leqq M_\lambda \|u\|_n \|v\|_n, \ B_\lambda(u, u) \geqq \delta_0 \|u\|_n^2.$$

For any $f \in L^2 \cap C^\infty$ the linear functional $(u, f)_0$ is a bounded functional defined on H_n since $|(u, f)_0| \leqq \|u\|_n \|f\|_0$. Hence, by the F. Riesz theorem, there exists an element $v(f) \in H_n$ such that $(u, f)_0 = (u, v(f))_n$ for all $u \in H_n$. Thus, by Theorem 6. 1, we get

$$(u, f)_0 = (u, v(f))_n = B_\lambda(u, S_\lambda v(f)) \qquad \text{for all } u \in H_n,$$

where S_λ is a bounded linear operator from H_n onto itself which is determined in Theorem 6. 1. Let $v_k \in H$ be a sequence such that

$$\lim_{k \to \infty} \|v_k - S_\lambda v(f)\|_n = 0.$$

Then, for $u \in \mathfrak{D}(E^m)$*) $\subset H$,

$$B_\lambda(u, S_\lambda v(f)) = \lim_{k \to \infty} B_\lambda(u, v_k) = \lim_{k \to \infty} (\lambda u - A^* u, v_k)_0$$
$$= (\lambda u - A^* u, S_\lambda v(f))_0,$$

so that

$$(u, f)_0 = (\lambda u - A^* u, S_\lambda v(f))_0 \qquad \text{for all } u \in \mathfrak{D}(E^m).$$

Thus $(\lambda - A) S_\lambda v(f) = f$ in $\mathfrak{D}'$(= the dual of $\mathfrak{D}$ = the space of distributions). $f(x)$ being any C^∞-function and $(\lambda - A)$ being an elliptic differential operator, we see, by the L. Schwartz theorem [6], that $u_f = S_\lambda v(f) \in H_n$ is a C^∞-solution.

LEMMA 6. 5. *Let λ be any fixed number such that $\lambda > \lambda_1$, where $\lambda_1 (> \lambda_0)$ is a constant. If $w \in L^2 \cap C^\infty$ and if $\lambda w - Aw = 0$, then $w(t) = 0$ for all $t \in E^m$.*

PROOF. Let $\mathfrak{S}$ be the space of all rapidly decreasing functions and let $\mathfrak{S}'$ be the dual of $\mathfrak{S}$. We now define

$$T_w(\varphi) = \int_{E^m} w(t) \varphi(t) \, dt$$

for all $\varphi \in \mathfrak{S}$. It is clear that $T_w \in \mathfrak{S}'$. According to L.Schwartz [6] we can

*) $\mathfrak{D}(E^m)$ denotes the space of C^∞-functions with compact carriers.

define the Fourier transform $F(T) \in \mathfrak{S}'$ for all $T \in \mathfrak{S}'$. It is well known that

$$(6.17) \qquad F(\partial T/\partial t_i) = 2\pi\sqrt{-1}\, t_i \cdot F(T),$$

$$(6.18) \qquad F^*(F(T)) = T,$$

where F^* denotes the conjugate Fourier transform.

By the hypothesis

$$\lambda T_w + (-1)^n \sum_{|\rho|, |\nu|=0}^{n} a^{\rho:\nu} D^{(\rho)} D^{(\nu)} T_w = 0,$$

so that by (6.17)

$$(6.19) \quad \lambda F(T_w) + (-1)^n \sum_{|\rho|, |\nu|=0}^{n} a^{\rho:\nu} (2\pi\sqrt{-1})^{(|\rho|+|\nu|)} t_1^{\rho_1+\nu_1} \cdots\cdots t_m^{\rho_m+\nu_m} \cdot F(T_w) = 0.$$

An elementary calculus shows that if $\lambda > \lambda_1 = \max(\lambda_0, C)$ then

$$|h_\lambda(t)| = |\lambda + (-1)^n \sum_{|\rho|, |\nu|=0}^{n} a^{\rho:\nu} (2\pi\sqrt{-1})^{|\rho|+|\nu|} t_1^{\rho_1+\nu_1} \cdots\cdots t_m^{\rho_m+\nu_m}|$$
$$\geqq \lambda - C > 0,$$

where $C = (2\pi)^{2n} \sum_{|\rho|, |\nu|=0}^{n} |a^{\rho:\nu}| \alpha_0^{|\rho|+|\nu|}$ and $\alpha_0 = \max(1, \varepsilon_0^{-1} \sum_{|\rho|, |\nu|=0}^{n} |a^{\rho:\nu}|)$. Then, for each $\lambda > \lambda_1$, $1/h_\lambda$ is a slowly increasing function, so that $\varphi/h_\lambda \in \mathfrak{S}$ for all $\varphi \in \mathfrak{S}$.

