ON DIFFERENTIABLE MANIFOLDS WITH CERTAIN STRUCTURES WHICH ARE CLOSELY RELATED TO ALMOST CONTACT STRUCTURE I

SHIGEO SASAKI

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1. Introduction. Let M^{2n} be a differentiable manifold. If there exists a tensor field ϕ of type $(1, 1)$ over M^{2n} such that

$$\phi^a_b\phi^b_c = -\delta^a_c, \qquad (a, b, c = 1, 2, \ldots\ldots, 2n)$$

then M^{2n} is said to be a differentiable manifold with almost complex structure. (Tensor fields of the form given above may exist only for some manifolds with even dimension.) We shall call ϕ the fundamental collineation of the almost complex structure. The set of differentiable manifolds with almost complex structure is wider than the set of complex manifolds.

Every differentiable manifold with almost complex structure ϕ admits a poistive definite Riemannian metric g such that

$$g_{ab}\phi^a_c\phi^b_d = g_{cd},$$

and the manifold is said to have Hermitian structure and to be a Hermitian manifold. Making use of the metric g and a skew symmetric tensor

$$\phi_{ab} = g_{ae}\phi^e_b,$$

we can reduce the structural group of the tangent bundle of any manifold with almost complex structure to the unitary group $U(n)$. The converse is also true.

Differentiable manifolds with almost complex structure or almost Hermitian structure were investigated by C. Ehresmann [1], B. Eckmann, A. Frölicher [2] and others and were interesting topics on differential geometry and topology in these fifteen years.

On the other hand, let M^{2n+1} be a $(2n+1)$-dimensional differentiable manifold. If there exists a tensor field ϕ^i_j, contravariant and covariant vector fields ξ^i and η_j over M^{2n+1} such that

$$\xi^i\eta_i = 1, \qquad (i, j, k = 1, 2, \ldots\ldots, 2n+1) \tag{1.1}$$

$$\text{rank } |\phi^i_j| = 2n, \tag{1.2}$$

$$\phi^i_j\xi^j = 0, \tag{1.3}$$

$$\phi^i_j \eta_i = 0, \tag{1.4}$$

$$\phi^i_j \phi^j_k = -\delta^i_k + \xi^i \eta_k, \tag{1.5}$$

then we say that M^{2n+1} has (ϕ, ξ, η)-*structure*. (ϕ, ξ, η)-structure may be regarded as an analogue of almost complex structure for odd dimensional manifolds.

In the same way as almost complex manifolds we can prove that every differentiable manifold M^{2n+1} with (ϕ, ξ, η)-structure admits a positive definite Riemannian metric g such that

$$g_{ij}\xi^j = \eta_i, \tag{1.6}$$

$$g_{ij}\phi^i_h \phi^j_k = g_{hk} - \eta_h \eta_k, \tag{1.7}$$

and M^{2n+1} is said to have (ϕ, ξ, η, g)-*structure*. It is an analogue of the almost Hermitian structure in almost complex manifold.

Now, an odd dimensional differentiable manifold M^{2n+1} is said to have *contact structure* if there exists a 1-form η over M^{2n+1} such that

$$\eta \wedge (d\eta)^n \neq 0. \tag{1.8}$$

The structural group of the tangent bundle of differentiable manifold with contact structure is reducible to $U(n) \times 1$. The set of differentiable manifolds such that the structural groups of their tangent bundles reduce to $U(n) \times 1$ is wider than the set of differentiable manifolds with contact structure and any one of the set is called differentiable manifold with *almost contact structure*. Differentiable manifolds with contact or almost contact structure were investigated by W. Gray [4] and W. M. Boothby-H. C. Wang [3] rather from topological point of view.

I got the idea of (ϕ, ξ, η)-structure and (ϕ, ξ, η, g)-structure in studying manifolds with contact structure. However, after I talked some results about these structures to Y. Hatakeyama, he proved that "The structural group of any differentiable manifold M^{2n+1} with (ϕ, ξ, η)-structure is reducible to $U(n) \times 1$, so the M^{2n+1} in consideration is a manifold with almost contact structure. Conversely, if M^{2n+1} is a differentiable manifold with almost contact structure, then we can endow to M^{2n+1} a (ϕ, ξ, η)-structure". (The author's proof is given in section 5 of the present paper). Therefore, our (ϕ, ξ, η)-structure is closely related to almost contact structure.

In the present paper, we shall study some algebraic properties on differentiable manifolds with (ϕ, ξ, η)-structure and (ϕ, ξ, η, g)-structure. Differential geometric properties will be studied in later papers.

2. Linear map ϕ and $\phi + \xi\eta$. Let M^{2n+1} be a differentiable manifold with (ϕ, ξ, η)-structure and suppose tensor fields ϕ^i_j, ξ^i and η_j satisfy the relations (1.1) to (1.5). First we remark that these five relations are not independent.

(1.2) shows that there exist (at least locally) vector fieds ξ^i and η_j which satisfy (1.3) and (1.4). In the second place, putting (1.5) into the associative law

$$(\phi_j^i\phi_k^j)\phi_l^k = \phi_j^i(\phi_k^j\phi_l^k),$$

we see that

$$\xi^l\phi_l^k\eta_k = \phi_j^i\xi^j\eta_i.$$

Hence (1.4) follows from (1.3) and (1.5), and (1.3) follows from (1.4) and (1.5). In the third place, the existence of solutions of (1.3) and (1.4) shows that the rank of $|\phi_j^i|$ is smaller than $2n+1$. However, if $\bar{\xi}^i$ is another solution of (1.3), then (1.5) multiplied by $\bar{\xi}^k$ shows us that $\bar{\xi}^i$ is proportional to ξ^i. Hence the rank of $|\phi_j^i|$ is equal to $2n$. Therefore, if we only assume that the rank of $|\phi_j^i|$ is smaller than $2n+1$, then (1.2) follows from (1.3), (1.4) and (1.5).

