ON REALIZATIONS OF SOME WHITEHEAD PRODUCTS

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Introduction. For any arcwise connected space B with a base point b_0, the sequence of homotopy groups of (B, b_0):

$$\pi_1, \pi_2, \cdots\cdots, \pi_n, \cdots\cdots$$

are defined. These groups except the first one are abelian and are written additively, while the fundamental group π_1 is in general non-abelian and is written multiplicatively. Among these groups there are two kinds of important operations defined topologically. The first one is the operations of π_1 on π_p with $p \geqq 2$ (for the definition see § 16 of [17][1]), i. e. π_p becomes a π_1-modules, namely, for $w \in \pi_1$ and $\alpha \in \pi_p$, $p \geqq 2$, a unique element $w \cdot \alpha$ is determined and

$$w \cdot (\alpha_1 + \alpha_2) = w \cdot \alpha_1 + w \cdot \alpha_2,$$
$$w_1 \cdot (w_2 \cdot \alpha) = (w_1 w_2) \cdot \alpha, \qquad 1 \cdot \alpha = \alpha.$$

The second one is so-called Whitehead products (for the definition see [24]), i. e. for $\alpha \in \pi_p$, $\beta \in \pi_q$ with $p, q \geqq 2$, a bilinear product $[\alpha, \beta] \in \pi_{p+q-1}$ is defined. Hence these products define homomorphisms from $\pi_p \otimes \pi_q$ into π_{p+q-1}, which will be denoted by $W_{p,q}$ or $W_{p,q}(B)$, where the tensor product is taken over the integer coefficients.

It is well-known that these operations satisfy the following properties ([24], [16]):

(1) The skew symmetric law:

$$[\alpha, \beta] = (-1)^{pq}[\beta, \alpha], \text{ or}$$
$$W_{p,q}(\alpha \otimes \beta) = (-1)^{pq} W_{q,p}(\beta \otimes \alpha),$$

(2)
$$w \cdot [\alpha, \beta] = [w \cdot \alpha, w \cdot \beta], \text{ or}$$
$$w \cdot W_{p,q}(\alpha \otimes \beta) = W_{p,q}((w \cdot \alpha) \otimes (w \cdot \beta)),$$

(3) The Jacobi identity:

$$(-1)^{p(r-1)}[\alpha, [\beta, \gamma]] + (-1)^{q(p-1)}[\beta, [\gamma, \alpha]]$$

1) Numbers in brackets refer to the references at the end of the paper.

$$+ (-1)^{r(q-1)}[\gamma, [\alpha, \beta]] = 0,$$

where $w \in \pi_1$, $\alpha \in \pi_p$, $\beta \in \pi_q$, $\gamma \in \pi_r$, $p, q, r \geqq 2$.

We define operations of π_1 on $\pi_p \otimes \pi_q$ by $w\cdot(\alpha \otimes \beta) = ((w\cdot\alpha) \otimes (w\cdot\beta))$, then the property (2) means that $W_{p,q}$ preserves operations of π_1, i. e. $W_{p,q}$ is a π_1-homomorphism.

The realization problem of Whitehead products is stated as follows. Let π_1 be a given multiplicative group and π_p with $p \geqq 2$ be given π_1-modules, and $T_{p,q}: \pi_p \otimes \pi_q \to \pi_{p+q-1}$ with $p, q \geqq 2$ be given π_1-homomorphisms which satisfy the properties corresponding to (1) and (3). The realization of this system π_n, $T_{p,q}$ with $n \geqq 1$, $p, q \geqq 2$ is to construct an arcwise connected space B with a base point b_0 satisfying the following conditions:

(i) there exists, for each $n \geqq 1$, an isomorphism

$$h_n : \pi_n(B, b_0) \approx \pi_n,$$

(ii) for arbitrary elements $w \in \pi_1(B, b_0)$, $\alpha \in \pi_p(B, b_0)$ with $p \geqq 2$,

$$h_p(w\cdot\alpha) = h_1(w)\cdot h_p(\alpha),$$

(iii) for arbitrary elements $\alpha \in \pi_p(B, b_0)$, $\beta \in \pi_q(B, b_0)$ with $p, q \geqq 2$,

$$h_{p+q-1}([\alpha, \beta]) = T_{p,q}(h_p(\alpha) \otimes h_q(\beta)), \text{ or}$$
$$h_{p+q-1} \circ W_{p,q}(B) = T_{p,q} \circ (h_p \otimes h_q).$$

At first J. H. C. Whitehead [25] succeeded to construct a CW-complex which realizes groups $\pi_1, \pi_2, \ldots\ldots, \pi_n, \ldots\ldots$ and operations of π_1 on π_p with $p \geqq 2$. Also, S. T. Hu [13] constructed a space B which realizes this system such that all $T_{p,q}$'s are trivial.

Recently P. J. Hilton explained in his paper [12] that all identical relations between Whitehead products follow from the skew symmetric law and the Jacobi identity by application of the laws of addition and the distributivity of the Whitehead product. Therefore, the properties which exist between π_1-homomorphisms $W_{p,q}$ are essentially (1) and (3). But the properties corresponding to (1) and (3) for $T_{p,q}$'s are not sufficient conditions in order that this system is realizable. Indeed we shall need to assume that[2)]

(4) $\qquad T_{p,p}(\alpha \otimes \alpha) = 0 \qquad$ for $p = 3$ or 7 $(\alpha \in \pi_p)$.

Moreover, we shall need to impose other conditions. For this end we shall consider the composition operations. The composition operation is a map $C_{r,n}$:

2) Recently J. F. Adams proved that there is no element of Hopf invariant one in $\pi_{2n-1}(S^n)$ unless $n = 2, 4$ or 8 (cf. Bull. Amer. Math. Soc. 64(1958), 279–282). Therefore necessary conditions imposed to $T_{p,p}(\alpha \otimes \alpha)$ are only (4).

$\pi_r \circ \pi_n(S^r) \to \pi_n$ for each n, $r \geqq 2$ which preserves operations of π_1 on π_r and π_n, but in general not homomorphic. The right distributivity holds and these operations are related to Whitehead products by the following formula (cf. [12]):

$$(5) \quad (\alpha + \beta)\circ\xi = \alpha\circ\xi + \beta\circ\xi + [\alpha, \beta]\circ H_0(\xi) + [\alpha, [\alpha, \beta]]\circ H_1(\xi) + [\beta, [\alpha, \beta]]\circ H_2(\xi) + \cdots\cdots,$$

where $\alpha, \beta \in \pi_r$, $\xi \in \pi_n(S^r)$ and $H_0: \pi_n(S^r) \to \pi_n(S^{2r-1})$, $H_1, H_2: \pi_n(S^r) \to \pi_n(S^{3r-2})$,...... are generalizations of Hopf invariants.

Thus there arises a question that "if $T_{p,q}$ and $C_{r,n}$ satisfy the conditions corresponding to (1), (3), (4) and (5), then is the system π_n, $T_{p,q}$ realizable?" To solve the realization problem of Whitehead products seems to be very difficult.

As the first step to attack the realization problem of Whitehead products we shall deal with the realizability of a $T_{p,q}$ with arbitrary preassigned p and q $(p \neq q)$ and that of $T_{p,p}$ for $p \leqq 8$.

In § 1 we shall summarize the method of S. T. Hu by which realization problems are reduced to construct a simply connected space with π_1 as transformation group which realizes π_n and $T_{p,q}$ $(n, p, q \geqq 2)$. In §§ 2 and 3 we shall give some topological and algebraic lemmas which will be used in later sections. Lemmas 1 and 2 in § 2 are generalizations of Lemmas 2 and 3 of [2] to the case of spaces on which a group operates. Replacing these lemmas in the construction of fibre space due to Cartan-Serre-G. W. Whitehead ([2], [22]) by our lemmas, we can give a sufficient condition for the Problem 11 of [15] (see Proposition 2 in § 2). In § 4 we treat with the realization of only one $T_{p,q}$ with $p \neq q$. In this case no condition for $T_{p,q}$ is needed. Also we obtain some results concerning the simultaneous realization of some $T_{p,q}$'s with $p \neq q$. §§5 and 6 are devoted to the realizations of $T_{p,p}$ for $p = 2$, 4 and § 7 is devoted to the simultaneous realization of $T_{2,2}$ and $T_{2,3}$ which is the lowest dimensional case where the Jacobi identity appears. The results concerning to $T_{6,6}$ and $T_{7,7}$ are stated in § 8 and also that of $T_{p,p}$ for $p = 3$, 5, 8 are stated in § 9.

Except the cases of $T_{p,q}$ for $p \neq q$ and $T_{p,p}$ for $p = 6$ or 7, our results are incomplete in the sense that some additional conditions are assumed. And it is desirable to remove these conditions.

1. The method of S. T. Hu. Let Y be an arcwise connected space on which a multiplicative group W operates as a transformation group. Such space will be called a *W-space*. By an invariant subspace of a W-space X we mean a subspace X_0 such that $w(X_0) \subset X_0$ for any $w \in W$. Hence X_0 itself will be regarded as a W-space, and if X_0 consists of only one point, this will be called a fixed point. If, for any $w \in W (w \neq 1)$ and for any $x \in X_0$, $wx \neq x$, then we say that W operates freely on X_0.

If X is a simply connected W-space, n-th homotopy groups $\pi_n(X, x)$ relative to every point $x \in X$ form a simple system of local groups (for examples see § 23 of [4]). Therefore the unique isomorphism $\phi(x_0, x): \pi_n(X, x) \approx \pi_n(X, x_0)$ is defined. Besides, each $w \in W$ induces the isomorphism $w_*: \pi_n(X, x) \approx \pi_n(X, wx_0)$. If we define an isomorphism $w: \pi_n(X, x_0) \approx \pi_n(X, x_0)$ by $w = \phi(x_0, wx_0) \circ w_*$, then $\pi_n(X, x_0)$ becomes a W-modules.

Throughout this paper homotopy groups of a simply connected W-space are understood as W-modules in this sense.

Let B be an arcwise connected space with a base point b_0. By $\widetilde{B}$ we denote the universal covering space constructed by usual method (cf. § 23 of [4]), by $\widetilde{b}_0$ denote the point of $\widetilde{B}$ represented by the constant path $I \to b_0$. Let $p: \widetilde{B} \to B$ be the projection. It is well known that $\widetilde{B}$ is simply connected and p induces the isomorphism $p_*: \pi_n(\widetilde{B}, \widetilde{b}_0) \approx \pi_n(B, b_0)$ for each $n \geqq 2$. Also $\pi_1(B, b_0)$ operates on $\widetilde{B}$ as the group of covering transformations. It is easily seen that p_* is an operator isomorphism, i. e. $\pi_1(B, b_0)$-isomorphism.

Let (X, X_0) be a pair of a W-space X and a simply connected invariant subspace $X_0 \subset X$. Then operations of W on $\pi_n(X, X_0; x_0)$ are similarly defined. In addition, if X is simply connected, then the homomorphism induced by inclusion $j_*: \pi_n(X, x_0) \to \pi_n(X, X_0; x_0)$ and the boundary homomorphism $\partial: \pi_n(X, X_0; x_0) \to \pi_{n-1}(X_0, x_0)$ are operator homomorphisms.

Let (π, n) be a pair of a group π and an integer $n \geqq 1$. For $n > 1$ we assume the commutativity of π. We shall denote by $P(\pi, n)$ the Giever-Hu's geometric realization ([8], [13]) of Eilenberg-MacLane complex $K(\pi, n)$ ([6]).

We recall some results on $P(\pi, n)$ (cf. [13]). If $n > 1$ and π is a π_1-module, then $P(\pi, n)$ is a π_1-space and the unique 0-cell is a fixed point, and $\pi_m(P(\pi, n)) = 0$ for $1 \leqq m \neq n$ and there exists a natural π_1-isomorphism $\pi_n(P(\pi, n)) \approx \pi$. For $P(\pi_1, 1)$, $\pi_1(P(\pi_1, 1)) \approx \pi_1$ and $\pi_m(P(\pi_1, 1)) = 0$ for $m > 1$. We shall identify $\pi_n(P(\pi, n))$ with π under this natural π_1-isomorphism for $n > 1$ and also $\pi_1(P(\pi_1, 1))$ with π_1.

Let $\mathfrak{B} = \{B, p, X, Y, G\}$ be a coordinate bundle in the sense of Steenrod [17]. We assume the following conditions:

(1) X and Y are arcwise connected,

(2) $\pi_i(X) = 0$ for $i > 1$, and $\pi_1(Y) = 1$,

(3) the structural group G is totally disconnected.