Now, by (6.19), $(F(T_w))(h_\lambda \varphi) = 0$ for all $\varphi \in \mathfrak{S}$. Hence if $\lambda > \lambda_1$, then $(F(T_w))(\varphi) = (F(T_w))(h_\lambda \dfrac{\varphi}{h_\lambda}) = 0$ for all $\varphi \in \mathfrak{S}$, that is, $F(T_w) = 0$. Thus we have $T_w = 0$ from (6.18). Then

$$\int_{E^m} w(t)\, \varphi(t)\, dt = 0$$

for all $\varphi \in \mathfrak{S}$, so that $w(t) = 0$ since $\mathfrak{S}$ is dense in L^2.

LEMMA 6.6. *Let λ be any fixed number such that $\lambda > \lambda_1$. Then, for any functions $f \in H$, the equation*

$$(\lambda - A)u = f$$

has a unique solution $u_f \in H$ and, for each semi-norm $p_D(\cdot)$,

$$p_D(u_f) \leqq \frac{1}{\lambda - C} p_D(f),$$

where λ_1 and C are constants in the preceding lemma.

PROOF. By Lemma 6.4, for each function $f \in H$ there exists a function $u_f \in H_n \cap C^\infty$ such that $(\lambda - A)u_f = f$. Operating any partial differential operator D, we have

(6.20) $$(\lambda - A)Du_f = Df.$$

Again it follows from Lemma 6.4 that there exists a function $u_{Df} \in H_n \cap C^\infty$ such that

(6.21) $$(\lambda - A)u_{Df} = Df.$$

We remark that $D^{(1)}u_f$ belongs to $L^2 \cap C^\infty$, where $D^{(1)}$ denotes a partial differential operator of the first order. If we put $w = D^{(1)} u_f - u_{D^{(1)}f}$, then $w \in L^2 \cap C^\infty$ and $(\lambda - A)w = 0$. Thus Lemma 6.5 shows that $D^{(1)}u_f = u_{D^{(1)}f} \in H_n \cap C^\infty \subset L^2 \cap C^\infty$. Repeating the same argument, for each partial differential operator D, we have $Du_f = u_{Df} \in H_n \cap C^\infty \subset L^2 \cap C^\infty$. Hence $u_f \in H$ and the uniqueness of the solution follows from Lemma 6.5.

Finally we get from (6.20)

$$h_\lambda(\boldsymbol{t})\, F(Du_f) = F(Df),$$

where F denotes the Fourier transform on L^2, and

$$|h_\lambda(\boldsymbol{t})| = |\lambda + (-1)^n \sum_{|\rho|, |\nu| = 0}^{n} a^{\rho;\nu}(2\pi\sqrt{-1})^{|\rho|+|\nu|} t_1^{\rho_1+\nu_1} \cdots\cdots t_m^{\rho_m+\nu_m}| \geqq \lambda - C$$

for all $\boldsymbol{t} \in E^m$. Hence

$$|F(Du_f)(\boldsymbol{t})| \leqq \frac{1}{\lambda - C} |F(Df)(\boldsymbol{t})|$$

for all $\boldsymbol{t} \in E^m$, so that by the Parseval theorem

$$\begin{aligned} p_D(u_f) &= \left(\int_{E^m} |Du_f(\boldsymbol{t})|^2 d\boldsymbol{t}\right)^{1/2} = \left(\int_{E^m} |F(Du_f)(\boldsymbol{t})|^2 d\boldsymbol{t}\right)^{1/2} \\ &\leqq \frac{1}{\lambda - C}\left(\int_{E^m} |F(Df)(\boldsymbol{t})|^2 d\boldsymbol{t}\right)^{1/2} = \frac{1}{\lambda - C}\left(\int_{E^m} |Df(\boldsymbol{t})|^2 d\boldsymbol{t}\right)^{1/2} \\ &= \frac{1}{\lambda - C} p_D(f). \end{aligned}$$

This concludes the proof of Lemma 6.6.