Let P be a point of M^{2n+1} and M_P be the tangent space of M^{2n+1} at P. In M_P, the set of vectors v^i such that

$$v^i\eta_i = 0$$

spans a $2n$-dimensional vector subspace V_P of M_P. If we vary P over M^{2n+1}, the set of such vector subspaces determines a distribution in M^{2n+1}.

Now, we define a linear map

$$\phi : M_P \longrightarrow M_P$$

by $v^i \rightarrow {'v^i}$, where

$$'v^i = \phi_j^i v^j, \quad (v^i \in M_P). \tag{2.1}$$

Then we see that

$$'v^i\eta_i = (\phi_j^i v^j)\eta_i = 0.$$

Hence the map ϕ is a singular map and

$$\phi : M_P \longrightarrow V_P.$$

However, if $v^i \in V_P$, we see that

$$\phi_j^i(\phi_k^j v^k) = (-\delta_k^i + \xi^i\eta_k)v^k,$$
$$\therefore \quad ''v^i = -v^i.$$

Accordingly, the map ϕ restricted to V_P behaves just like the fundamental collineation of an almost complex structure. We shall call ϕ *the fundamental singular collineation* of the (ϕ, ξ, η)-structure.

Contrary to the map ϕ, the map

$$\phi + \xi\eta : M_P \longrightarrow M_P$$

defined by $v^i \rightarrow \bar{v}^i$, where

$$\bar{v}^i = (\phi^i_j + \xi^i\eta_j)v^j, \quad (v^i \in M_P) \tag{2.2}$$

is a non-singular transformation. For,

$$(\phi^i_j + \xi^i\eta_j)(-\phi^j_k + \xi^j\eta_k) = \delta^i_k, \tag{2.3}$$

as we can easily verify it by (1.1)$\sim$(1.5). The two matrices $\phi + \xi\eta$ and $-\phi + \xi\eta$ are inverse to each other.

In order to clarify the geometrical meaning of the map $\phi + \xi\eta$, we put

$$\begin{cases} v^i_\xi = (v^h\eta_h)\xi^i, \\ v^i_\eta = v^i - (v^h\eta_h)\xi^i. \end{cases} \tag{2.4}$$

Then

$$v^i = v^i_\xi + v^i_\eta \tag{2.5}$$

and

$$v^i_\eta\eta_i = 0. \tag{2.6}$$

We call that v^i_ξ is the *ξ-component* of the vector v^i and v^i_η is the *η-component* of the vector v^i.

Now, we can easily see that

$$v^i_\xi + \phi^i_j v^j_\eta = (\phi^i_j + \xi^i\eta_j)v^j,$$

hence the linear manp $\phi + \xi\eta$ may also be defined as $v^i \rightarrow \bar{v}^i$, where

$$\bar{v}^i = v^i_\xi + \phi^i_j v^j_\eta.$$

It has the following properties:

1) Any vector with the direction ξ^i is fixed under the map.

2) Any vector which is contained in the distribution η is transformed in the same way as by the fundamental collineation of an almost complex structure.

More generally, we see that

$$\begin{aligned} \bar{\bar{v}}^i &= (\phi^i_j + \xi^i\eta_j)(\phi^j_k + \xi^j\eta_k)v^k \\ &= (-\delta^i_k + \xi^i\eta_k + \xi^i\eta_k)v^k \end{aligned}$$

$$\bar{\bar{v}}^i = v^i_\xi - v^i_\eta. \tag{2.7}$$

Accordingly, $\bar{\bar{v}}^i$ is the difference of the ξ-component and the η-component of the original vector v^i.

3. Associated Riemannian metric g. Let M^{2n+1} be a differentiable manifold with (ϕ, ξ, η)-structure. We shall show that M^{2n+1} admits Riemannian metric which stands analogous situation to almost Hermitian metric for any differentiable manifold with almost complex structure. We begin with a lemma.

LEMMA. *Suppose ξ and η be contravariant and covariant vector field on a differentiable manifold M^{2n+1} such that*

$$\xi^i \eta_i = 1. \tag{3.1}$$

Then M^{2n+1} admits a positive definite Riemannian metric h such that

$$\eta_i = h_{ij}\xi^j. \tag{3.2}$$

N.B. (3.1) and (3.2) imply that ξ^i is a unit vector field over M^{2n+1} with respect to the metric h.

PROOF. First we take an arbitrary positive definite Riemannian metric on M^{2n+1}. Let $\{U_\alpha\}$ be an open covering of M^{2n+1} by coordinate neighborhoods U_α. In every U_α we take $2n$ unit vector fields $\xi^i_{(a)}$ with respect to the metric in consideration so that they are orthogonal to each other and contained in the distribution η. Then $2n+1$ vector fields $\xi^i_{(a)}$ and

$$\xi^i_{(\Delta)} \equiv \xi^i \tag{3.3}$$

constitute frames over U_α, where Δ is an abbreviation of $2n+1$. We put

$$h^{ij} = \sum_{a=1}^{2n} \xi^i_{(a)}\xi^j_{(a)} + \xi^i_{(\Delta)}\xi^j_{(\Delta)}, \tag{3.4}$$

then h^{ij} is a new positive definite contravariant metric tensor of U_α.