Let $b_0 \in B$ be a point and put $x_0 = p(b_0)$, $Y_0 = p^{-1}(x_0)$. By the exactness of the homotopy sequence of $\mathfrak{B}$ and the assumption (2), the inclusion map $(Y_0, b_0) \subset (B, b_0)$ induces isomorphisms $i_n^*: \pi_n(Y_0, b_0) \approx \pi_n(B, b_0)$ for $n > 1$, and the projection $p: B \to X$ induces the isomorphism $p_1^*: \pi_1(B, b_0) \approx \pi_1(X, x_0)$.

Let $\xi: Y_0 \to Y$ be an admissible map and

$$\chi: \pi_1(X, x_0) \to G$$

be the homomorphism of the characteristic class $\chi(\mathfrak{B})$ determined by ξ (for the definition cf. § 13 of [17]).

Under these assumptions and notations we have following

PROPOSITION 1. *For arbitrary elements* $w \in \pi_1(B, b_0)$, $\alpha \in \pi_n(B, b_0)$ *with* $n > 1$, *we have*

$$h_n(w \cdot \alpha) = h_1(w) \cdot h_n(\alpha),$$

where

$$h_n = \xi_n^* \circ i_n^{*-1}: \pi_n(B, b_0) \approx \pi_n(Y, y_0), \ (n > 1),$$
$$h_1 = \chi \circ p_1^*: \pi_1(B, b_0) \to G.$$

This proposition is essentially proved in Theorem 3 and 4 in [13]. In that proof it is used that Y has a fixed point, but it is easily seen that this restriction can be removed.

By this proposition, the realization of $\pi_1, \pi_2, \ldots\ldots, \pi_n, \ldots\ldots$ and $T_{p,q}$'s is reduced to construct a simply connected π_1-space with π_2, π_3, and $T_{p,q}$'s as homotopy groups and Whitehead products.

2. Topological lemmas. First we shall prove the following lemmas.

LEMMA 1. *Let X be a simply connected W-space, and n be a fixed integer > 1. Let $\rho: \pi_n(X) \to G$ be a given W-homomorphism from $\pi_n(X)$ into a W-module G. Then there exists a W-space Z such that Z contains X as an invariant subspace and W operates freely on $Z - X$, and the inclusion map $i: X \subset Z$ induces the W-isomorphisms $i_r^*: \pi_r(X) \approx \pi_r(Z)$ for $1 \leqq r < n$ and i_n^* is onto and the kernel of i_n^* coincides with the kernel of ρ.*

LEMMA 2. *Let X be a simply connected W-space. Then, for an integer $n > 1$, there exists a W-space Z which contains X as an invariant subspace, and the inclusion $i: X \subset Z$ induces the W-isomorphisms $i_r^*: \pi_r(X) \approx \pi_r(Z)$ for $1 \leqq r < n$ and $\pi_r(Z) = 0$ for $r \geqq n$.*

PROOF OF LEMMA 1. Let Γ be the kernel of the given W-homomorphism $\rho: \pi_n(X) \to G$, and for each element γ of Γ, let $f_\gamma: \dot{E}^{n+1} \to X$ be a fixed map which represents the element γ, where E^{n+1} and $\dot{E}^{n+1}$ denote an

$(n+1)$-element and its boundary respectively. For any pair (w, γ) for $w \in W$, $\gamma \in \Gamma$, we consider the set $E^{n+1}_{(w,\gamma)} = \{(x, w, \gamma) | x \in E^{n+1}\}$. We introduce a topology to $E^{n+1}_{(w,\gamma)}$ such that the map $\lambda_{(w,\gamma)}: E^{n+1}_{(w,\gamma)} \to E^{n+1}$ defined by $\lambda_{(w,\gamma)}(x, w, \gamma) = x$ becomes a homeomorphism. $E^{n+1}_{(w,\gamma)}$ are mutually disjoint. Define a map $\psi_{(w,\gamma)}: \dot{E}^{n+1}_{(w,\gamma)} \to X$ by $\psi_{(w,\gamma)} = w \circ f_\gamma \circ \lambda_{(w,\gamma)}$.

Let Z be a space obtained from X by attaching each $E^{n+1}_{(w,\gamma)}$ $(w \in W, \gamma \in \Gamma)$, by the map $\psi_{(w,\gamma)}$. The operation of W on Z is defined as follows: For any element $v \in W$, we shall define a map $v: Z \to Z$ by

$$v(z) = v(z) \qquad \text{if } z \in X,$$
$$v(x, w, \gamma) = (x, vw, \gamma) \qquad \text{if } (x, w, \gamma) \in E^{n+1}_{(w,\gamma)}.$$

It is easily verified that the map v is well-defined and continuous and Z becomes a W-space which contains X as an invariant subspace. The characteristic map $\psi_{(w,\gamma)}: \dot{E}^{n+1}_{(w,\gamma)} \to X$ represents the element $w \cdot \gamma$ of $\pi_n(X)$ and the set of elements $w \cdot \gamma$ generates Γ. Therefore, by Theorem 18 of [23], i^*_n is onto and the kernel of i^*_n = the kernel of ρ. It is obvious that i^*_r $(1 \leqq r < n)$ are W-isomorphisms. Thus Z has the required properties.

PROOF OF LEMMA 2. Applying Lemma 1 with $G = \{0\}$, $\rho: \pi_n(X) \to G$, then we obtain a W-space Z_n which contains X as an invariant subspace and $i^*_r: \pi_r(X) \approx \pi_r(Z_n)$ for $r < n$ and $\pi_n(Z_n) = 0$. Next, applying Lemma 1 with $G = \{0\}$, $\rho: \pi_{n+1}(Z_n) \to G$, we have a W-space $Z_{n+1} \supset Z_n$ such that $\pi_r(Z_{n+1}) \approx \pi_r(Z_n)$ for $r < n+1$ and $\pi_{n+1}(Z_{n+1}) = 0$. If we continue this process, we have a sequence of W-space $X = Z_{n-1} \subset Z_n \subset Z_{n+1} \subset \ldots\ldots$ such that Z_i is an invariant subspace of Z_{i+1} and i^*_r: $\pi_r(Z_s) \approx \pi_r(Z_{s+1})$ for $r < s-1$, and $\pi_{s+1}(Z_{s+1}) = 0$ $(s \geqq n-1)$. Therefore the limit space $Z = \lim Z_i$ has the required properties.

As described in the introduction, these lemmas are generalizations of Lemma 2, 3 of [2]. Replacing lemmas in the construction of fibre spaces due to Cartan-Serre-G.W.Whitehead ([2], [22]) by the above lemmas, we can give the following solution for Problem 11 of [15].

PROPOSITION 2. *Let X be an $(n-1)$-connected W-space $(n \geqq 2)$. If X has a fixed point, there exist an n-connected W-space X' and a fibre map $p: X' \to X$ such that p commutes with the operations of W, and the induced homomorphisms $p_*: \pi_i(X') \to \pi_i(X)$ are onto W-isomorphisms for $i > n$.*

We shall need the following lemmas in later sections.

LEMMA 3. *For any arcwise connected space X with a base point x_0 and for any integer $n > 1$, there exists a $\pi_1(X, x_0)$-space E having the following properties:*

(1) *$\pi_i(E)$ is trivial for $1 \leqq i < n$,*

(2) *there exists $\pi_1(X, x_0)$-isomorphisms $h_i: \pi_i(E) \approx \pi_i(X, x_0)$ $(i \geqq n)$*

such that

$$h_{p+q-1} \circ W_{p,q}(E) = W_{p,q} \circ (h_p \otimes h_q), \ (p, q \geqq n).$$

This lemma is easily obtained from the following lemma and Proposition 2.

LEMMA 4. *For any arcwise connected space X with a base point x_1, there exists a simply connected $\pi_1(X, x_1)$-space B which satisfies the following conditions:*

(1) *there exists a fixed point $b_0 \in B$,*

(2) *there exist $\pi_1(X, x_1)$-isomorphisms $h_n: \pi_n(B, b_0) \approx \pi_n(X, x_1)$ $(n \geqq 2)$ such that $h_{p+q-1} \circ W_{p,q}(B) = W_{p,q}(X) \circ (h_p \otimes h_q)$ for $p, q > 1$.*

PROOF. Let $\widetilde{X}$ be the universal covering space of X and $p: \widetilde{X} \to X$ be the projection, and $\widetilde{x}_1$ be a point of $\widetilde{X}$ represented by the constant map $I \to x_1$. $\widetilde{X}$ is a $\pi_1(X, x_1)$-space and p induces $\pi_1(X, x_1)$-isomorphisms p_*: $\pi_n(\widetilde{X}, \widetilde{x}_1) \approx \pi_n(X, x_1)$ for $n > 1$.

For any $w \in \pi_1(X, x_1)$, the covering transformation $w: \widetilde{X} \to \widetilde{X}$ induces an isomorphism $w_{\#}: S \to S$, where S denotes the total singular complex of $\widetilde{X}$. Thus $\pi_1(X, x_1)$ operates on S.

We shall consider a minimal subcomplex M_1 of $\widetilde{X}$ relative to the base point $\widetilde{x}_1$ [5], and we define operations of $\pi_1(X, x_1)$ on M_1. The image subcomplex $M_w = w_{\#}(M_1)$ is obviously a minimal subcomplex relative to the base point $w(\widetilde{x}_1)$. Since $\widetilde{X}$ is simply connected, the isomorphism φ_w: $M_w \to M_1$ introduced in § 7 of [5] is uniquely determined, i. e. φ_w is independent upon the choice of a path joining $\widetilde{x}_1$ and $w(\widetilde{x}_1)$ used in definition of φ_w.

We define an isomorphism w: $M_1 \to M_1$ by $w = \varphi_u \circ w^0_{\#}$, where $w^0_{\#} = w_{\#} | M_1$. We shall consider following diagrams:

$$\begin{array}{ccc} M_w & \xrightarrow{\varphi_w} & M_1 \\ \downarrow {\scriptstyle w^1_{\#}} & & \downarrow {\scriptstyle v^0_{\#}} \\ M_{vw} & \xrightarrow{\varphi} & M_v \end{array} \qquad \begin{array}{ccc} M_{vw} & \xrightarrow{\varphi} & M_v \\ & {\scriptstyle \varphi_{vw}} \searrow & \downarrow {\scriptstyle \varphi_v} \\ & & M_1 \end{array}$$

where $v^1_\# = v_\# | M_w$, and φ is the similar isomorphism as φ_w. By the uniqueness of maps φ, φ_v, φ_w, φ_{vw}, commutativities hold in these diagrams, i. e.

$\varphi \circ v^1_\# = v_\# \circ \varphi_w$ and $\varphi_{vw} = \varphi_v \circ \varphi$. Since $(vw)^0_\# = v^1_\# \circ w^0_\#$ we have

$$\varphi_{vw} \circ (vw)^0_\# = \varphi_v \circ \varphi \circ v^1_\# \circ w^0_\# = (\varphi_v \circ v^0_\#) \circ (\varphi_w \cdot w^0_\#).$$

Thus, under this definition, $\pi_1(X, x_1)$ operates on M_1 as a group of isomorphisms.

Therefore realization polytopes $P(M_1)$ and $P(S)$ are $\pi_1(X, x_1)$-spaces and the unique 0-cell b_0 of $B = P(M_1)$ is a fixed point. Let $i: B \to P(S)$ be the map induced by the inclusion $M_1 \subset S$ and $q: P(S) \to \tilde{X}$ be the projection. It is well-known that i and q induce isomorphisms $i_*: \pi_n(B) \approx \pi_n(P(S))$, $q_*: \pi_w(P(S)) \approx \pi_n(X)$ for $n > 1$. Since q is a $\pi_1(X, x_1)$-map, q_* is a $\pi_1(X, x_1)$-isomorphism. We define $h_n: \pi_n(B) \approx \pi_n(X)$ $(n > 1)$ by $h_n = p_* \circ q_* \circ i_*$.

Let $M^0_1 \subset M_1$ be the subcomplex consisting of all collapsed simplexes, then $P(M^0_1)$ is a contractible invariant subspace. Therefore $j_*: \pi_n(P(M_1)) \to \pi_n(P(M_1), P(M^0_1))$ are $\pi_1(X, x_1)$-isomorphisms. On the other hand $\pi_n(P(M^0_1))$ is generated by n-cells corresponding to n-simplexes of M_1 with collapsed faces. Therefore, from the definition of φ it is easily seen that i_* is a $\pi_1(X, x_1)$-isomorphism, and by the naturality of Whitehead products, the condition (2) is satisfied. Thus the proof is complete.

3. Algebraic lemmas. Let W be a multiplicative group, and H be a W-module. If H is a free abelian group and there exists a set $B \subset H$ such that the element $w \cdot b$, for all $w \in W$, $b \in B$, are pairwise distinct and form a basis for H, then H is said to be *W-free*. This set B is called a *W-basis*.