It is clear that A is a continuous linear operator from H into itself. If we put $R(\lambda; A)f = u_f$, then we obtain from Lemma 6.6 that $R(\lambda; A)$ $(\lambda > \lambda_1)$ is a continuous linear operator from H onto itself, that

$$(\lambda - A)R(\lambda; A)f = R(\lambda; A)(\lambda - A)f = f \qquad \text{for all } f \in H,\ \lambda > \lambda_1,$$

and that

$$p_D([R(\lambda; A)]^k f) \leqq \frac{1}{(\lambda - C)^k} p_D(f)$$

for each semi-norm p_D, $k = 1, 2, 3, \ldots\ldots$ and $\lambda > \lambda_1$. Hence the set

$$\{[(\lambda - \sigma)R(\lambda; A)]^k f;\ \lambda > \sigma,\ k = 0, 1, 2, \cdots\cdots\}$$

is bounded in H for each $f \in H$, where $\sigma = \max(\lambda_1, C)$. Thus Theorem 5.2 shows that the differential operator A generates a semi-group of operators satisfying the following conditions;

$$T(\xi + \eta) = T(\xi)T(\eta),\ T(0) = I, \tag{6.22}$$

$$p_D(T(\xi)f) \leqq e^{\sigma\xi}p_D(f) \quad \text{for all } f \in H, \tag{6.23}$$

$$\lim_{\xi \downarrow 0} T(\xi)f = f \quad \text{for all } f \in H. \tag{6.24}$$

In this case $D(A) = H$, so that

$$\frac{d^l T(\xi)f}{d\xi^l} = T(\xi)A^l f = A^l T(\xi)f \tag{6, 25}$$

for all $f \in H$ and $l = 1, 2, 3, \cdots\cdots\cdots$.

It follows from Lemma 6.1 that $dT(\xi)f / d\xi$ is equal to the ordinary derivative $\partial(T(\xi)f)(t)/\partial\xi$ and that $\lim_{\xi \downarrow 0} (T(\xi)f)(t) = f(t)$ for all $t \in E^m$. Thus if we put $u(\xi, t; f) = T(\xi)f$, then

$$\frac{\partial u(\xi, t; f)}{\partial \xi} = Au(\xi, t; f) \quad \text{for all } \xi \geqq 0, t \in E^m, \tag{6.26}$$

$$\lim_{\xi \downarrow 0} u(\xi, t; f) = f(t) \quad \text{for all } t \in E^m. \tag{6.27}$$

Furthermore, for each partial differential operator D_t with respeet to t, we have

$$\left(\int_{E^m} |D_t u(\xi, t; f)|^2 dt\right)^{1/2} \leqq e^{\sigma\xi}\left(\int_{E^m} |D_t f(t)|^2 dt\right)^{1/2} \tag{6.28}$$

for all $\xi \geqq 0$. It is clear that $u(\xi, t; f)$ is a C^∞-function with respect to ξ and $u(\xi, t; f) \in H$ for each $\xi \geqq 0$.

Finally we shall prove that $u(\xi, t; f)$ is a C^∞-function with respect to (ξ, t). Since $D_t A = AD_t$, we have $D_t R(\lambda; A) = R(\lambda; A)D_t$. Then, by the continuity of D_t and the representation theorem of $T(\xi)$, we have

$$T(\xi)D_t = D_t T(\xi).$$

Therefore we obtain from (6.25) that

$$\frac{\partial^l}{\partial \xi^l} D_t u(\xi, t; f) = D_t \frac{\partial^l}{\partial \xi^l} u(\xi, t; f) = u(\xi, t; A^l D_t f)$$

for $l = 1, 2, 3, \cdots\cdots\cdots$.