Now, assume $U_\alpha \cap U_\beta$ is not empty. We denote $2n$ vector fields over U_β constructed in the same way as $\xi^i_{(a)}$ over U_α by $\overline{\xi}^i_{(a)}$. Then, over $U_\alpha \cap U_\beta$ there exist relations

$$\overline{\xi}^i_{(a)} = \sum_{b=1}^{2n} c_{ab}\xi^i_{(b)},$$

where (c_{ab}) is an orthogonal martix, because $\overline{\xi}^i_{(a)}$'s and $\xi^i_{(a)}$'s are both orthonormal vector fields. Hence, we see that

$$\sum_{a=1}^{2n} \overline{\xi}^i_{(a)}\overline{\xi}^j_{(a)} = \sum_{a=1}^{2n} \xi^i_{(a)}\xi^j_{(a)}.$$

Accordingly, the metric tensors above over U_α and U_β coincide over $U_\alpha \cap U_\beta$. Consequently, the metric tensor defined above over every U_α of $\{U_\alpha\}$ constitutes a single metric tensor h of M^{2n+1}.

In the next place, we define $\xi^{(h)}_j$ over U_α by

$$\xi^i_{(h)} = h^{ij}\xi^{(h)}_j. \tag{3.5}$$

Then, we get

$$\sum_{k=1}^{2n+1} \xi^i_{(k)}(\delta^h_k - \xi^j_{(k)}\xi^{(h)}_j) = 0.$$

Hence, we see that

$$\xi^j_{(k)}\xi^{(h)}_j = \delta^h_k. \tag{3.6}$$

By virtue of the last relation, we can easily verify that

$$h^{ij}\xi^{(h)}_i\xi^{(k)}_j = \delta^{hk}, \tag{3.7}$$

so $\xi^{(h)}_j$'s are covariant orthonormal vector fields over U_α with respect to the metric h. Therefore, by virtue of (3.5), $\xi^i_{(h)}$'s are contravariant orthonormal vector fields over U_α.

From (3.6) we get especially

$$\xi^i_{(a)}\xi^{(\Delta)}_i = 0, \ \xi^i\xi^{(\Delta)}_i = 1.$$

However, by assumption there exist the relations

$$\xi^i_{(a)}\eta_i = 0, \ \xi^i\eta_i = 1.$$

Comparing these equations we get

$$\eta_i = \xi^{(\Delta)}_i = h_{ij}\xi^j_{(\Delta)},$$

$$\therefore \quad \eta_i = h_{ij}\xi^j.$$

Consequently, the metric h is the required Riemannian metric.

THEOREM 1. *Let M^{2n+1} be a differentiable manifold with (ϕ, ξ, η)-structure. Then there exists a positive definite Riemannian metric g such that*

$$\eta_i = g_{ij}\xi^j, \tag{3.8}$$

$$g_{ij}\phi^i_h\phi^j_k = g_{hk} - \eta_h\eta_k. \tag{3.9}$$

N. B. (3.1) and (3.8) imply that ξ is a unit vector field with respect to the metric g.

PROOF. Let h be a Riemannian metric over M^{2n+1} which has the properties stated in the last Lemma and put

$$g_{ij} = \frac{1}{2}(h_{ij} + h_{lm}\phi^l{}_i\phi^m_j + \eta_i\eta_j).$$

Then we can easily verify that

$$g_{ij}\xi^j = \eta_i,$$

$$g_{ij}\xi^i\xi^j = \eta_i\xi^i = 1.$$

In the next place we see that

$$\frac{1}{2}(h_{ij} + h_{lm}\phi^l_i\phi^m_j + \eta_i\eta_j)\phi^i_h\phi^j_k$$

$$= \frac{1}{2}\{ h_{ij}\phi^i_h\phi^j_k + h_{lm}(- \delta^l_h + \xi^l\eta_h)(- \delta^m_k + \xi^m\eta_k)\}$$

$$= \frac{1}{2}\{h_{hk} + h_{ij}\phi^i_h\phi^j_k - \eta_h\eta_k\},$$

that is

$$g_{ij}\phi^i_h\phi^j_k = g_{hk} - \eta_h\eta_k.$$

Hence, the theorem is proved.

We shall say that the metric which has the property stated in the last theorem *an associated Riemannian metric* to the given (ϕ, ξ, η)-structure. If a differentiable manifold M^{2n+1} admits tensor fields (ϕ, ξ, η, g) such that g is an associated Riemannian metric of the (ϕ, ξ, η)-structure, then we say that M^{2n+1} has (ϕ, ξ, η, g)-structure. In this case ξ^i is nothing but contravariant components of η_j, so we may denote it also (ϕ, η, g)-structure for brevity.

The following theorem gives another analogue of the Hermitian condition for almost complex manifold.

THEOREM 2. *Let M^{2n+1} be a differentiable manifold with (ϕ, ξ, η, g)-structure, then the relations*

$$g_{ij}(\phi^i_h + \xi^i\eta_h)(\phi^j_k + \xi^j\eta_k) = g_{hk}, \tag{3.10}$$

$$g_{ij}(- \phi^i_h + \xi^i\eta_h)(- \phi^j_k + \xi^j\eta_k) = g_{hk} \tag{3.11}$$

hold good.

PROOF. By virtue of (1. 1), (1. 4), (3. 8) and (3. 9) we can easily verify the first relation. As $\phi + \xi\eta$ and $- \phi + \xi\eta$ are inverse matrices, (3. 11) follows immediately from (3. 10).