By the same way as the proof of Lemma 6. 3 of [7] we have the following

LEMMA 5. *If H_0 is a submodule of W-module H and the factor W-module H/H_0 is W-free, then H_0 is a direct summand of H.*

LEMMA 6. *Let H be an abelian group and H_0 be a subgroup of H such that H/H_0 is decomposed to a direct sum $F + A$, where F is a free group and A is a direct sum of finite cyclic groups. For any abelian group G, in order that any homomorphism $\theta: H_0 \to G$ is extendable to a homomorphism $\theta^*: H \to G$, it is necessary and sufficient that for any element $h \in H$ and for any integer m such that $mh \in H_0$, the element $\theta(mh)$ is divisible by m.*

PROOF. The necessity is obvious, so we shall prove the sufficiency. By the assumption,

$$H/H_0 = F + \sum A_\alpha \qquad \text{(direct sum decomposition)},$$

where A_α is a finite cyclic group of order $r_\alpha(>1)$ with a generator a_α.

Let H_1 be a subgroup of H such that $H/H_1 = F$. Then, by Lemma 5, H_1 is a direct summand of H, hence any homomorphism $H_1 \to G$ is extendable over H. Thus we may assume that $F = 0$, $H/H_0 = \sum A_\alpha$.

Let $p: H \to H/H_0$ be the projections and for each a_α we select an element $h_\alpha \in H$ such that $ph_\alpha = a_\alpha$. Since $r_\alpha h_\alpha \in H_0$, by the assumption for θ, there exists an element $g_\alpha \in G$ such that $\theta(r_\alpha h_\alpha) = r_\alpha g_\alpha$.

Now, any element h of H can be written as

$$h = h_0 + \sum m_\alpha h_\alpha,$$

where $h_0 \in H_0$ and m_α are integers which are zero except finite numbers. We define a map

$$\theta^* : H \to G$$

by

$$\theta^*(h) = \theta(h_0) + \sum m_\alpha g_\alpha.$$

First we must show that θ^* is a well defined homomorphism. If h has another representation $h = h_0' + \sum n_\alpha h_\alpha$, then

$$h_0 - h_0' = \sum (n_\alpha - m_\alpha) h_\alpha.$$

Hence $0 = p(h_0 - h_0') = \sum (n_\alpha - m_\alpha) p h_\alpha = \sum (n_\alpha - m_\alpha) a_\alpha$, thus we have

$$n_\alpha - m_\alpha = r_\alpha t_\alpha \qquad (r_\alpha : \text{integers}).$$

Therefore $h_0 - h_0' = \sum r_\alpha t_\alpha h_\alpha$, and we have

$$\theta(h_0) - \theta(h_0') = \theta\left(\sum r_\alpha t_\alpha h_\alpha\right) = \sum t_\alpha \theta(r_\alpha h_\alpha)$$
$$= \sum t_\alpha r_\alpha g_\alpha = \sum (n_\alpha - m_\alpha) g_\alpha.$$

Thus we have $\theta(h_0) + \sum m_\alpha g_\alpha = \theta(h_0') + \sum n_\alpha g_\alpha$, which shows that θ^* is well-defined. In virtue of the definition of θ^*, it is obvious that θ^* is a homomorphism and an extension of θ. Thus the proof is complete.

LEMMA 7. *Let H and H_0 be the same groups as in Lemma 6. If, for any element $h \in H$ and for any interger m such that $mh \in H_0$, the element mh is divisible by m in H_0, then H_0 is a direct summand of H.*

PROOF. By Lemma 6, the identity $H_1 \to H_0$ is extendable to a homomorphism $\psi : H \to H_0$. Therefore we have $H = H_1 +$ kernel of ψ, which proves the lemma.

In the Lemma 7, the assumptions for H/H_0 can be removed. Namely we have the following

LEMMA 8. *Let H be any abelian group and H_0 be any subgroup of H. If, for any element $h \in H$ and for any integer m such that $mh \in H_0$, the element mh is divisible by m in H_0, then H_0 is a direct summand of H.*

PROOF. Let $F = \{f_\alpha\}$ be a set of generators of H, and we consider the family $\mathfrak{S}$ of all finite subsets of F. For $S \in \mathfrak{S}$, let H_s denote the subgroup of H generated by H_0 and elements of S. Then H_s/H_0 is finitely generated. Therefore all assumptions in Lemma 7 is satisfied for H_s and H_0. Hence there exists a subgroup $V_s \subset H$ such that $H_s \cap V_s = 0$ and $H_s = H_0 + V_s$. Let V be the smallest subgroup which contains V_s for all $S \in \mathfrak{S}$. Then it is easily verified that $V \cap H_0 = 0$ and $H = H_0 + V$. Therefore H_0 is a direct summand of H.

LEMMA 9. *Let π be an abelian group and π^* be a free abelian group with generators $\sigma(\alpha)$ corresponding to each element α of π. Then the kernel of the homomorphism $\theta: \pi^* \to \pi$ defined by $\theta(\sigma(\alpha)) = \alpha$ is generated by elements of the form $\sigma(\alpha + \beta) - \sigma(\alpha) - \sigma(\beta)$ for $\alpha, \beta \in \pi$.*

PROOF. Let $\Gamma \subset \pi^*$ be a subgroup generated by elements $\sigma(\alpha + \beta) - \sigma(\alpha) - \sigma(\beta)$ $(\alpha, \beta \in \pi)$. It is obvious that $\Gamma \subset$ kernel of θ, and it remains to show that $\Gamma \supset$ kernel of θ.

To show it, we shall first prove that $\sum_{i=1}^{n} \sigma(\alpha_i) - \sigma\left(\sum_{i=1}^{n} \alpha_i\right) \in \Gamma$, $(\alpha_i \in \pi)$. For $n = 1$ and $n = 2$, this fact is true. We assume that our assertion is true for $n - 1$. If we put

$$\tau = \sigma\left(\sum_{i=1}^{n} \alpha_i\right) - \sigma\left(\sum_{i=1}^{n-1} \alpha_i\right) - \sigma(\alpha_n),$$

then $\tau \in \Gamma$ and

$$\sum_{i=1}^{n} \sigma(\alpha_i) - \sigma\left(\sum_{i=1}^{n} \alpha_i\right) = \sum_{i=1}^{n-1} \sigma(\alpha_i) - \sigma\left(\sum_{i=1}^{n-1} \alpha_i\right) - \tau.$$

Therefore, by the assumption that $\sum_{i=1}^{n-1} \sigma(\alpha_i) - \sigma\left(\sum_{i=1}^{n-1} \alpha_i\right) \in \Gamma$, we know that $\sum_{i=1}^{n} \sigma(\alpha_i) - \sigma\left(\sum_{i=1}^{n} \alpha_i\right) \in \Gamma$.

Now, if $a = \sum_{i=1}^{n} r_i\ \sigma(\alpha_i)$ (r_i : integers) belongs to the kernel of θ, then $\sum_{i=1}^{n} r_i\alpha_i = 0$. By the above fact, $r_i\sigma(\alpha_i) - \sigma(r_i\alpha_i) \in \Gamma$ for each i, hence $a - \sum_{i=1}^{n} \sigma(r_i\alpha_i) \in \Gamma$. On the other hand, it is easily seen that

$$\Gamma \ni \sum_{i=1}^{n} \sigma(r_i\alpha_i) - \sigma\left(\sum_{i=1}^{n} r_i\alpha_i\right) = \sum_{i=1}^{n} \sigma(r_i\alpha_i) - \sigma(0)$$

$$= \sum_{i=1}^{u} \sigma(r_i\alpha_i) - [\sigma(0) - \sigma(0) + \sigma(0)],$$

hence $\sum_{i=1}^{n} \sigma(r_i\alpha_i) \in \Gamma$. Therefore $a \in \Gamma$.

4. The realization of $T : \pi_p \otimes \pi_q \to \pi_{p+q-1}$ for $1 < p < q$. In this section we prove the following

THEOREM 1. *Let π_1 be any multiplicative group and π_n ($n \geqq 2$) be any π_1-modules. For fixed integers p, q with $1 < p < q$, let $T: \pi_p \otimes \pi_q \to \pi_{p+q-1}$ be an arbitrary given π_1-homomorphism. Then the system $\pi_1, \pi_2, \ldots\ldots, \pi_n, \ldots$ $\ldots, T$ is realizable. Namely, there exist a space (B, b_0) and isomorphisms $h_n : \pi_n(B, b_s) \approx \pi_n (n \geqq 1)$ with the following properties:*

(1) *$h_n(w\cdot\alpha) = h_1(w)\cdot h_n(\alpha)$ for $\alpha \in \pi_n(B, b_0)$ ($n \geqq 2$), $w \in \pi_1(B, b_s)$,*

(2) *$h_{p+q-1} \circ W_{p,q}(B) = T \circ (h_p \otimes h_q)$,*

(3) *if one of integers p', $q' > 1$ is different from p and q, then $W_{p',q'}(B) = 0$.*

PROOF. We put $P_p = P(\pi_p, p)$, $P_q = P(\pi_q, q)$ and let $P = P_p \vee P_q$ be a space obtained from the union $P_p \cup P_q$ by identifying 0-cells of P_p and P_q to a point p_0. Since 0-cells of P_p and P_q are fixed points, P is naturally a π_1-space. By a theorem of Whitehead-Chang ([11])

$$\pi_i(P) = \begin{cases} 0 & \text{if } i < p \text{ or } p < i < q \text{ or } q < i < p+q-1 \\ \pi_p & \text{if } i = p \\ \pi_q & \text{if } i = q \\ \pi_p \otimes \pi_q & \text{if } i = p+q-1, \end{cases}$$

where $\pi_i(P_i)$ ($= \pi_i$) for $i = p$ or q is embedded in $\pi_i(P)$ by the inclusion map $P_i \subset P$, and $\pi_p \otimes \pi_q = \pi_p(P_p) \otimes \pi_q(P_q)$ is embedded in $\pi_{p+q-1}(P)$ by the Whitehead product. Also these embedding isomorphisms commute with operations of π_1.

Let (S^{p+q-1}, s_0) be a pair of a $(p+q-1)$-sphere S^{p+q-1} and a point s_0 on it. For each pair (w, α) with $w \in \pi_1$, $\alpha \in \pi_{p+q-1}$, let $(S^{p+q-1}_{(w,\alpha)}, s_{(w,\alpha)})$ be disjoint copies of (S^{p+q-1}, s_0) and we attach these spheres to P by identifying $s_{(w,\alpha)}$ with P_0. The space thus obtained will be denoted by Q. As in the proof of Lemma 1, this space Q may be regarded as a π_1-space. Obviously Q has the same homotopy groups as P in dimensions $< p+q-1$, and $\pi_{p+q-1}(Q) = \pi_p \otimes \pi_q + \pi^*_{p+q-1}$, where π^*_{p+q-1} is the free abelian group generated by the elements $\iota_{(w,\alpha)}$ represented by $S^{p+q-1}_{(w,\alpha)}$. And operations of π_1 on π^*_{p+q-1} is such as $v\cdot(\iota_{(w,\alpha)}) = \iota_{(vw,\alpha)}$.

We define a π_1-homomorphism λ from $\pi_{p+q-1}(Q)$ onto π_{p+q-1} by

$$\begin{cases} \lambda \mid \pi_p \otimes \pi_q = T, & \\ \lambda(\iota_{(w,\alpha)}) = w\cdot\alpha & \text{for a generator } \iota_{(w,\alpha)} \text{ of } \pi^*_{p+q-1}. \end{cases}$$

We apply Lemma 1 with $G = \pi_{p+q-1}(Q)$, $\rho = \lambda$, then there exists a π_1-space Q^* such that Q^* contains Q as an invariant subspace, and

$$i^*_r : \pi_r(Q) \approx \pi_r(Q^*) \text{ for } r < p+q-1,$$

and the kernel of i^*_{p+q-1} = the kernel of λ. Hence there exists a π_1-isomomorphism $h_{p+q-1} : \pi_{p+q-1}(Q^*) \approx \pi_{p+q-1}$ such that $h_{p+q-1} \circ i^*_{p+q-1} = \lambda$. Since $\pi_p \otimes \pi_q$ is embedded in $\pi_{p+q-1}(Q)$ by the Whitehead product which is natural, we have the following commutative diagram:

$$\begin{array}{ccc} \pi_p(Q^*) \otimes \pi_q(Q^*) & \xrightarrow{\;W\;} & \pi_{p+q-1}(Q^*) \\ \downarrow{\scriptstyle h_p \otimes h_q} & & \downarrow{\scriptstyle h_{p+q-1}} \\ \pi_p \otimes \pi_q & \xrightarrow{\;T\;} & \pi_{p+q-1} \end{array}$$

where $h_i : \pi_i(Q^*) \approx \pi_i$ for $i = p$ or q are the inverses of isomorphisms induced by the inclusion maps $P_i \subset Q^*$, and W denotes the Whitehead product in Q^*. Therefore the space Q^* is a π_1-space which realizes π_1-modules π_p, π_q, π_{p+q-1} and π_1-homomorphism T. Hence, by Lemma 2, there exists a π_1-space Y_0 which realizes π_p, π_q, π_{p+q-1}, T and $\pi_i(Y_0) = 0$ for $i \neq p, q, p+q-1$.