Now $T(\xi)A^l D_t f = u(\xi, \cdot; A^l D_t f)$ is a continuous function of $\xi \geqq 0$ with values in H, so that it follows from Lemma 6.1 that $u(\xi, t; A^l D_t f)$ is a continuous function of (ξ, t). Hence $u(\xi, t; f)$ is a C^∞-function with respect to (ξ, t). Thus we have the following

THEOREM 6.2. *The Cauchy problem for the parabolic equation in m-*

dimensional euclidean space E^m

$$\begin{cases} \dfrac{\partial u(\xi, t)}{\partial \xi} = (Au)(\xi, t), & \xi \geqq 0, t \in E^m, \\ u(0, t) = f(t), & t \in E^m, \end{cases}$$

is solvable in the following sense. For any given $f \in H$ *the above parabolic equation admits a* C^∞*(with respect to* (ξ, t)*) solution* $u(\xi, t) = u(\xi, t; f)$ *satisfying the conditions*

(i) $u(\xi + \eta, t; f) = u(\xi, t; u(\eta, \cdot; f))$ *for all* $\xi, \eta \geqq 0$ *and* $t \in E^m$,

(ii) *there exists a constant* $\sigma > 0$ *such that*

$$\left(\int_{E^m} |D_t u(\xi, t; f)|^2 dt\right)^{1/2} \leqq e^{\sigma\xi} \left(\int_{E^m} |D_t f(t)|^2 dt\right)^{1/2}$$

for all $\xi \geqq 0$ *and for all partial differential operators* D_t,

(iii) $\lim_{\xi \downarrow 0} u(\xi, \cdot; f) = f(\cdot)$ *in H and* $du(\xi, \cdot; f)/d\xi = Au(\xi, \cdot; f)$ *in H.*

Furthermore the solution $u(\xi, t; f)$ *such that the conditions* (ii) *and* (iii) *satisfy, is uniquely determined for* $f \in H$.

PROOF OF UNIQUENESS. We suppose $u_1(\xi, t; f)$ and $u_2(\xi, t; f)$ satisfy the conditions (ii) and (iii). Then $v(\xi, t; f) = u_1(\xi, t; f) - u_2(\xi, t; f)$ implies the followings;

$$\begin{cases} \lim_{\xi \downarrow 0} v(\xi, \cdot; f) = 0 \ \text{in } H, \\ \dfrac{dv(\xi, \cdot; f)}{d\xi} = Av(\xi, \cdot; f). \ \text{in } H, \\ p_D(v(\xi, \cdot; f)) \leqq 2e^{\sigma\xi} p_D(f). \end{cases}$$

Hence

$$L(\lambda; v) = \int_0^\infty e^{-\lambda\xi} v(\xi, \cdot; f)\, d\xi$$

exists for each $\lambda > \sigma$ and

$$\begin{aligned} AL(\lambda; v) &= \int_0^\infty e^{-\lambda\xi} Av(\xi, \cdot; f)\, d\xi = \int_0^\infty e^{-\lambda\xi} \frac{dv(\xi, \cdot; f)}{d\xi}\, d\xi \\ &= \lambda L(\lambda; v), \end{aligned}$$

that is, $(\lambda - A)L(\lambda; v) = 0$. Thus $L(\lambda; v) = 0$ for all $\lambda > \sigma$. Hence $v(\xi, \cdot; f) = 0$ for all $\xi \geqq 0$, so that $u_1(\xi, t; f) = u_2(\xi, t; f)$.

6.3. Wave equation. Let A be a partial differential operator of the second order in m-dimensional euclidean space E^m with constant coefficients satisfying (6.9) and (6.10) (with $n = 1$).

We now consider the Cauchy problem for the wave equation in m-

dimensional euclidean space E^m;

$$(6.29)\quad \begin{cases} \dfrac{\partial^2 u(\xi,t)}{\partial\xi^2} = Au(\xi,t), & -\infty<\xi<\infty, \\ u(0,t)=f(t),\ u_\xi(0,t) = \dfrac{\partial}{\partial\xi}u(0,t) = g(t). \end{cases}$$

The problem is equivalent to the matrical equation

$$(6.30)\quad \begin{cases} \dfrac{\partial}{\partial\xi}\begin{pmatrix} u(\xi,t) \\ v(\xi,t)\end{pmatrix} = \begin{pmatrix} 0 & I \\ A & 0\end{pmatrix}\begin{pmatrix} u(\xi,t) \\ v(\xi,t)\end{pmatrix}, \\ \begin{pmatrix} u(0,t) \\ v(0,t)\end{pmatrix} = \begin{pmatrix} f(t) \\ g(t)\end{pmatrix}. \end{cases}$$

Let λ_0 be a fixed positive number such that the Gårding inequality (6.13) holds, and let D be a partial differential operator of the form $\partial^{k_1+\cdots+k_m}/\partial t_1^{k_1}\cdots\partial t_m^{k_m}$. We define q_D by

$$(6.31)\quad q_D(f) = ((\lambda_0 - A)Df, Df)_0^{1/2} \qquad \text{for all } f\in H.$$

The following lemma is easily proved from the Gårding inequalities.