The relation (3. 10) shows that the linear map

$$\bar{v}^i = (\phi^i_j + \xi^i\eta_j)v^j$$

on any tangent space of M^{2n+1} is an orthogonal transformation with respect to the Euclidean metric induced on the tangent space by g.

4. Associated 2-forms. Let M^{2n+1} be a differentiable manifold with (ϕ, ξ, η, g)-structure. We put

$$\phi_{ij} = g_{ih}\phi^h_j. \tag{4.1}$$

Then, the tensor ϕ_{ij} is skew-symmetric with respect to i and j. To prove it, we notice the associative law

$$(g_{ij}\phi^i_h\phi^j_k)\phi^k_l = g_{ij}\phi^i_h(\phi^j_k\phi^k_l).$$

Putting (3.9) and (1.5) into the last equation, we get

$$(g_{hk} - \eta_h\eta_k)\phi^k_l = g_{ij}\phi^i_h(-\delta^j_l + \xi^j\eta_l),$$

which, by virtue of (1.4) and (3.8), reduces to

$$\phi_{hl} = -\phi_{lh}. \tag{4.2}$$

Of course, the rank of the matrix (ϕ_{ij}) is $2n$. We call ϕ_{ij} *the associated skew symmetric tensor* of the (ϕ, ξ, η, g)-structure, and the exterior 2-form $\frac{1}{2}\phi_{ij}dx^i \wedge dx^j$ over M^{2n+1} *the associated 2-form* of the (ϕ, ξ, η, g)-structure.

In the next place we shall study the converse problem. Let M^{2n+1} be a differentiable manifold which admits a 2-form $\frac{1}{2}\phi_{ij}dx^i \wedge dx^j$ such that the rank of the matrix (ϕ_{ij}) is everywhere $2n$ over M^{2n+1}. We shall show, under the assumption that M^{2n+1} is simply connected, that it admits (ϕ, ξ, η, g)-structure. (We can remove the assumption of simply-connectedness by a slight modification of (ϕ, ξ, η, g)-structure so that ξ^i and η_j are not globally defined vector fields. However, we do not want to digress in such direction.)

We introduce first an arbitrary positive definite Riemannian metric h over M^{2n+1}. As M^{2n+1} is simply connected the tensor field ϕ_{ij} admits a vector field ξ^i over M^{2n+1} such that

$$\phi_{ij}\xi^j = 0. \tag{4.3}$$

Although ξ^i is determined only within scalar factor, we take ξ^i so that

$$h_{ij}\xi^i\xi^j = 1 \tag{4.4}$$

and put

$$\eta_i = h_{ij}\xi^j. \tag{4.5}$$

Now, we take the symmetric tensor $\phi_{ih}h^{hk}\phi_{kj}$ and consider its characteristic equation

$$|\phi_{ih}h^{hk}\phi_{kj} + \rho h_{ij}| = 0, \tag{4.6}$$

where ρ is an unknown variable. As h is positive definite, all characteristic roots are real. We see that 0 is a simple characteristic root and the corresponding characteristic vector is ξ^i. Moreover, all non-zero characteristic roots are positive. To see it we assume that ρ_1 is a non-zero characteristic root and X^i is a characteristic vector corresponding to ρ_1. Then,

$$(\phi_{ih}h^{hk}\phi_{kj} + \rho_1 h_{ij})X^j = 0. \tag{4.7}$$

If we contract X^i with the last equation we see that

$$(h_{ij}X^iX^j)\cdot\rho_1 = h^{hk}(\phi_{hi}X^i)(\phi_{kj}X^j),$$

so ρ_1 is positive.

We denote all different non-zero characteristic roots by $\rho_1, \rho_2, \ldots\ldots, \rho_l$, their multiplicities by $\nu_1, \nu_2, \ldots\ldots, \nu_l$ and the characteristic spaces corresponding to 0, $\rho_1, \ldots\ldots, \rho_l$ by $V_0, V_1, \ldots\ldots, V_l$, then

$$\dim V_\lambda = \nu_\lambda, \qquad \lambda = 0, 1, \ldots\ldots, l$$

where we have put $\nu_0 = 1$.

Now, we wish to change the metric h over M^{2n+1} so that the new metric g and the tensor

$$\phi_j^i = g^{ih}\phi_{hj}$$

defined by the new metric g play the roles of g and ϕ of (ϕ, ξ, η, g)-structure. To this purpose in mind we consider linear map of the tangent space M_P of M^{2n+1} at an arbitrary point P into itself defined by

$$\widetilde{X}^i = h^{ih}\phi_{hj}X^j, \quad X^i \in M_P. \tag{4.8}$$

If $X^i \in V_1$, then

$$\phi_{ih}h^{hk}\phi_{kj}X^j = -\rho_1 h_{ij}X^j$$

holds good. So, we get

$$\begin{aligned}&(\phi_{ih}h^{hk}\phi_{kj} + \rho_1 h_{ij})h^{jl}\phi_{lm}X^m \\ &= \phi_{ih}h^{hk}(-\rho_1 h_{km}X^m) + \rho_1\phi_{im}X^m \\ &= 0.\end{aligned}$$

Hence, we see that if $X^i \in V_1$, then $\widetilde{X}^i \in V_1$ too. Moreover, we get

$$\begin{aligned}\widetilde{\widetilde{X}}^i &= h^{ih}\phi_{hj}(h^{jl}\phi_{lm}X^m) \\ &= h^{ih}(-\rho_1 h_{hm}X^m).\end{aligned}$$

Therefore, we see that if $X^i \in V_1$, then

$$\widetilde{\widetilde{X}}^i = -\rho_1 X^i. \tag{4.9}$$

Analogous facts hold good also for vectors of $V_2, V_3, \ldots\ldots, V_l$ too.