From now we proceed in the same way as § 6 of [13]. We construct the product space

$$Y = Y_0 \times (\Pi P_i),$$

where $P_i = P(\pi_i, i)$ and in the product ΠP_i of P_i indices i run over integers $i > 1$, $i \neq p, q, p+q-1$. This space Y is naturally a π_1-space and $\pi_i(Y)$ are π_i-isomorphic to π_i for $i > 1$.

Let π^0_1 be the subgroup of π_1 consisting of all elements which operate on

Y as the identity. Let $\chi: \pi_1 \to \pi_1/\pi_1^0 = G$ be the projection. Let $\mathfrak{B} = (B, P, X, Y, G)$ be a fibre bundle such that the base space X is $P(\pi_1, 1)$ and the fibre is Y and the structure group is G with the discrete topology, and the characteristic map is $\chi: \pi_1(X) = \pi_1 \to G$. Such a bundle certainly exists by Theorem in § 13. 8 of [17]. It is easily seen by Proposition 1 that the total space B is the required one (cf. [13]).

We next give the following

THEOREM 2. *Let π_1 be a multiplicative group and $\pi_n (n \geqq 2)$ be π_1-modules. Let π_1-homomorphisms $T_{p,q}: \pi_p \otimes \pi_q \to \pi_{p+q-1}$ with $p, q > 1$, $p + q - 1 \leqq r$ and $T: \pi_{p_0} \otimes \pi_{q_0} \to \pi_{p_0+q_0-1}$ with $1 < p_0 < q_0$ be given. If $\pi_1, \pi_2, \ldots \ldots, \pi_r$ and $T_{p,q}$ for $p + q - 1 \leqq r$ are realizable, then $\pi_1, \pi_2, \ldots\ldots$, and $T_{p,q}$ for $p + q - 1 < \min (q_0, r + 1)$, $p + q - 1 \neq p_0$ and T are simultaneously realizable in a space B such that all Whitehead products vanish except W_{p_0,q_0}, W_{q_0,p_0} and $W_{p,q}$ for $p + q - 1 < \min (q_0, r + 1)$, $p + q - 1 \neq p_0$.*

PROOF. Case (i): $r < p_0$. By the assumption there exists an arcwise connected space A which realizes $\pi_1, \pi_2, \ldots\ldots, \pi_r$ and $T_{p,q}$ for $p + q - 1 < r$ $(p, q > 1)$. By Lemma 2 we can assume that $\pi_i(A) = 0$ for $i > r$. On the other hand, by Theorem 1, there exists a simply connected π_1-space C which realizes $\pi_{p_0}, \pi_{q_0}, \pi_{p_0+q_0-1}$ and T such that $\pi_i(C) = 0$ for $1 < i \neq p_0, q_0, p_0 + q_0 - 1$.

We construct the product space

$$Y = \widetilde{A} \times C \times (\Pi P_i),$$

where $\widetilde{A}$ is the universal covering space of A and in the product ΠP_i the index i runs over integers $i > r$ except $p_0, q_0, p_0 + q_0 - 1$. Then Y is the simply connected π_1-space which realizes $\pi_2, \pi_3, \ldots\ldots$, and T, $T_{p,q}$ for $p + q - 1 \leqq r$, and the other Whitehead products vanish. Thus by the same process as the last step in the proof of Theorem 1, we obtain a required space.

Case (ii): $p_0 \leqq r$. Let A be a space which realizes $\pi_1, \ldots\ldots, \pi_r$ and $T_{p,q}$ for $p + q - 1 \leqq r$. We apply Lemma 3 with $X = A$, $n = p_0$, then we know that there exists a π_1-space A_1 having a fixed point such that $\pi_i(A_1) = 0$ for $i < p_0$ and A_1 realizes $\pi_{p_0}, \ldots\ldots, \pi_r$ and $T_{p,q}$ for $p + q - 1 \leqq r$ $(p, q \geqq p_0)$. Hence again by Lemma 2 there exists a π_1-space A_2 having a fixed point a_0 such that $\pi_i(A_2) = 0$ for $i < p_0$ and $i \geqq s$ and A_2 realizes π_{p_0}, π_{s-1}, $T_{p,q}$ for $p + q - 1 < s$ $(p, q \geqq p_0)$, where $s = \min (q_0, r + 1)$,.

We construct the π_1-space $A_2 \vee P(\pi_{q_0}, q_0)$, where only the fixed point $a_0 \in A_2$ and the fixed point of $P(\pi_{q_0}, q_0)$ are identified. Hence this is a π_1-space. For this π_1-space we can apply the same process used in the first step in the proof of Theorem 1 and we obtain a π_1-space A_3 such that $\pi_i(A_3) = 0$

for $i < p_0$ and $i \geqq s$, $(i \neq q_0,\ p_0 + q_0 - 1)$ and A_3 realizes $\pi_{p_0}, \ldots\ldots, \pi_s, \pi_{q_0}, \pi_{p_0+q_0-1}$ and T, $T_{p,q}$ for $p + q - 1 < s$ $(p,\ q \geqq p_0)$.

Also, let B be a space which realizes $\pi_1, \ldots\ldots, \pi_{p_0-1}$, $T_{p,q}$ for $p + q - 1 < p_0$, $(p,\ q > 1)$, and $\pi_i(B) = 0$ for $i \geqq p_0$. We construct $Y = A_3 \times \widetilde{B} \times (\Pi P_i)$, where the index i in the product ΠP_i runs over integers $\geqq s$, $\neq p_0, q_0, p_0 + q_0 - 1$. Then in the same way as in (i), we have a required space. Thus the proof is complete.

By repeated applications of Theorem 2 we directly obtain some results concerning the simultaneous realization of some π_1-homomorphisms of type $T_{p,q} : \pi_p \otimes \pi_q \to \pi_{p+q-1}$ with $p \neq q$. To formulate these we need following terminologies.

Pairs of integers (p_0, q_0) and (p_1, q_1) will be called *distinct* if any two of integers $p_0, q_0, p_1, q_1, p_0 + q_0 - 1, p_1 + q_1 - 1$ are distinct. Pairs of integers (s, t_1) and (s, t_2) such that $s < t_1 < t_2$ will be called to be *separated* if $s + t_1 - 1 < t_2$.

COROLLARY 1. *Let $\pi_n (n > 1)$ be π_1-modules. Let $A = \{(p_i, q_i)\}$ be a given set of pairs of integers (p_i, q_i) such that $1 < p_i < q_i$. For any pair $(p_i, q_i) \in A$, let $T_i : \pi_{p_i} \otimes \pi_{q_i} \to \pi_{p_i+q_i-1}$ be a given π_1-homomorphism. If any two elements of A are distinct, then the system $\pi_1, \pi_2, \ldots\ldots$ and $\{T_i\}$ is realizable.*

COROLLARY 2. *Let $\pi_n (n > 1)$ be π_1-modules. Let $B = \{(s, t_j)\}$ be a given set of pairs of integers (s, t_i) such that $1 < s < t_1 < t_2 \ldots\ldots$ For any pair $(s, t_j) \in B$, let $T'_j : \pi_s \otimes \pi_{t_j} \to \pi_{s+t_j-1}$ be the given π_1-homomorphism. If any two elements of B are separated then the system $\pi_1, \pi_2, \ldots\ldots$ and $\{T'_j\}$ is realizable.*

COROLLARY 3. *Let A, B, T_i, T'_j be the same as in Corollaries* 1 *and* 2. *If any (s, t_j) and (p_i, q_i) are distinct, then the system $\pi_1, \pi_2, \ldots\ldots$ and $\{T\}$, $\{T'_j\}$ is realizable.*

5. The realization of $T_2 : \pi_2 \otimes \pi_2 \to \pi_3$. If $\alpha \in \pi_n(S^r)$ and $\beta_1, \beta_2 \in \pi_r(B)$ and if $1 < n < 3r - 3$, then by a theorem due to G. Whitehead we have

$$(\beta_1 + \beta_2) \circ \alpha = \beta_1 \circ \alpha + \beta_2 \circ \alpha + [\beta_1, \beta_2] \circ H(\alpha),$$

where $\circ$ denotes the composition operation and $H(\alpha)$ is the Hopf invariant of α (cf. § 5 of [20]).

Let $\eta \in \pi_3(S^2) + Z$[3] be the element represented by Hopf fibre map, then $H(\eta) = 1$. Thus we have

3) In the following Z denotes the group of integers, and for an integer $m>1$, Z_m denotes the cyclic group of the order m.

$$[\alpha, \beta] = (\alpha + \beta)\circ\eta - \alpha\circ\eta - \beta\circ\eta,$$

where $\alpha, \beta \in \pi_2(B)$.

This formula was also proved by H. Whitney in [26] and he showed that $\alpha\circ\eta = (-\alpha)\circ\eta$. Since $[\alpha, \beta]$ is bilinear, $\alpha\circ\eta$ satisfies the relation

$$(\alpha + \beta + \gamma)\circ\eta - (\alpha + \beta)\circ\eta - (\beta + \gamma)\circ\eta - (\gamma + \alpha)\circ\eta + \alpha\circ\eta + \beta\circ\eta + \gamma\circ\eta = 0.$$

From this relation, using $\alpha\circ\eta = (-\alpha)\circ\eta$, we have

$$(2\alpha)\circ\eta = 4(\alpha\circ\eta).$$

Therefore we have

$$[\alpha, \alpha] = 2(\alpha\circ\eta).$$

Also the correspondence $\alpha \to \alpha\circ\eta$ preserves operations of π_1.

Now we state the following theorems.

THEOREM 3. *Let $\pi_n (n > 1)$ be given π_1-modules and $T_2 : \pi_2 \otimes \pi_2 \to \pi_3$ be a given π_1-homomorphism. We assume that there exists an exact sequence of π_1-modules and π_1-homomorphisms*

$$0 \to F_0 \to F_1 \overset{\phi}{\to} \pi_2 \to 0$$

such that F_0 and F_1 are π_1-free. In order that the system $\pi_1, \pi_2, \ldots\ldots$ and T_2 is realizable[4], it is necessary and sufficient that there exists a π_1-map $\eta : \pi_2 \to \pi_3$ such that

$$\begin{cases} T_2(\alpha \otimes \beta) = \eta(\alpha + \beta) - \eta(\alpha) - \eta(\beta) \\ \eta(\alpha) = \eta(-\alpha) \qquad \text{for } \alpha, \beta \in \pi_2. \end{cases}$$

THEOREM 3′. *Let $\pi_n (n > 1)$ be given π_1-modules and we assume that π_1 operates trivially on π_2 and π_3. Let $T_2 : \pi_2 \otimes \pi_2 \to \pi_3$ be a given homomorphism. In order that the system $\pi_1, \pi_2, \ldots\ldots$ and T_2 is realizable[4], it is necessary and sufficient that there exists a map $\eta : \pi_2 \to \pi_3$ such that*

$$\begin{cases} T_2(\alpha \otimes \beta) = \eta(\alpha + \beta) - \eta(\alpha) - \eta(\beta), \\ \eta(\alpha) = \eta(-\alpha) \qquad \text{for } \alpha, \beta \in \pi_2. \end{cases}$$

PROOF OF THEOREM 3. The necessity is stated above. To prove the sufficiency, by Proposition 1 and Theorem 2, it is sufficient to show the existence of a simply connected π_1-space realizing π_2, π_3 and T_2.

Let B_0 be a π_1-basis for F_1, and we put $B = \{w\cdot b \mid w \in \pi_1, b \in B_0\}$.

4) This system is realizable in a space B such that $W_{p,q}(B) = 0$ if $p \neq 2$ or $q \neq 2$.

For each element $a \in B$, let (S_a^2, s_a) be a topological image of the pair (S^2, s_0) of a 2-sphere S^2 and its point s_0, and we assume that (S_a^2, s_a) are mutually disjoint. Let $f_a : (S^2, s_0) \to (S_a^2, s_a)$ be fixed homeomorphisms. We consider a CW-complex

$$K^2 = \bigvee_{a \in B} S_a^2$$

which is obtained from the union $\bigcup_{a \in B} S_a^2$ by identifying points s_a to a point p_0.

