THE QUOTIENT OF FINITE EXPONENTIAL SUMS*

QAZI IBADUR RAHMAN

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A finite sum of the form

$$(1) \qquad \sum_{j=1}^{m} a_j(z)e^{\mu_j z}, \ \mu_k \neq \mu_j \text{ for } k \neq j, \ \mathrm{Re}\,\mu_1 \geqq \mathrm{Re}\,\mu_2 \geqq \cdots\cdots \geqq \mathrm{Re}\,\mu_m,$$

will hereafter be called a function of the *class* P if $a_1(z)$, $a_2(z)$, ..., $a_m(z)$ are polynomials in z. If $a_1(z)$, $a_2(z)$, ..., $a_m(z)$ are rational functions then we shall say that (1) belongs to the *class* R. (1) is called an *exponential polynomial* if $a_1(z)$, $a_2(z)$, ..., $a_m(z)$ are all constants.

The following theorem was proved by Lax [2].

THEOREM A. *If $A(z)$ and $B(z)$ are two exponential polynomials*

$$A(z) = \sum_{j=1}^{m} a_j e^{\mu_j z}, \quad B(z) = \sum_{j=1}^{n} b_j e^{\nu_j z},$$

and if $f(z) = A(z)/B(z)$ is an entire function then $f(z)$ too is an exponential polynomial.

The example $(e^z - 1)/ze^z$ shows that if $A(z)$ and $B(z)$ both belong to the class P and the quotient $f(z) = A(z)/B(z)$ is an entire function then $f(z)$ need not be a function of the class P. The following theorem is however true.

THEOREM 1. *If*

$$A(z) = \sum_{j=1}^{m} p_j(z)e^{\mu_j z}, \quad B(z) = \sum_{j=1}^{n} q_j(z)e^{\nu_j z}$$

(where $p_j(z)$ and $q_j(z)$ are polynomials) are two functions of the class P and the quotient $f(z) = A(z)/B(z)$ is an entire function then $f(z)$ belongs to the class R

Since the quotient of two functions of the class R can be written as the quotient of two functions of the class P we can state the following more general

THEOREM 2. *If $A(z)$ and $B(z)$ are two functions of the class R and*

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if $f(z) = A(z)/B(z)$ is an entire function then $f(z)$ too belongs to the class R.

PROOF OF THEOREM 1. Our method of proof is very much similar to the one adopted by Lax in proving Theorem A. We acknowledge our indebtedness to Prof. Lax in reproducing a number of ideas from his paper.

$A(z)$ and $B(z)$ are obviously of order one and normal type. By the method of Lax [2, p. 967] we can show that $f(z)$ too is of order one and exponential type.

We consider the indicator $h(\theta)$ of $f(z)$:

$$h(\theta) = \overline{\lim_{r\to\infty}}\ (1/r) \log|f(re^{i\theta})|.$$

The indicator of exponential type which is not identically zero has a number of simple properties [1, Chapter V] of which we shall use the following ones:

(I) $h(\theta)$ is a bounded and continuous function of θ.

(II) $h(\theta) + h(\theta + \pi) \geqq 0$.

(III) If $h_1(\theta)$, $h_2(\theta)$ and $h(\theta)$ are the indicators of $g_1(z)$, $g_2(z)$ and $g_1(z) + g_2(z)$, then

$$h(\theta) \leqq \max\ \{h_1(\theta),\ h_2(\theta)\}.$$

The indicator $h(\theta)$ of the function $f(z) = A(z)/B(z)$ can be written explicitly:

$$h(\theta) = \mathrm{Re}\ [(\mu_k - \nu_l)e^{i\theta}] \tag{2}$$

where

$$\mathrm{Re}\ [\mu_k e^{i\theta}] = \max_j \mathrm{Re}\ [\mu_j e^{i\theta}],\ \mathrm{Re}\ [\nu_l e^{i\theta}] = \max_j \mathrm{Re}\ [\nu_j e^{i\theta}]. \tag{3}$$

(2) holds obviously for those values of θ for which the maxima (3) occur for one value of j only, and by (I) it follows for all other values of θ.

Next we choose a direction θ for which the maximum in the second part of (3) is reached for one value of j only; without loss of generality we may take this direction as that of the positive real axis. This means that $l = k = 1$ and

$$\mathrm{Re}\ (\nu_j - \nu_1) < -d < 0 \qquad (j > 1). \tag{4}$$

We construct

$$f_1(z) = f(z)\ q_1(z) - p_1(z)e^{(\mu_1 - \nu_1)z}.$$

If $f_1(z) \equiv 0$, the theorem is proved.