Now, let $\{U_\alpha\}$ be sufficiently fine open covering of M^{2n+1}. We take, over any one of U_α, frames $e_1, \ldots\ldots, e_{2n}, e_\Delta$ so that $e_1, \ldots\ldots, e_{\nu_1}$ span V_1, $e_{\nu_1+1}, \ldots\ldots, e_{\nu_1+\nu_2}$ span V_2, $\ldots\ldots$, $e_{\nu_1+\cdots+\nu_{l-1}+1}, \ldots\ldots, e_{\nu_1+\cdots+\nu_l}$ span V_l and $e_\Delta = \xi$. We call such frame an adapted frame. Then we can easily see that the two matrices h_{ij} and ϕ_{ij} have the form

$$(4.10)\qquad h=\begin{pmatrix} h_1 & & & & 0\\ & h_2 & & & \\ & & \ddots & & \\ & & & h_l & \\ 0 & & & & 1\end{pmatrix},\quad \phi=\begin{pmatrix} \phi_1 & & & & 0\\ & \phi_2 & & & \\ & & \ddots & & \\ & & & \phi_l & \\ 0 & & & & 0\end{pmatrix},$$

where h_λ, $\phi_\lambda(\lambda = 1, 2, \ldots\ldots, l)$ are matricens of ν_λ columns and ν_λ rows. Therefore, if we refer to adapted frames, the linear map (4. 8) decomposes into

$$\widetilde{X}^{\alpha_\lambda} = h^{\alpha_\lambda\beta_\lambda}\phi_{\beta_\lambda\gamma_\lambda}X^{\gamma_\lambda},$$

$$\widetilde{X}^{\Delta} = 0,$$

where $\alpha_\lambda, \beta_\lambda, \gamma_\lambda$ run the range of values $\nu_1 + \cdots\cdots + \nu_{\lambda-1} + 1, \ldots\ldots, \nu_1 + \cdots\cdots + \nu_\lambda$, $\lambda = 1, 2, \ldots\ldots, l$, and

$$\widetilde{\widetilde{X}}^{\alpha_\lambda} = -\rho_\lambda X^{\alpha_\lambda}.$$

Now, we introduce a new Riemannian metric g over U_α such that

$$(4.11)\qquad g=\begin{pmatrix} g_1 & & & & 0\\ & g_2 & & & \\ & & \ddots & & \\ & & & g_l & \\ 0 & & & & 1\end{pmatrix}$$

with respect to the adapted frames, where we have put

$$(4.12)\qquad g_\lambda = \sqrt{\rho_\lambda}\, h_\lambda, \quad \lambda = 1, \ldots\ldots, l.$$

As ρ_λ's are scalar functions over M^{2n+1} and do not depend upon the choice of adapted frames, we see that g for each U_α defines globally a Riemannian metric over M^{2n+1}.

If we define a linear map of the tangent space M_P of M^{2n+1} at any point P of U_α by

$$(4.13)\qquad {}'X^i = g^{ih}\phi_{hj}X^j,$$

then it decomposes to

$$(4.14)\qquad \begin{cases} {}'X^{\alpha_\lambda} = g^{\alpha_\lambda\beta_\lambda}\phi_{\beta_\lambda\gamma_\lambda}X^{\gamma_\lambda}, \\ {}'X^{\Delta} = 0 \end{cases}$$

with respect to adapted frames. Hence, we can easily see that

$$(4.15)\qquad \begin{cases} {}''X^{\alpha_\lambda} = -X^{\alpha_\lambda}, \\ {}''X^{\Delta} = 0. \end{cases}$$

Consequently, the linear map (4. 13) induces an almost complex structure on V_1, $V_2, \ldots\ldots, V_l$.

$$\phi_j^i = g^{ih}\phi_{hj} \tag{4.16}$$

is a globally defined tensor field over M^{2n+1}. Denoting the components of ϕ with respect to an adapted frame by ϕ_β^α we see, by virtue of (4. 15), that

$$(\phi_\beta^\alpha)(\phi_\gamma^\beta) = \begin{pmatrix} -\delta_b^a & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.17}$$

Now, let us denote the transformation of components X^α of a vector with respect to the adapted frame on U_α and those X^t of the same vector with respect to a natural frame on U_α by

$$X^t = A_\alpha^i X^\alpha, \quad X^\alpha = B_i^\alpha X^t.$$

Of course, A_α^i and B_i^α are inverse matrices. Then, for components $\phi_j^i\phi_k^j$ and $\phi_\beta^\alpha\phi_\gamma^\beta$ of the tensor $\phi\phi$ there exists the relation

$$\phi_j^i\phi_k^j = A_\alpha^i\phi_\beta^\alpha\phi_\gamma^\beta B_k^\beta.$$

Hence we see that

$$\begin{aligned}(\phi_j^i)(\phi_k^j) &= \begin{pmatrix} A_b^a & A_\Delta^a \\ A_b^\Delta & A_\Delta^\Delta \end{pmatrix} \begin{pmatrix} -\delta_e^b & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_c^e & B_\Delta^e \\ B_c^\Delta & B_\Delta^\Delta \end{pmatrix} \\ &= \begin{pmatrix} -A_e^a B_c^e & -A_e^a B_\Delta^e \\ -A_e^\Delta B_c^e & -A_e^\Delta B_\Delta^e \end{pmatrix} \\ &= \begin{pmatrix} -\delta_c^a + A_\Delta^a B_c^\Delta & -\delta_\Delta^a + A_\Delta^a B_\Delta^\Delta \\ -\delta_\Delta^c + A_\Delta^\Delta B_c^\Delta & -\delta_\Delta^\Delta + A_\Delta^\Delta B_\Delta^\Delta \end{pmatrix}.\end{aligned}$$