For any element $w \in \pi_1$, we define a map $w : K^2 \to K^2$ by

$$w \mid S_a^2 = f_{w \cdot a} \circ f_a^{-1} \qquad (a \in B),$$

then K^2 is a simply connected π_1-space on which π_1 operates freely. The group $\pi_2(K^2)$ is a free abelian group generated by elements ι_a represented by maps f_a for $a \in B$ and π_1 operates on $\pi_2(K^2)$ so that $w \cdot (\iota_a) = \iota_{w \cdot a}$ for $w \in \pi_1$. Therefore $\pi_2(K^2)$ is π_1-isomorphic to F_1 under the correspondence $\iota_a \to a$ $(a \in B)$, and these groups are identified by this π_1-isomorphism.

We assume the axiom of choice, and may therefore suppose that the elements of B are well ordered. Then, by Theorem A of [12]

$$\pi_3(K^2) = \sum_{a \in B} \pi_3(S_a^2) + \sum_{\substack{a, b \in B \\ a < b}} Z(a, b),$$

where $\pi_3(S_a^2)$ is embedded in $\pi_3(K^2)$ by the inclusion map $S_a^2 \subset K^2$ and hence is the free group generated by $\iota_a \circ \eta$, and $Z(a, b)$ is the free group with the generator $z(a, b) = [\iota_a, \iota_b]$.

Since $[\iota_a, \iota_a] = 2(\iota_a \circ \eta)$, the Whitehead product

$$W_2 : \pi_2(K^2) \otimes \pi_2(K^2) \to \pi_3(K^2)$$

is represented as follows:

$$W_2(\iota_a \otimes \iota_b) = \begin{cases} z(a, b) & a < b, \\ z(b, a) & a > b, \ (a, b \in B). \\ 2(\iota_a \circ \eta) & a = b, \end{cases}$$

We define a homomorphism $\lambda : \pi_3(K^2) \to \pi_3$ by

$$\begin{cases} \lambda(\iota_a \circ \eta) = \eta(\phi a), \\ \lambda(z(a, b)) = T_2(\phi a \otimes \phi b) \qquad \text{for } a, b \in B, \ a < b. \end{cases}$$

It is easily seen that λ is a π_1-homomorphism. We notice that since π_1-map $\eta : \pi_2 \to \pi_3$ satisfies $\eta(\alpha) = \eta(-\alpha)$ and $\eta(\alpha + \beta + \gamma) - \eta(\alpha + \beta) - \eta(\beta + \gamma) - \eta(\gamma + \alpha) + \eta(\alpha) + \eta(\beta) + \eta(\gamma) = 0$, we have $T_2(\alpha \otimes \alpha) = 2\,\eta(\alpha)$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} \pi_2(K^2)\otimes\pi_2(K^2) & \xrightarrow{W_2} & \pi_3(K^2) \\ \Big\downarrow \rho\otimes\rho & & \Big\downarrow \lambda \\ \pi_2\otimes\pi_2 & \xrightarrow{T_2} & \pi_3 \end{array}$$

where $\rho : \pi_2(K^2)\to\pi_2$ is an onto π_1-homomorphism defined by $\rho(\iota_a) = \phi(a)$.

Now we apply Lemma 1 with $X = K^2$, $G = \pi_3$, then we have a π_1-space K^3 obtained by attaching 3-cells to K^2 so that π_1 operates freely on K^3 and the kernel of i_2 coincides the kernel of ρ and i_2 is onto, where $i_2 : \pi_2(K^2)\to\pi_2(K^3)$ is the π_1-homomorphism induced by the inclusion map $K^2\subset K^3$. Hence there exists a π_1-isomorphism $h_2 : \pi_2(K^3)\approx\pi_2$ such that $h_2\circ i_2 = \rho$.

Next, we shall show that there exists a π_1-homomorphism $\lambda^* : \pi_3(K^3)\to\pi_3$ such that $\lambda^*\circ i_3 = \lambda$, where $i_3 : \pi_3(K^2)\to\pi_3(K^3)$ is the π_1-homomorphism induced by the inclusion map $K^2\subset K^3$. We set $\Gamma = i_3\,\pi_3(K^2)$ and consider the exact sequence

$$\pi_3(K^2)\xrightarrow{i_3}\pi_3(K^3)\xrightarrow{j_3}\pi_3(K^3, K^2)\xrightarrow{\partial_3}\pi_2(K^2)\xrightarrow{i_2}\pi_2(K^3)\longrightarrow 0.$$

Since $i_2^{-1}(0)\approx F_0$, by the assumption, $i_2^{-1}(0)$ is π_1-free, hence $\partial_3\pi_3(K^3, K^2)$ $(\approx i_2^{-1}(0))$ is π_1-free. Since $\pi_3(K^3, K^2)/j_3\pi_3(K^3)\approx\partial_3\pi_3(K^3, K^2)$, by Lemma 5, $j_3\pi_3(K^3)$ is a direct summand of the π_1-module $\pi_3(K^3, K^2)$. On the other hand, since π_1 operates freely on K^3, $\pi_3(K^3, K^2)$ is π_1-free. It is obvious that a direct summand of a π_1-free module is π_1-free. Therefore $j_3(\pi_3(K^3))$ is π_1-free. Since $\pi_3(K^3)/i_3\pi_3(K^2)\approx j_3\pi_3(K^3)$, again by Lemma 5, $i_3\pi_3(K^2)$ is a direct summand of $\pi_3(K^3)$.

Therefore to show the existence of λ^*, it is sufficient to show the existence of $\lambda' : \Gamma\to\pi_3$ such that $\lambda'\circ i_3 = \lambda$. To show this we prove λ (kernel of i_3) $= 0$. To this end we consider a CW-complex $K_0^2 = \bigvee_{\alpha\in\pi_2} S_\alpha^2$. Then $\pi_2(K^2)$ is the free group generated by ι_α for $\alpha\in\pi_2$. Therefore, by Lemma 9, the kernel of the homomorphism $\rho_0 : \pi_2(K_0^2)\to\pi_2$ defined by $\rho_0(\iota_\alpha) = \alpha$ is generated by the elements of the form $\iota_\alpha - \iota_\beta + \iota_\gamma$ for $\alpha, \beta, \gamma\in\pi_2$, $\alpha-\beta+\gamma = 0$.

For each $\iota_\alpha - \iota_\beta + \iota_\gamma$ $(\alpha-\beta+\gamma = 0)$, we attach a 3-cell E^3 to K_0^2 by a map $E^3\to K_0^2$ which represents the element $\iota_\alpha - \iota_\beta + \iota_\gamma$. Then we have a CW-complex K_0^3 such that $\pi_2(K_0^3) = \pi_2$. Define a map $g_0 : K^2\to K_0^2$ by $g_0\,|\,S_a^2 = f_{\phi(a)}\circ f_a^{-1}(a\in B)$. Then g_0 can be extended to a map $g : K^3\to K_0^3$. And we shall consider the following diagram:

$$\begin{array}{ccccc} & & \pi_3(K^2) & \xrightarrow{\;i_3\;} & \pi_3(K^3) \\ & \overset{\lambda}{\swarrow} & \Big\downarrow g_* & & \Big\downarrow g_* \\ \pi_3 & \underset{\lambda_0}{\nwarrow} & \pi_3(K_0^2) & \xrightarrow{\;i_{3,0}\;} & \pi_3(K_0^3) \end{array}$$

where λ_0 is defined in the same way as λ, and g_* and $i_{3,0}$ are the homomorphisms induced by g and the inclusion map $K_0^2 \subset K_0^3$ respectively. It is easily verified that commutativities hold in this diagram. Thus, to show that $\lambda(i_3^{-1}(0)) = 0$, it is sufficient to show that $\lambda_0(i_{3,0}^{-1}(0)) = 0$.

By Theorem 4 of [24] the kernel of $i_{3,0}$ is generated by elements of the forms $f\circ g$ and $[f, h]$ where $f \in \pi_2(K^2)$ is an element represented by the characteristic map of an attaching 3-cell of K^3 and $g \in \pi_3(S^2)$, $h \in \pi_3(K_0^2)$. Therefore it is sufficient to show that $\lambda_0((\iota_\alpha - \iota_\beta + \iota_\gamma)\circ\eta) = 0$ and $\lambda_0([\iota_\alpha - \iota_\beta + \iota_\gamma, \iota_\delta]) = 0$ for $\alpha, \beta, \gamma, \delta \in \pi_2$, $\alpha - \beta + \gamma = 0$. Since $(\iota_\alpha - \iota_\beta + \iota_\gamma)\circ\eta = \iota_\alpha\circ\eta + \iota_\beta\circ\eta + \iota_\gamma\circ\eta - [\iota_\alpha, \iota_\beta] + [\iota_\alpha, \iota_\gamma] - [\iota_\beta, \iota_\gamma]$, this is verified by straightforward computations using corresponding properties of $\eta: \pi_2 \to \pi_3$ and the definition of λ_0.

Thus we have a π_1-homomorphism $\lambda^* : \pi_3(K^3) \to \pi_3$ such that $\lambda^*\circ i_3 = \lambda$.

Again we construct a CW-complex

$$L = K^3 \bigvee_{\xi\in\pi_3} S_\xi^3,$$

where S_ξ^3 is a copy of 3-sphere corresponding to each $\xi \in \pi_3$ and only one point of S_ξ^3 is attached to the fixed point p_0 of K^3. Hence L is naturally a π_1-space and $\pi_2(L) = \pi_2(K^3)$, $\pi_3(L) = \pi_3(K^3) + \sum_{\xi\in\pi_3} Z(\xi)$, where $Z(\xi)$ is the infinite cyclic group with the generators $z(\xi)$ represented by S_ξ^3. The operations of π_1 on $\sum_{\xi\in\pi_3} Z(\xi) \subset \pi_3(L)$ is such as $w: z(\xi) \to z(w\cdot\xi)$. Hence we can define an onto π_1-homomorphism $\mu: \pi_3(L) \to \pi_3$ by

$$\begin{cases} \mu\,|\,\pi_3(K^3) = \lambda^* \\ \mu(z(\xi)) = \xi \qquad \text{for } \xi \in \pi_3. \end{cases}$$

We apply Lemma 1 with $X = L$, $G = \pi_3$, $\rho = \mu$ and we obtain a simply connected π_1-space L^* which contains L as an invariant subspace and has following properties:

$$i_2^*: \pi_2(L) \approx \pi_2(L^*), \qquad i_3^*: \pi_3(L) \to \pi_3(L^*)$$

is onto, and the kernel of i_3^* = the kernel of μ. Therefore there exists a π_1-isomorphism $h_3: \pi_3(L^*) \approx \pi_3$ such that $h_3\circ i_3^* = \mu$. By the naturality of Whitehead product, this simply connected π_1-space L^* realizes π_2, π_3, T_2. Hence the theorem is proved.

We note that the proof of Theorem 3$'$ is essentially contained in that of Theorem 3.

Combining Theorem 3 with Theorem 2 we have the following

COROLLARY 4. *Let* $\pi_n (n \leqq 2)$ *be* π_1*-modules. Let* $A = \{(p_i, q_i)\}$, $B = \{(s, t_j)\}$ *and* $C = \{(2, r_k)\}$ *be given sets of pairs of integers such that* $2 < p_i < q_i$, $2 < s < t_1 < t_2 < \ldots\ldots$, *and* $r_1 = 2$, $4 < r_2 < r_3 < \ldots\ldots$ *Let* T_i : $\pi_{p_i} \otimes \pi_{q_i} \to \pi_{p_i+q_i-1}$, $T'_j : \pi_s \otimes \pi_{t_j} \to \pi_{s+t_j-1}$, $T''_k : \pi_2 \otimes \pi_{r_k} \to \pi_{r_k+1}$ *be* π_1*-homomorphisms. If any two elements of* A *are distinct and any two elements of* B *(and* C*) are separated, and any element of* A *and any element of* B *or* C *are distinct, and if* T''_1 *satisfies the condition of Theorem* 3, *then the system* π_1, $\pi_2, \ldots\ldots$ *and* $\{T_i\}$, $\{T'_j\}$, $\{T''_k\}$ *is realizable.*

6. The realization of $T_4 : \pi_4 \otimes \pi_4 \to \pi_7$. First we formulate necessary conditions for T_4. For $\alpha, \beta \in \pi_4(B)$ and $\xi \in \pi_7(S^4)$,

$$(\alpha + \beta)\circ\xi = \alpha\circ\xi + \beta\circ\xi + [\alpha, \beta]\circ H(\xi)$$

holds. By J. P. Serre [18] and H. Toda [19], $\pi_7(S^4) \approx Z + Z_{12}$ and its generators are ν and a, where ν is the element represented by the so-called Hopf fibre map $S^7 \to S^4$ and a is the suspension $E(a_3)$ of the generator $a_3 \in \pi_6(S^3)$ which is defined by Blakers and Massey. Also it is shown that $[\iota, \iota] = 2\nu - a$ for the element $\iota \in \pi_4(S^4)$ represented by the identity map $S^4 \to S^4$. Since $H(\nu) = 1$, we have

(i) $\qquad [\alpha, \beta] = (\alpha + \beta)\circ\nu - \alpha\circ\nu - \beta\circ\nu \qquad$ for $\alpha, \beta \in \pi_4(B)$.