If $f_1(z) \not\equiv 0$, call $h_1(\theta)$ the indicator of $f_1(z)$; $h_1(\theta)$ will satisfy the following two inequalities:

(a) $$h_1(0) \leqq h(0),$$

(b) $$h_1(\pi) \leqq h(\pi).$$

PROOF. $f_1(z)$ is the sum of two functions $f(z)q_1(z)$ and $-p_1(z)e^{(\mu_1-\nu_1)z}$. Both these functions have the same indicator for $\theta = 0$, namely $h(0)$. Hence by (III) we infer (A). For $\theta = \pi$ the indicator of $f(z)$ is $h(\pi)$ while the indicator of $-p_1(z)e^{(\mu_1-\nu_1)z}$ is Re $(\nu_1 - \mu_1) = -h(0)$.

By (III)

(5) $$h_1(\pi) \leqq \max\{h(\pi), \ -h(0)\}.$$

From (II) it follows that $h(\pi) \geqq -h(0)$, hence (b) follows from (5).

$f_1(z)$ can be written as the quotient of the functions of the class P, $f_1(z) = A_1(z)/B_1(z)$ where

$$B_1(z) = B(z), \ A_1(z) = A(z)q_1(z) - B(z)p_1(z)e^{(\mu_1-\nu_1)z}.$$

Since $f_1(z)$ is of the same form as $f(z)$, we can define $f_2(z)$ in a similar manner, i. e.

$$\begin{aligned} f_2(z) &= f_1(z)q_1(z) - p_2(z)q_1(z)e^{(\mu_2-\nu_1)z} \\ &= f(z)\{q_1(z)\}^2 - p_1(z)q_1(z)e^{(\mu_1-\nu_1)z} - p_2(z)q_1(z)e^{(\mu_2-\nu_1)z}. \end{aligned}$$

If $f_2(z) \equiv 0$, the theorem is proved; if not, we define $f_3(z)$, etc. We shall show that there exists N for which $f_N(z) \equiv 0$.

Assume that $f_N(z) \equiv 0$ holds for no N, i.e. that the reduction process can be carried out indefinitely. We write:

$$A(z) = \sum_{j=1}^{m} p_j(z)e^{\mu_j z} = \sum_{j=1}^{s} p_j(z)e^{\mu_j z} + \sum_{j=s+1}^{m} p_j(z)e^{\mu_j z},$$

where

(6) $$\text{Re}\,\mu_j \geqq \text{Re}\,\mu_1 - d \text{ for } j \leqq s, \ \text{Re}\,\mu_j < \text{Re}\,\mu_1 - d \text{ for } j > s.$$

Then

$$\begin{aligned} f_s(z) &= \{q_1(z)\}^{s-1}\Big[f(z)\ q_1(z) - \sum_{j=1}^{s} p_j(z)e^{(\mu_j-\nu_1)z}\Big] \\ &= \{q_1(z)\}^{s-1}\Big[A(z)\ q_1(z) - B(z)\sum_{j=1}^{s} p_j(z)e^{(\mu_j-\nu_1)z}\Big]/B(z) \\ &= \{q_1(z)\}^{s-1}\Big[q_1(z)\sum_{j=s+1}^{m} p_j(z)e^{\mu_j z} - \sum_{j=1}^{s}\sum_{k=2}^{n} q_k(z)p_j(z)e^{(\mu_j-\nu_1+\nu_k)z}\Big]/B(z) \end{aligned}$$

$$= A_s(z)/B_s(z).$$

It follows from (4) and (6) that all the exponents in $A_s(z)$ have a real part $< \operatorname{Re} \mu_1 - d$; consequently $h_s(0) < h(0) - d$.

In the same way we can find $s_1, s_2, \ldots\ldots, s_M$ with the property that

$$h_s = h_{s+s_1+\cdots+s_M} < h(0) - M \cdot d. \tag{7}$$

We choose M large enough so that

$$M \cdot d > D = h(0) + h(\pi) \geqq 0. \tag{8}$$

From (b) it follows that $h_s(\pi) \leqq h(\pi)$. Combining this with (7) and (8) it follows that

$$h_s(0) + h_s(\pi) < h(0) + h(\pi) - M \cdot d < 0. \tag{9}$$

But $h_s(\theta)$ is the indicator of $f_s(z)$, a function of exponential type, assumed to be $\not\equiv 0$. Therefore (9) contradicts (II), which means that the assumption that the reduction process can be carried out indefinitely is false.

COROLLARY. *The proof can be carried out under the weaker hypothesis that $A(z)/B(z)$ is regular in an angular region with an opening $> \pi$, since* (I), (II), *and* (III) *also hold for functions which are entire and of exponential type in an angular region.*

REFERENCES

[1] R. P. BOAS, Jr., Entire functions, New York, (1954).
[2] P. D. LAX, The quotient of exponential polynomials, Duke Math. J., 15(1948), 967–970.

NORTHWESTERN UNIVERSITY.