Therefore, there exists a relation

$$\phi_j^i\phi_k^j = -\delta_k^i + A_\Delta^i B_j^\Delta$$

over every U_α. However, the vector which has components A_Δ^i with respect to the natural frame has $(0, \ldots\ldots, 0, 1)$ as its components with respect to the adapted frames, so it is nothing but the vector ξ^i. In the same way the vector which has components B_i^Δ with respect to the natural frame is nothing but the vector η_i. We can easily verify that

$$\eta_i = g_{ij}\xi^j \tag{4.18}$$

holds good. Hence we see that

$$\phi_j^i\phi_k^j = -\delta_k^i + \xi^i\eta_k \tag{4.19}$$

holds globally over M^{2n+1}. We can also see that

$$\phi^i_j \xi^j = 0, \tag{4.20}$$

$$\phi^i_j \eta_i = 0, \tag{4.21}$$

$$\xi^i \eta_i = 1 \tag{4.22}$$

are true, because they hold good with respect to adapted frames.

From (4.16) we see that

$$g_{ih}\phi^h_j = - g_{jh}\phi^h_i = \phi_{ij} \tag{4.23}$$

holds good. So we get

$$\begin{aligned} g_{ij}\phi^i_h\phi^j_k = \phi_{jh}\phi^j_k &= - \phi_{hj}\phi^j_k \\ &= - g_{hi}\phi^i_j\phi^j_k \\ &= - g_{hi}(- \delta^i_k + \xi^i\eta_k). \end{aligned}$$

Hence, we see, by virtue of (4.18), that

$$g_{ij}\phi^i_h\phi^j_k = g_{hk} - \eta_h\eta_k \tag{4.24}$$

is true. Accordingly, the four tensor fields ϕ^i_j, ξ^i, η_j, g_{ij} constitute a (ϕ, ξ, η, g)-structure of the given manifold M^{2n+1}.

Summarizing the above results, we get the following

THEOREM 3. *If a differentiable manifold M^{2n+1} admits (ϕ, ξ, η, g)-structure, then M^{2n+1} admits a skew symmetric tensor ϕ_{ij} (in other words, 2-form $\frac{1}{2}\phi_{ij}dx^i \wedge dx^j$) whose rank is $2n$. If M^{2n+1} is simply connected, the converse is also true.*

5. Contact structure and almost contact structure. Let M^{2n+1} be a differentiable manifold with contact structure and let

$$\eta = \eta_i dx^i \tag{5.1}$$

be the 1-form which defines the contact structure. Then we have

$$\eta \wedge (d\eta)^n \neq 0, \tag{5.2}$$

where $d\eta$ means the exterior derivative of η and the symbol $\wedge$ means the exterior multiplication. $d\eta$ can be written as

$$d\eta = \frac{1}{2}\phi_{ij}dx^i \wedge dx^j, \tag{5.3}$$

where we have put

$$\phi_{ij} = \partial_i\eta_j - \partial_j\eta_i \tag{5.4}$$

provided that ∂_i means $\dfrac{\partial}{\partial x^i}$.

By virtue of the condition (5. 2), it follows that $d\eta$ is a 2-form of rank $2n$ everywhere over M^{2n+1} and ϕ_{ij} is a matrix whose rank is everywhere $2n$ over M^{2n+1}. We can easily verify that (5. 2) is equivalent to

$$\eta_{[1}\phi_{23}\phi_{45}\cdots\cdots\phi_{2n\ 2n+1]} \neq 0, \tag{5. 5}$$

where [ ] means a determinant divided by the factorial of the number of indices.

As the rank of ϕ_{ij} is $2n$, there exists at least locally a vector field ξ^i such that

$$\phi_{ij}\xi^j = 0. \tag{5. 6}$$

However, as ξ^i is given by

$$\begin{aligned} \xi^1 &= \frac{1}{\lambda}\phi_{[23}\phi_{45}\cdots\cdots\phi_{2n\ 2n+1]}, \\ \xi^2 &= \frac{1}{\lambda}\phi_{[34}\phi_{56}\cdots\cdots\phi_{2n+1\ 1]}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \xi^{2n+1} &= \frac{1}{\lambda}\phi_{[12}\phi_{34}\cdots\cdots\phi_{2n-1\ 2n]}, \end{aligned} \tag{5. 7}$$

where we have put

$$\lambda = (2n+1)\eta_{[1}\phi_{23}\phi_{45}\cdots\cdots\phi_{2n\ 2n+1]}, \tag{5. 8}$$

ξ^i is a vector field globally defined over M^{2n+1}.