LEMMA 6.7. *q_D is a semi-norm of the vector space H and H becomes a Frechét space under the topology defined by the totality of semi-norms q_D corresponding to all partial differential operators. Further this topology is equivalent to the previous topology determined by $\{p_D; D\}$.*

Let us put

$$r_D\begin{pmatrix} f \\ g\end{pmatrix} = (q_D^2(f) + p_D^2(g))^{1/2}$$

for each $\begin{pmatrix} f \\ g\end{pmatrix} \in H\times H$.

It is a semi-norm of the product vector space $H\times H$ and the totality of those semi-norms corresponding to all partial differential operators defines a topology of $H\times H$. We shall again denote by $H\times H$ the topological vector space $H\times H$ provided with this topology. Then it is clear that the product space $H\times H$ is a Fréchet space.

From Yosida's arguments [8] and Lemma 6.6 we can prove the following

LEMMA 6.8. *There exists a positive number λ_2 such that if λ is a real number with $|\lambda|>\lambda_2$, then the equation*

$$\left(\lambda\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} - \begin{pmatrix} 0 & I \\ A & 0\end{pmatrix}\right)\begin{pmatrix} v_1 \\ v_2\end{pmatrix} = \begin{pmatrix} f \\ g\end{pmatrix}$$

has a unique solution $\begin{pmatrix} v_1 \\ v_2\end{pmatrix} = \begin{pmatrix} v_f \\ v_g\end{pmatrix} \in H\times H$ for each $\begin{pmatrix} f \\ g\end{pmatrix}\in H\times H$.

Further

(6. 32) $$r_D\begin{pmatrix} v_f \\ v_g \end{pmatrix} \leqq \frac{1}{|\lambda| - \beta} r_D\begin{pmatrix} f \\ g \end{pmatrix}$$

for each semi-norm $r_D, |\lambda| \geqq \lambda_2$ *and* $\begin{pmatrix} f \\ g \end{pmatrix} \in H \times H$, *where* β *is a positive constant independent of* λ, $\begin{pmatrix} f \\ g \end{pmatrix}$ *and* r_D.

Using the same method in the parabolic case, we can prove the following theorem.

THEOREM 6. 3. *The Cauchy problem for* (6. 29) *is solvable in the following sense*: *For any given pair* $\begin{pmatrix} f \\ g \end{pmatrix}$ *of H-functions the equation* (6. 29) *admits a* C^∞ (*with respect to* (ξ, t)) *solution* $u(\xi, t) = u(\xi, t; \binom{f}{g})$ *satisfying the conditions*

(i′) *there exists a constant* $\sigma > 0$ *such that*

$$[((\lambda_0 - A)D_t u(\xi, \cdot), D_t u(\xi, \cdot))_0 + (D_t u_\xi(\xi, \cdot), D_t u_\xi(\xi, \cdot))_0]^{1/2} \leqq e^{\sigma|\xi|}[((\lambda_0 - A)D_t f, D_t f)_0 + (D_t g, D_t g)_0]^{1/2}$$

for all ξ *and for each partial differential operator* D_t,

(ii′) $$\begin{cases} \lim_{\xi \to 0} \begin{pmatrix} u(\xi, \cdot) \\ u_\xi(\xi, \cdot) \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \text{ in } H \times H, \\ \frac{d}{d\xi} \begin{pmatrix} u(\xi, \cdot) \\ u_\xi(\xi, \cdot) \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \begin{pmatrix} u(\xi, \cdot) \\ u_\xi(\xi, \cdot) \end{pmatrix} \text{ in } H \times H. \end{cases}$$

Further the solution $u(\xi, t)$ *which satisfies the conditions* (i′) *and* (ii′) *is uniquely determined for* $\begin{pmatrix} f \\ g \end{pmatrix} \in H \times H$.

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