From $[\iota, \iota] = 2\nu - a$, by the naturality of Whitehead products and the definition of the composition operation, we have

(ii) $\qquad [\alpha, \alpha] = 2(\alpha\circ\nu) - \alpha\circ a \qquad$ for $\alpha \in \pi_4(B)$.

The bilinearity of $[\alpha, \beta]$ is equivalent to $(\alpha + \beta + \gamma)\circ\nu - (\alpha + \beta)\circ\nu - (\beta + \gamma)\circ\nu - (\gamma + \alpha)\circ\nu + \alpha\circ\nu + \beta\circ\nu + \gamma\circ\nu = 0$ and this relation, using (ii), implies

(iii) $\qquad (-\alpha)\circ\nu = \alpha\circ\nu - \alpha\circ a \qquad$ and

(iv) $\qquad (2\alpha)\circ\nu = 4(\alpha\circ\nu) - \alpha\circ a.$

The correspondence $\alpha \to \alpha\circ\nu$ is not homomorphic, but the correspondence $\alpha \to \alpha\circ a$ is homomorphic, since a is a suspended element.

We shall prove following

THEOREM 4. *Let* π_4, π_7 *be given abelian groups and* $T_4 : \pi_4 \otimes \pi_4 \to \pi_7$

be a given homomorphism. In order that the system π_4, π_7 and T_4 is realizable[5]*, it is necessary and sufficient that there exist a map $\nu: \pi_4 \to \pi_7$ and a homomorphism $a: \pi_4 \to \pi_7$ satisfying the following conditions:*

(1) $T_4(\alpha \otimes \beta) = \nu(\alpha + \beta) - \nu(\alpha) - \nu(\beta)$,

(2) $\nu(-\alpha) = \nu(\alpha) - a(\alpha)$,

(3) $12\, a(\alpha) = 0, \qquad$ for $\alpha, \beta \in \pi_4$.

PROOF. We shall prove the sufficiency. We construct a space $K^4 = \bigvee_{\alpha \in \pi_4} S_\alpha^4$ in the same way as in § 5. The group $\pi_4(K^4)$ is a free abelian group and is generated by generators ι_α represented by the inclusion maps $S_\alpha^4 \subset K^4$. We define a homomorphism $\rho: \pi_4(K^4) \to \pi_4$ by $\rho(\iota_\alpha) = \alpha$. By Lemma 9 the kernel of ρ is generated by element of the form $\iota_\alpha - \iota_\beta + \iota_\gamma$ for $\alpha, \beta, \gamma \in \pi_4$, $\alpha - \beta + \gamma = 0$. For each $\iota_\alpha - \iota_\beta + \iota_\gamma$ $(\alpha - \beta + \gamma = 0)$, we attach 5-cells to K^4, by a map which represents $\iota_\alpha - \iota_\beta + \iota_\gamma$. Then we obtain a CW-complex K^5 such that i_4 is onto and the kernel of i_4 = the kernel of ρ holds, where $i_4: \pi_4(K^4) \to \pi_4(K^5)$ is the homomorphism induced by the inclusion map. Hence there exists an isomorphism $h_4: \pi_4(K^5) \approx \pi_4$ such that $h_4 \circ i_4 = \rho$. By Theorem A of [12], the group $\pi_7(K^4)$ is

$$\pi_7(K^4) = \sum_{\alpha \in \pi_4} \pi_7(S_\alpha^4) + \sum_{\substack{\alpha, \beta \in \pi_4 \\ \alpha < \beta}} Z(\alpha, \beta),$$

where $\pi^7(S_\alpha^4)$ is embedded in $\pi_7(K^4)$ by an isomorphism induced by the inclusion map. Hence its generators are $\nu_\alpha = \iota_\alpha \circ \nu$ of infinite order and $a_\alpha = \iota_\alpha \circ a$ of order 12, and $Z(\alpha, \beta)$ for $\alpha < \beta$ is the infinite cyclic group with the generator $z(\alpha, \beta) = [\iota_\alpha, \iota_\beta]$.

By (3) we can define a homomorphism $\lambda: \pi_7(K^4) \to \pi_7$ by

$$\begin{cases} \lambda(\nu_\alpha) = \nu(\alpha), & \lambda(a_\alpha) = a(\alpha), \\ \lambda(z(\alpha, \beta)) = T_4(\alpha \otimes \beta) & \text{for } \alpha < \beta. \end{cases}$$

We shall consider the following diagram:

$$(D) \qquad \begin{array}{ccc} \pi_4(K^4) \otimes \pi_4(K^4) & \xrightarrow{W_4} & \pi_7(K^4) \\ \downarrow{\scriptstyle \rho \otimes \rho} & & \downarrow{\scriptstyle \lambda} \\ \pi_4 \otimes \pi_4 & \xrightarrow{T_4} & \pi_7 \end{array}$$

5) By Theorem 2, this system is realizable in a space B such that $\pi_i(B)$ with $i<4$ or $i>7$ are arbitrary given abelian groups, and $W_{p,q}(B)=0$ for $p+q-1<4$ or $p+q-1>7$. But it seems that $\pi_i(B)$ for $4<i<7$ are not arbitrary. This situation occurs in the cases of the realizations of $T_{p,q}$ for $p \geqq 3$.

where W_4 denotes the homomorphism defined by the Whitehead product. By assumptions (1) and (2), as we remarked in the first of this section, we have $T_4(\alpha \otimes \alpha) = 2\nu(\alpha) - a(\alpha)$, hence (D) is commutative.

We consider the exact sequence

$$\pi_8(K^5, K^4) \xrightarrow{\partial_8} \pi_7(K^4) \xrightarrow{i_7} \pi_7(K^5) \xrightarrow{j_7} \pi_7(K^5, K^4)$$

and put $\Gamma = i_7(K^4)$.

Applying Theorem III of [1] with $X = K^4$, $n = 5$, $X^* = K^5$, we know that $\pi_8(K^5, K^4)$ is generated by the subgroup $\zeta[\pi_4(K^4) \otimes \pi_5(K^5, K^4)]$ and by elements of the form $\beta \circ \alpha$ for $\beta \in \pi_5(K^5, K^4)$, $\alpha \in \pi_8(E^5, \dot{E}^5)$, where ζ and $\circ$ denote the generalized Whitehead product and the composition respectively ([14]). Therefore, the kernel of $i_7 = \partial_8 \pi_8(K^5, K^4)$ is generated by elements of the forms $[\iota_\alpha - \iota_\beta + \iota_\gamma, \iota_\delta]$, $(\iota_\alpha - \iota_\beta + \iota_\gamma)\circ \nu$, $(\iota_\alpha - \iota_\beta + \iota_\gamma)\circ a$ for $\alpha, \beta, \gamma, \delta \in \pi_4$ and $\alpha - \beta + \gamma = 0$. It is obvious that $\lambda(\xi) = 0$ if ξ is an element of the first or the third type stated above. Since $\xi = (\iota_\alpha - \iota_\beta + \iota_\gamma)\circ \nu = \iota_\alpha \circ \nu + \iota_\beta \circ \nu + \iota_\gamma \circ \nu - \iota_\beta \circ a - [\iota_\alpha, \iota_\beta] + [\iota_\alpha, \iota_\gamma] - [\iota_\beta, \iota_\gamma]$ and corresponding formula for $\nu(\alpha - \beta + \gamma)$ holds, it is easily verified that $\lambda(\xi) = 0$. Thus $\lambda(\text{kernel of } i_7) = 0$. Hence there exists a homomorphism $\lambda' : \Gamma \to \pi_7$ such that $\lambda' \circ i_7 = \lambda$. If λ' has an extension $\lambda^* : \pi_7(K^5) \to \pi_7$, then the remaider of the proof is quite similar to that of Theorem 3.

Thus it remains only to prove that λ' has an extension. In fact we can prove that Γ is a direct summand of $\pi_7(K^5)$.

By Lemma 8, if the following condition (A) is satisfied, then Γ is a direct summand of $\pi_7(K^5)$:

(A) for any $\alpha \in \pi_7(K^5)$ and any integer m such that $m\alpha \in \Gamma$, there exists $\alpha_0 \in \Gamma$ such that $m\alpha = m\alpha_0$.

For such α and m we can find a finite subcomplex $K_0^5 \subset K^5$, which has the property that there exists an element $\bar{\alpha} \in \pi_7(K_0^5)$ such that $l(\bar{\alpha}) = \alpha$ and $m\bar{\alpha} \in i\pi_7(K_0^4)$, where $l: \pi_7(K_0^5) \to \pi_7(K^5)$ and $i: \pi_7(K_0^4) \to \pi_7(K_0^5)$ denote the homomorphisms induced by the inclusion maps. Hence to prove (A) it is sufficient to show (A'):

(A') for any finite subcomplex K_0^5 of K^5, and for any element $\alpha \in \pi_7(K_0^5)$ and any integer m such that $m\alpha \in i\pi_7(K_0^4)$, there exists an element $\alpha_0 \in i\pi_7(K_0^4)$ such that $m\alpha = m\alpha_0$.

Now consider any CW-complex L which has the same homotopy type as K_0^5. Let $\lambda: K_0^5 \to L$ be a homotopy equivalence and $\mu: L \to K_0^5$ be a homotopy inverse of λ, i. e. $\lambda\circ\mu \simeq 1$, $\mu\circ\lambda \simeq 1$. We may assume that λ, μ are cellular maps.

We shall consider the following commutative diagram:

$$\begin{array}{ccccc}
\pi_7(K_0^4) & \xrightarrow{\ i\ } & \pi_7(K_0^5) & \xrightarrow{\ j\ } & \pi_7(K_0^5,\ K_0^4) \\
\lambda_4 \Big\downarrow \Big\uparrow \mu_4 & & \lambda_5 \Big\downarrow \Big\uparrow \mu_5 & & \lambda_6 \Big\downarrow \Big\uparrow \mu_6 \\
\pi_7(L^4) & \xrightarrow{\ i\ } & \pi_7(L^5) & \xrightarrow{\ j\ } & \pi_7(L^5,\ L^4)
\end{array}$$

where λ_i, $\mu_i (i = 4, 5, 6)$ are homomorphisms induced by λ and μ respectively.

For $\alpha \in \pi_7(K^5)$, we put $\bar{\alpha} = \lambda_5(\alpha)$. If $m\,\alpha \in i\pi_7(K_0^4)$, then $j(m\,\bar{\alpha}) = j\circ\lambda_5 (m\,\alpha) = \lambda_6\circ j(m\,\alpha) = 0$. Hence $m\,\bar{\alpha} \in i\pi_7(L^4)$. If there exists an $\bar{\alpha}_0 \in i\,\pi_7(L^4)$ such that $m\,\bar{\alpha} = m\,\bar{\alpha}_0$, then $\alpha_0 = \mu_5\bar{\alpha}_0 \in i\,\pi_7(K^5)$, and since $\mu_5\circ\lambda_5 =$ identity, $m\,\alpha = \mu_5\circ\lambda_5(m\,\alpha) = \mu_5(m\,\bar{\alpha}) = m\,\bar{\alpha}_0$. Therefore, again by Lemma 8, to show (A') it is sufficient to prove that for some L of the same homotopy type as K_0^5, $i\,\pi_7(L^4)$ is a direct summand of $\pi_7(L^5)$. This is shown as follows.