To prove ξ^i given by (5. 7) satisfies (5. 6), it is sufficient to verify the case $i=1$. We see that

$$\begin{aligned} \phi_{1j}\xi^j = \frac{1}{\lambda}\{&\phi_{12}\phi_{[34}\phi_{56}\cdots\cdots\phi_{2n+1\ 1]} \\ &+ \phi_{13}\phi_{[45}\phi_{67}\cdots\cdots\phi_{12]} \\ &+ \cdots\cdots \\ &+ \phi_{12n+1}\phi_{[12}\phi_{34}\cdots\cdots\phi_{2n-1\ 2n]}\}. \end{aligned}$$

On the right hand side of the last equation, the terms which contain $\phi_{12}\phi_{13}$ as a factor are contained only in the first and second terms and their sum is easily seen to be

$$\begin{aligned} &(-1)^{2n-1}\phi_{12}\phi_{13}\phi_{[45}\cdots\cdots\phi_{2n\ 2n+1]} \\ +\ &(-1)^{2(2n-2)}\phi_{13}\phi_{12}\phi_{[45}\cdots\cdots\phi_{2n\ 2n+1]} = 0. \end{aligned}$$

In the same way all the other terms cancel to each other. Hence

$$\phi_{1j}\xi^j = 0.$$

Consequently

$$\phi_{ij}\xi^j = 0.$$

We can easily see also that

$$(5.9) \qquad \xi^i\eta_i = 1$$

holds good.

Now, by the Lemma of § 3, there exists a positive definite Riemannian metric h over M^{2n+1} such that

$$(5.10) \qquad \eta_i = h_{ij}\xi^j.$$

Taking this metric h and ξ^i, η_j, ϕ_{ij} as those of § 4, we get the following

THEOREM 4. *Let M^{2n+1} be a differentiable manifold with contact structure. If $\eta = \eta_i dx^i$ is the 1-form which defines the contact structure, then we can find a (ϕ, ξ, η, g)-structure in M^{2n+1} such that the vector field η_j is the one given by the coefficients of the 1-form η and*

$$(5.11) \qquad g_{ih}\phi^h_j = \phi_{ij} = \partial_i\eta_j - \partial_j\eta_i.$$

Now, a differentiable manifold M^{2n+1} is said to have almost contact structure if the structural group of the tangent bundle of the manifold is reducible to $U(n) \times 1$, where $U(n)$ is the unitary group of n complex variables. We shall prove the following

THEOREM 5. *Let M^{2n+1} be a differentiable manifold with (ϕ, ξ, η, g)-structure, then the structure induces an almost contact structure to M^{2n+1}. The converse is also true.*

PROOF. Let $\{U_\alpha\}$ be an open covering of M^{2n+1} by coordinate neighborhoods. We shall determine orthogonal frames in every U_α in the following way. First we put

$$(5.12) \qquad \xi^i_{(\Delta)} = \xi^i, \qquad (\Delta = 2n+1)$$

and take a unit vector field $\xi^i_{(1)}$ over U_α so that it is orthogonal to $\xi^i_{(\Delta)}$. We define $\xi^i_{(1^*)}$ by

$$\xi^i_{(1^*)} = \phi^i_j\xi^j_{(1)}. \qquad (1^* = n+1)$$

Then, we can easily verify that $\xi^i_{(1)}$, $\xi^i_{(1^*)}$, $\xi^i_{(\Delta)}$ are unit vector fields over U_α orthogonal to each other.

Next, we take a unit vector field $\xi^i_{(2)}$ over U_α so that it is orthogonal to three vector field $\xi^i_{(1)}$, $\xi^i_{(1^*)}$, $\xi^i_{(\Delta)}$, and put

$$\xi^i_{(2^*)} = \phi^i_j\xi^j_{(2)}. \qquad (2^* = n+2)$$

Then, we can easily verify that $\xi^i_{(1)}$, $\xi^i_{(2)}$, $\xi^i_{(1^*)}$, $\xi^i_{(2^*)}$, $\xi^i_{(\Delta)}$ are unit vector fields over U_α orthogonal to each other.

Proceeding in the same way, we finally get orthonormal frames $(\xi^i_{(\lambda)}, \xi^i_{(\lambda^*)}, \xi^i_{(\Delta)})$ such that

$$\xi^i_{(\lambda^*)} = \phi^i_j \xi^j_{(\lambda)}. \qquad (\lambda^* = n + \lambda) \tag{5.13}$$

We can easily verify that

$$\xi^i_{(\lambda)} = -\phi^i_j \xi^j_{(\lambda^*)}, \tag{5.14}$$

by virtue of (1.5). We call such frame an adapted frame.

From our construction, we see easily that the matrix whose elements are components of the fundamental tensor g with respect to any adapted frame takes the following form:

$$g = \begin{pmatrix} \delta^\lambda_\mu & 0 & 0 \\ 0 & \delta^\lambda_\mu & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.15}$$

By virtue of (5.13) and (5.14), we see also that the matrix whose elements are components of the tensor ϕ with respect to the same adapted frame is

$$\phi = \begin{pmatrix} 0 & -\delta^\lambda_\mu & 0 \\ \delta^\lambda_\mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.16}$$

Now, let U_α, U_β be coordinate neighborhoods such that their intersection $U_\alpha \cap U_\beta$ is not empty. At every point of $U_\alpha \cap U_\beta$ we can take an adapted frame $(\xi^i_{(\lambda)}, \xi^i_{(\lambda^*)}, \xi^i_{(\Delta)})$ of U_α and an adapted frame $(\overline{\xi}^i_{(\lambda)}, \overline{\xi}^i_{(\lambda^*)}, \overline{\xi}^i_{(\Delta)})$ of U_β. If we denote by $(v^\lambda, v^{\lambda^*}, v^\Delta)$, $(\bar{v}^\lambda, \bar{v}^{\lambda^*}, \bar{v}^\Delta)$ and ϕ^i_j, $\overline{\phi}^i_j$ the components of the same vector and the fundamental singular collineation respectively with respect to the two adapted frames, we see that the relations