Since K_0^5 is an A_n^2-polyhedron with $n = 3$, K_0^5 is of the same homotopy type as $L = M_1 \vee M_2 \vee \cdots\cdots \vee M_k$, where M_i are elementary complexes ([3], [10]). But

$$\pi_7(L^4) = \sum_{1\leqq r\leqq k} \pi_7(M_r^4) + \sum_{1\leqq r<s\leqq k} \pi_4(M_r^4)\otimes\pi_4(M_s^4),$$

$$\pi_7(L^5) = \sum_{1\leqq r\leqq k} \pi_7(M_r^5) + \sum_{1\leqq r<s\leqq k} \pi_4(M_5^r)\otimes\pi_4(M_5^r),$$

and $i:\ \pi_7(L^4)\to\pi_7(L^5)$ is represented under these direct sum decompositions by

$$i_r:\ \pi_7(M_r^4)\to\pi_7(M_r^5),$$

$$j_r\otimes j_s:\ \pi_4(M_r^4)\otimes\pi_4(M_s^4)\to\pi_4(M_r^5)\otimes\pi_4(M_s^5),$$

where i_r, j_r are homomorphisms induced by the inclusion maps, and obviously j_r are onto. On the other hand, since $H_i(K, G) = 0$ for $i > 5$ and any coefficient group G, $H_i(L_r, G) = 0$ for $i > 5$. Hence the types of each M_r are 1, 4 or 5 in § 3 of [10].

By Theorem 6. 2, and 6. 3 of [10], $i_r(\pi_7(M_r^4))$ is one of the direct summands of $\pi_7(M_r^5)$ for M_r^5 of types 4 or 5. Thus Γ is a direct summand of $\pi_7(K^5)$. Hence λ' has an extension $\lambda^*:\ \pi_7(K^5)\to\pi_7$. The proof of Theorem 4 is complete.

7. The realization of $T_2: \pi_2\otimes\pi_2\to\pi_3$ and $T_3: \pi_2\otimes\pi_3\to\pi_4$. First we shall formulate necessary conditions for T_2 and T_3. Let $\eta\in\pi_3(S^2)$ be the element represented by the Hopf fibre map. As stated in § 5

(i) $\quad [\alpha, \beta] = (\alpha+\beta)\circ\eta - \alpha\circ\eta - \beta\circ\eta,$

(ii) $\quad \alpha\circ\eta = (-\alpha)\circ\eta, \qquad$ for $\alpha, \beta \in \pi_2(B)$

holds. For α, β, $\gamma \in \pi_2(B)$, the Jacobi identity

(iii) $$[\gamma, [\alpha, \beta]] + [\alpha, [\beta, \gamma]] + [\beta, [\gamma, \alpha]] = 0$$

holds. G. Whitehead proved in [21] that the suspension homomorphism E: $\pi_4(S^2) \to \pi_5(S^3)$ is isomorphic, while $E[\iota, \eta] = 0$, hence $[\iota, \eta] = 0$, where ι is a generator of $\pi_2(S^2)$. Therefore, for $\alpha \in \pi_2(B)$

(iv) $$[\alpha, \alpha\circ\eta] = 0$$

holds. Further we show that the following relation holds:

(v) $$[\alpha, [\alpha, \beta]] = -[\alpha\circ\eta, \beta] \qquad \text{for } \alpha, \beta \in \pi_2(B).$$

For, if $\alpha = \beta$, then both sides are equal to zero. If $\alpha \neq \beta$, we consider the space $S_1^2 \vee S_2^2$ and let $\iota_i \in \pi_2(S_1^2 \vee S_2^2) \approx \pi_2(S_1^2) + \pi_2(S_2^2)$ be the generator of $\pi_2(S_i^2)$. From the Jacobi identity we have $2[\iota_1, [\iota_1, \iota_2]] + [\iota_2, [\iota_1, \iota_1]] = 0$, hence $2[\iota_1, [\iota_1, \iota_2]] + 2[\iota_2, \iota_1\circ\eta] = 0$. By Theorem A in [12], $\pi_4(S_1^2 \vee S_2^2) \approx \pi_4(S_1^2) + \pi_4(S_1^2) + \pi_4(S^4) + \pi_4(S^4) \approx Z_2 + Z_2 + Z + Z$ while the element $[\iota_1, [\iota_1, \iota_2]] + [\iota_2, \iota_1\circ\eta]$ obviously belongs to the free part, hence $[\iota_1, [\iota_1, \iota_2]] + [\iota_2, \iota_1\circ\eta] = 0$, which proves (v).

Therefore, in order that T_2, T_3 are realizable it is necessary that T_2, T_3 satisfy the conditions correspond to (i) — (v).

Now we can prove the following

THEOREM 5. *Let π_1-modules $\pi_2, \pi_3, \pi_4, \ldots\ldots$ and homomorphisms T_2: $\pi_2 \otimes \pi_2 \to \pi_3$ and $T_3 : \pi_2 \otimes \pi_3 \to \pi_4$ be given. We assume*[6] *that π_1 operates trivially on π_2, π_3 and π_4, and π_2 is free. In order that the system $\pi_1, \pi_2, \ldots$ … and T_2, T_3 is realizable*[7]*, it is necessary and sufficient that there exists a map $\eta: \pi_2 \to \pi_3$ such that T_2, T_3 and η satisfy the following relations:*

(1) $T_2(\alpha \otimes \beta) = \eta(\alpha + \beta) - \eta(\alpha) - \eta(\beta)$,

(2) $\eta(\alpha) = \eta(-\alpha)$,

(3) $T_3(\gamma \otimes T_2(\alpha \otimes \beta)) + T_3(\alpha \otimes T_2(\beta \otimes \gamma)) + T_3(\beta \otimes T_2(\gamma \otimes \alpha)) = 0$,

(4) $T_3(\alpha \otimes \eta(\alpha)) = 0$,

(5) $T_3(\alpha \otimes T_2(\alpha \otimes \beta)) = -T_3(\beta \otimes \eta(\alpha)) \qquad$ for $\alpha, \beta, \gamma \in \pi_2$.

PROOF. The necessity is stated above, so we shall prove the sufficiency. Let B be a basis for the free group π_2, and introduce an ordering "$<$" in this set B. For each element $\alpha \in B$, let f_α: $(S^2, s_0) \to (S_\alpha^2, s_\alpha)$ be a fixed homeomorphism. We construct a CW-complex

6) It is desired to remove these assumptions.

7) We remark that this system is realized in a space B such that $W_{p,q}(B) = 0$ except $W_{2,2}(B)$, $W_{2,3}(B)$.

$$K^2 = \bigvee_{\alpha \in B} S^2_\alpha$$

which is obtained from the disjoint union $\bigcup_{\alpha \in B} S^2_\alpha$ by identifying points s_α to a point p_0. Then K^2 is a simply connected space and $\pi_2(K^2)$ is the free group with the basis $\{\iota_\alpha | \alpha \in B\}$, where ι_α is the element represented by the map $f_\alpha: S^2_\alpha \subset K^2$. We can define an isomorphism $h_2: \pi_2(K^2) \approx \pi_2$ by $h_2(\iota_\alpha) = \alpha$. The homotopy groups $\pi_3(K^2)$ and $\pi_4(K^2)$ are computed by Theorem A of [12] as follows:

$$\pi_3(K^2) = \sum_{\alpha \in B} \pi_3(S^2_\alpha) + \sum_{\substack{\alpha, \beta \in B \\ \alpha < \beta}} Z(\alpha, \beta),$$

$$\pi_4(K^2) = \sum_{\alpha \in B} \pi_4(S^2_\alpha) + \sum_{\substack{\alpha, \beta \in B \\ \alpha < \beta}} G(\alpha, \beta) + \sum_{\substack{\alpha, \beta, \gamma \in B \\ \beta < \gamma}} Z(\alpha, \beta, \gamma),$$

where $Z(\alpha, \beta)$ $(\alpha < \beta)$ is the free group with the generator $z(\alpha, \beta) = [\iota_\alpha, \iota_\beta]$, and $\pi_3(S^2_\alpha)$ is embedded in $\pi_3(K^2)$ by the inclusion map and so it is the free group with the generator $\iota_\alpha \circ \eta$, and $G(\alpha, \beta)$ $(\alpha < \beta)$ is the cyclic group of order 2 with the generator $g(\alpha, \beta) = [\iota_\alpha, \iota_\beta] \circ \zeta$ (ζ denotes a generator of $\pi_4(S^3) \approx Z_2$), and $Z(\alpha, \beta, \gamma)$ $(\beta < \gamma)$ is the free group with the generator $z(\alpha, \beta, \gamma) = [\iota_\alpha, [\iota_\beta, \iota_\gamma]]$.

By properties (i), (ii), (iv), (v), Whitehead products in K^2

$$W_2: \pi_2(K^2) \otimes \pi_2(K^2) \to \pi_3(K^2)$$
$$W_3: \pi_2(K^2) \otimes \pi_3(K^2) \to \pi_4(K^2)$$

are represented as follows:

$$W_2(\iota_\alpha \otimes \iota_\beta) = \begin{cases} z(\alpha, \beta) & \text{if } \alpha < \beta \\ z(\beta, \alpha) & \text{if } \alpha > \beta \\ 2(\iota_\alpha \circ \eta) & \text{if } \alpha = \beta, \end{cases}$$

$$\begin{cases} W_3(\iota_\alpha \otimes (\iota_\beta \circ \eta)) = \begin{cases} -z(\beta, \alpha, \beta) & \text{if } \alpha < \beta \\ -z(\beta, \beta, \alpha) & \text{if } \alpha > \beta \\ 0 & \text{if } \alpha = \beta \end{cases} \\ W_3(\iota_\alpha \otimes z(\beta, \gamma)) = z(\alpha, \beta, \gamma) \qquad \text{for } \beta < \gamma. \end{cases}$$

We can define a homomorphism

$$\lambda_3: \pi_3(K^2) \to \pi_3$$

by

$$\begin{cases} \lambda_3(\iota_\alpha \circ \eta) = \eta(\alpha), \\ \lambda_3(z(\alpha, \beta)) = T_2(\alpha \otimes \beta) \qquad \text{for } \alpha < \beta. \end{cases}$$

Then, we have the following commutative diagram:

$$(D_1)\qquad \begin{array}{ccc} \pi_2(K^2)\otimes\pi_2(K^2) & \xrightarrow{W_2} & \pi_3(K^2) \\ \big\downarrow h_2\otimes h_2 & & \big\downarrow \lambda_3 \\ \pi_2\otimes\pi_2 & \xrightarrow{T_2} & \pi_3 \end{array} .$$

Next we construct $K^3 = K_2 \vee_{\xi\in\pi_3} S^3_\xi$. We know by Whitehead-Chang's theorem that

$$\pi_2(K^3) = \pi_2(K^2) \approx \pi_2$$
$$\pi_3(K^3) = \pi_3(K^2) + \pi_3(\vee_{\xi\in\pi_3} S^3_\xi) = \pi_3(K^2) + \pi_3^*$$
$$\pi_4(K^3) = \pi_4(K^2) + \sum_{\xi\in\pi_3}\pi_4(S^3_\xi) + \pi_2(K^3)\otimes\pi_3(\vee_{\xi\in\pi_3} S^3_\xi)$$
$$= \pi_4(K^2) + \sum_{\xi\in\pi_3}\pi_4(S^3_\xi) + \pi_2\otimes\pi_3^*,$$

where π_3^* is the free group with generators $\sigma(\xi)$ represented by S^3_ξ, and $\pi_i(K^2)$ with $i = 2, 3, 4$ are embedded in $\pi_i(K^3)$ by the isomorphisms induced by the inclusion maps and $\pi_2(K^3)\otimes\pi_3(\vee S^3_\xi)$ is embedded in $\pi_4(K^3)$ by the Whitehead product.

We define a homomorphism $\lambda_3^*: \pi_3(K^3)\to\pi_3$ by

$$\lambda_3^* = \begin{cases} \lambda_3 & \text{on } \pi_3(K^2) \\ p & \text{on } \pi_3^* \end{cases}$$

where $p: \pi_3^*\to\pi_3$ is given by $p(\sigma(\xi)) = \xi$. Then, from (D_1) we have the following commutative diagram:

$$(D_2)\qquad \begin{array}{ccc} \pi_2(K^3)\otimes\pi_2(K^3) & \xrightarrow{W_2} & \pi_3(K^3) \\ \big\downarrow h_2\otimes h_2 & & \big\downarrow \lambda_3^* \\ \pi_2\otimes\pi_2 & \xrightarrow{T_2} & \pi_3 \end{array}$$

where W_2 denotes the Whitehead product in K^3. The Whitehead product

$$W_3^*: \pi_2(K^3)\otimes\pi_3(K^3)\to\pi_4(K^3)$$

is represented in following way under the above direct sum decompositions:

$$W_3^* = \begin{cases} W_3 & \text{on } \pi_2\otimes\pi_3(K^2) \\ \text{identity} & \text{on } \pi_2\otimes\pi_3^*. \end{cases}$$

We define a homomorphism

$$\lambda_4 : \pi_4(K^2) \to \pi_4$$

by

$$\lambda_4 = \begin{cases} T_3(\alpha, T_2(\beta, \gamma)) & \text{on } z(\alpha, \beta, \gamma) \qquad (\beta < \gamma), \\ 0 & \text{on } \sum_{\alpha \in B} \pi_4(S_\alpha^2) + \sum_{\substack{\alpha, \beta \in B \\ \alpha < \beta}} G(\alpha, \beta). \end{cases}$$

Then, using the Jacobi identities it is easily verified that in the diagram

$$(D_3) \qquad \begin{array}{ccc} \pi_2(K^2) \otimes \pi_3(K^2) & \xrightarrow{W_3} & \pi_4(K^2) \\ \downarrow h_2 \otimes \lambda_3 & & \downarrow \lambda_4 \\ \pi_2 \otimes \pi_3 & \xrightarrow{T_3} & \pi_4 \end{array}$$

the commutativity $\lambda_4 \circ W_3 = T_3 \circ (h_2 \otimes \lambda_3)$ holds good. We also define a homomorphism

$$\lambda_4^* : \pi_4(K^3) \to \pi_4$$

by

$$\begin{cases} \lambda_4^* = \begin{cases} \lambda_4 & \text{on } \pi_4(K^2), \\ 0 & \text{on } \pi_4(S_\xi^3) \qquad (\xi \in \pi_3), \end{cases} \\ \lambda_4^*(z(\alpha, \beta, \gamma)) = T_3(\alpha \otimes T_2(\beta \otimes \gamma)) \ (\beta < \gamma, \ \alpha, \beta, \gamma \in B). \end{cases}$$

Then, from (D_3) we have the following commutative diagram:

$$(D_4) \qquad \begin{array}{ccc} \pi_2(K^3) \otimes \pi_3(K^3) & \xrightarrow{W_3^*} & \pi_4(K^3) \\ \downarrow h_2 \otimes \lambda_3^* & & \downarrow \lambda_4^* \\ \pi_2 \otimes \pi_3 & \xrightarrow{T_3} & \pi_4. \end{array}$$

Now we apply Lemma 1 with $X = K^3$, $\rho = \lambda_3^*$, then we obtain $K^4 \supset K^3$ such that the kernel of λ_3^* = the kernel of i_3 and i_3 is onto, where $i_3 : \pi_3(K^3) \to \pi_3(K^4)$ is the homomorphism induced by the identity map. Therefore there exists an isomorphism $h_3 : \pi_3(K^4) \approx \pi_3$ such that $h_3 \circ i_3 = \lambda_3^*$. By the naturality of Whitehead products and the commutativity of the diagram (D_2), we have the following commutative diagram:

$$\begin{array}{ccc} \pi_2(K^4) \otimes \pi_2(K^4) & \xrightarrow{W_2} & \pi_3(K^4) \\ \Big\downarrow h_2 \otimes h_2 & & \Big\downarrow h_3 \\ \pi_2 \otimes \pi_2 & \xrightarrow{T_2} & \pi_3 \end{array}$$

where W_2 denotes the Whitehead product in K^4.

We consider the exact sequence:

$$\pi_5(K^4,\ K^3) \xrightarrow{\partial_5} \pi_4(K^3) \xrightarrow{i_4} \pi_4(K^4) \xrightarrow{j_4} \pi_4(K^4,\ K^3).$$

Since $\pi_4(K^4, K^3)$ is free, $\Gamma = i_4\pi_4(K^3)$ is one of the direct summands of $\pi_4(K^4)$. Therefore, if we assume that $\lambda_4^*(\text{kernel of } i_4) = 0$, then there exists a homomorphism

$$\lambda_4^{**}: \pi_4(K^4) \rightarrow \pi_4$$

such that $\lambda_4^{**} \circ i_4 = \lambda_4^*$. Therefore, from the commutative diagram (D_4) we have the following commutative diagram

$$\begin{array}{ccc} \pi_2(K^4) \otimes \pi_3(K^4) & \xrightarrow{W_3} & \pi_4(K^4) \\ \Big\downarrow h_2 \otimes h_3 & & \Big\downarrow \lambda_3^{**} \\ \pi_2 \otimes \pi_3 & \xrightarrow{T_3} & \pi_4. \end{array}$$

Therefore by the same process as in the proof of Theorem 3 we have a simply connected space which realizes π_2, π_3, π_4, T_2, T_3. Thus the proof of our theorem is complete.

Now we shall prove that $\lambda_4^*(\text{kernel of } i_4) = 0$

By Lemma 4 of [24], the kernel of i_4 is generated by subgroups $(i_3^{-1}(0)) \otimes \pi_2(K^3)$ and $(i_3^{-1}(0)) \circ \pi_4(S^3)$ of $\pi_4(K^3)$, where $\otimes$ and $\circ$ mean the Whitehead product and the composition respectively.

Since $i_3^{-1}(0) = \lambda_3^{*-1}(0)$ and $\pi_3(K^3) = \pi_3(K^2) + \pi_3^*$, any element $\tau \in i_3^{-1}(0)$ may be represented as $\tau = \rho + \sigma$ where $\rho \in \pi_3(K^2) \subset \pi_3(K^3)$ and $\sigma \in \pi_3^* \subset \pi_3(K^3)$ and $\lambda_3\rho + p\sigma = 0$. Moreover, the element ρ is represented as

$$\rho = \sum m_i(\iota_{\alpha_i} \circ \eta) + \sum n_j[\iota_{\beta_j},\ \iota_{\gamma_j}]$$

where m_i and n_j are integers, and α_i, β_j, $\gamma_j \in B$, $\beta_j < \gamma_j$.

We consider the element $\sigma' = -\sum m_i\sigma(\eta(\alpha_i)) - \sum n_j\sigma(T_2(\beta_j \otimes \gamma_j))$ and set $\tau' = \rho + \sigma'$, $\sigma_0 = \sigma - \sigma'$. Then we have

$$\tau = \tau' + \sigma_0 \text{ and } \sigma_0 \in p^{-1}(0).$$

Since $E: \pi_3(S^2) \to \pi_4(S^3)$ is onto, to show that $\lambda_4^*(\text{kernel of } i_4) = 0$, it is sufficient to show that

$$\lambda_4^*(\tau' \otimes \pi_2(K^3)) = 0, \qquad \lambda_4^*(\tau' \circ \pi_4(S^3)) = 0,$$
$$\lambda_4^*(p^{-1}(0) \otimes \pi_2(K^3)) = 0, \qquad \lambda_4^*(p^{-1}(0) \circ \pi_4(S^3)) = 0,$$

where τ' is an element of the form $\iota_\alpha \circ \eta - \iota_{\eta(\alpha)}$ or $[\iota_\beta, \iota_\gamma] - \iota_{T_2(\beta \otimes \gamma)}$ $(\beta < \gamma)$. And this is proved by straightforward computations. Thus the proof is complete.

8. The realizations of $T_6: \pi_6 \otimes \pi_6 \to \pi_{11}$, and $T_7: \pi_7 \otimes \pi_7 \to \pi_{13}$. By H. Toda [19], the following results has been obtained:

(i) $$\pi_{11}(S^6) \approx Z$$

and its generators is $[\iota_6, \iota_6]$, where ι_6 is a generators of $\pi_6(S^6)$, and $\pi_{10}(S^6) = 0$,

(ii) $$\pi_{13}(S^7) \approx Z_2$$

and its generator is $\nu_7 \circ \nu_{10}$ and $\pi_{12}(S^7) = 0$, where ν_n denotes $(n-4)$-fold suspension of the element $\nu_4 \in \pi_7(S^4)$ which represented by the Hopf map.

Therefore, for any CW-complex K such that K^{n-1} consists of only one 0-cell for $n = 6$ or 7, by Theorem 1. 3 of [10], we have

$$\pi_{2n-1}(K^{n+1}, K^n) \approx \pi_{2n-1}(E^{n+1}, S^n) \otimes \pi_{n+1}(K^{n+1}, K^n)$$
$$\approx \pi_{2n-2}(S^n) \otimes \pi_{n+1}(K^{n+1}, K^n) = 0.$$

Hence $i: \pi_{2n-1}(K^n) \to \pi_{2n-1}(K^{n+1})$ is onto. Therefore, by the same way as the proof of Theorem 3, we have the following

THEOREM 6. *Let π_6, π_{11} be π_1-modules and $T_6: \pi_6 \otimes \pi_6 \to \pi_{11}$ be an arbitrary π_1-homomorphism such that $T_6(\alpha \otimes \beta) = T_6(\beta \otimes \alpha)$*[8] *for $\alpha, \beta \in \pi_6$. Then, the system π_1, π_6, π_{11} and T_6 is realizable.*

THEOREM 7. *Let π_7, π_{13} be π_1-modules and $T_7: \pi_7 \otimes \pi_7 \to \pi_{13}$ be an arbitrary π_1-homomorphism such that $T_7(\alpha \otimes \alpha) = 0$*[8] *for $\alpha \in \pi_7$. Then the system π_1, π_7, π_{13} and T_7 is realizable.*

9. The realizations of $T_p: \pi_p \otimes \pi_p \to \pi_{2p-1}$ for $p = 3, 5, 8$. For $T_3: \pi_3 \otimes \pi_3 \to \pi_5$, since $[\iota, \iota] = 0$ $(\iota \in \pi_3(S^3))$ it is necessary that $T_3(\alpha \otimes \alpha) = 0$ for $\alpha \in \pi_3$. For the elementary complex $S^3 \cup e^4$, where e^4 attached by a map of degree 2, $i(\pi_5(S^3))$ is not a direct summand of $\pi_6(S^3 \cup e^4)$([9]). Therefore, for a 2-connected CW-complex K, $i(\pi_5(K^3))$ is not necessarily a direct summand of $\pi_5(K^4)$, but $\pi_5(K^4)/i\pi_5(K^3) \subset \pi_5(K^4, K^3) \approx \pi_4(K^4, K^3) \otimes \pi_5(E^4, S^3) \approx \pi_4(K^4, K^3) \otimes Z_2$.

8) Of course this is a necessary condition.

Therefore we can state the following

THEOREM 8. *Let π_3, π_5 be finitely generated abelian groups and T_3: $\pi_3 \otimes \pi_3 \to \pi_5$ be a homomorphism such that $T_3(\alpha \otimes \alpha) = 0$ and $T_3(\pi_3 \otimes \pi_3) \subset 2\pi_5$*[9]*. Then, the system π_3, π_5, T_3 is realizable.*

With respect to the realizability of T_6, since $\pi_9(S^5)$ is the group of order 2 generated by $[\iota, \iota]$ for a generator $\iota \in \pi_5(S^5)$ and $\pi_8(S^5) \approx Z_{24}$, we have the following

THEOREM 9. *Let π_5, π_9 be finitely generated abelian groups and T_5: $\pi_5 \otimes \pi_5 \to \pi_9$ be a homomorphism such that $2\,T_5(\alpha \otimes \alpha) = 0$ and $T_5(\pi_5 \otimes \pi_5) \subset 24\,\pi_9$*[10]*. Then, the system π_5, π_9 and T_5 is realizable.*

H. Toda proved that $\pi_{15}(S^8) \approx Z + Z_{120}$, and $\pi_{14}(S^8) \approx Z_2$ (see Appendix p. 66 of [27]). Z and Z_{120} have generators ρ and a such that $[\iota_8, \iota_8] = 2\rho - a$.
Therefore we have the following

THEOREM 10. *Let π_8, π_{15} be given finitely generated abelian groups and $T_8 : \pi_8 \otimes \pi_8 \to \pi_{15}$ be a given homomorphism. If there exist a map ρ: $\pi_8 \to \pi_{15}$ and a homomorphism $a : \pi_8 \to \pi_{15}$ such that*

(1) $T_8(\alpha \otimes \beta) = \rho(\alpha + \beta) - \rho(\alpha) - \rho(\beta)$,

(2) $\rho(-\alpha) = \rho(\alpha) - a(\alpha)$,

(3) $120\, a(\alpha) = 0$,

(4)[11] $\rho(\pi_8) \subset 2\,\pi_{15}$, $a(\pi_8) \subset 2\pi_{15}$, *for* $\alpha, \beta \in \pi_8$,

then, the system π_8, π_{15} and T_8 is realizable.

In the proofs of Theorems stated in this section, Lemma 6 is used, but proofs are similar to that of theorems of preceding sections and so we shall omit the details.

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9) For an abelian group G and an integer n, nG denotes the subgroup of G consisting of elements ng for $g \in G$. The condition $T_3(\pi_3 \otimes \pi_3) \subset 2\pi_5$ is not a necessary condition.

10) The condition $T_5(\pi_5 \otimes \pi_5) \subset 24\pi_9$ is not a necessary condition.

11) The condition (4) is not a necessary condition.

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