$$\bar{v} = \begin{pmatrix} a^\lambda_\mu & b^\lambda_\mu & 0 \\ c^\lambda_\mu & d^\lambda_\mu & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v^\lambda \\ v^{\lambda^*} \\ v^\Delta \end{pmatrix} \tag{5.17}$$

and

$$\overline{\phi}^i_j \bar{v}^j = \overline{\phi^i_j v^j} \tag{5.18}$$

hold good. Of course the first matrix of the right hand side of (5.17) is an orthogonal matrix. As ϕ and $\overline{\phi}$ have the same form (5.16) with respect to the two adapted frames, we can easily see from (5.18) that

$$a_\mu^\lambda = d_\mu^\lambda, \quad b_\mu^\lambda = -c_\mu^\lambda \tag{5.19}$$

hold good. Hence we see that our matrix belongs to $U(n) \times 1$. Consequently, the (ϕ, ξ, η, g)-structure induces uniquely an almost contact structure to M^{2n+1}.

Conversely, let M^{2n+1} be a differentiable manifold with almost contact structure and let $\{U_\alpha\}$ be an open covering of M^{2n+1} by coordinate neighborhoods. Then, by definition, we can take frames over every U_α so that, if $U_\alpha \cap U_\beta$ is not empty, the transformation of components of the same vector with respect to frames of U_α and U_β is given by a matrix which is a real representation of $U(n) \times 1$, i. e. an orthogonal matrix of the form

$$\begin{pmatrix} a_\mu^\lambda & b_\mu^\lambda & 0 \\ -b_\mu^\lambda & a_\mu^\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We call such frames adapted frames.

If we take a symmetric tensor field g over U_α whose components are given by

$$\begin{pmatrix} \delta_\mu^\lambda & 0 & 0 \\ 0 & \delta_\mu^\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with respect to adapted frames, all such tensor fields g over U_α's are unified to a single positive definite tensor field over M^{2n+1}. We take it as the tensor field which defines a Riemannian metric over M^{2n+1}. In the same way, all tensor fields over U_α's of type (1, 1) whose components with respect to adapted frames are given by

$$\begin{pmatrix} 0 & -\delta_\mu^\lambda & 0 \\ \delta_\mu^\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

constitute a single tensor field ϕ over M^{2n+1}.

The same is true for a contravariant vector field ξ with components (0,... ..., 0, 1) and covariant vector field η with components (0,......, 0, 1) with respect to adapted frames. We can easily verify that

$$\phi_j^i \xi^j = 0, \quad \phi_j^i \eta_i = 0,$$
$$\xi^i \eta_i = 1, \qquad \eta_i = g_{ij}\xi^j,$$
$$\phi_j^i \phi_k^j = -\delta_k^i + \xi^i \eta_k$$

hold good with respect to adapted frames.

However, these equations are all tensor equations. So, they hold good for every natural frames too. Consequently, any differentiable manifold with almost contact structure is a differentiable manifold with (ϕ, ξ, η)-structure.

N. B. Hatakeyama's theorem stated in the introduction follows immediately from Theorem 1 of § 3 and Theorem 5 of this section.

6. The form $\eta\wedge(d\eta)^n$ of differentiable manifold with contact structure. Let M^{2n+1} be a differentiable manifold with contact structure η. Then, by virtue of Theorem 4, we can find a (ϕ, ξ, η, g)-structure such that

$$g_{ih}\phi^h_j = \phi_{ij} = \partial_i\eta_j - \partial_j\eta_i. \tag{6.1}$$

Now, we take an open covering $\{U_\alpha\}$ of M^{2n+1} by coordinate neighborhoods and let $\xi^i_{(\lambda)}$, $\xi^i_{(\lambda^*)}$, $\xi^i_{(\Delta)}$ be a field of orthonormal frames over U_α. We put

$$\eta^{(h)}_i = g_{ij}\xi^j_{(h)}, \tag{6.2}$$

then $\eta^{(h)}_i$ is the inverse matrix of $\xi^i_{(n)}$.

As $\xi^i_{(h)}$'s and $\eta^{(h)}_j$'s constitute bases of contravariant vector fields and covariant vector fields over U_α respectively, we can easily verify, by virtue of (1. 1), (1. 3), (1. 4), (5. 13) and (5. 14), that

$$\phi^i_j = \xi^i_{(\mu^*)}\eta^{(\mu)}_j - \xi^i_{(\mu)}\eta^{(\mu^*)}_j. \tag{6.3}$$

Hence we get

$$\phi_{ij} = \sum_{\mu=1}^{n}(\eta^{(\mu^*)}_i\eta^{(\mu)}_j - \eta^{(\mu)}_i\eta^{(\mu^*)}_j). \tag{6.4}$$

Putting (6. 4) into $\eta_{[1}\phi_{23}\phi_{45}\cdots\cdots\phi_{2n\ 2n+1]}$ we see that

$$\begin{aligned}
&\eta_{[1}\phi_{23}\phi_{45}\cdots\cdots\phi_{2n\ 2n+1]}\\
&= \phi_{[12}\phi_{34}\cdots\cdots\phi_{2n-1\ 2n}\eta_{2n+1]}\\
&= \sum_{\mu,\nu,\ldots,\rho=1}^{n} (\eta^{(\mu^*)}_{[1}\eta^{(\mu)}_2 - \eta^{(\mu)}_{[1}\eta^{(\mu^*)}_2)(\eta^{(\nu^*)}_3\eta^{(\nu)}_4 - \eta^{(\nu)}_3\eta^{(\nu^*)}_4)\cdots\cdots\\
&\qquad\qquad \cdots\cdots(\eta^{(\rho^*)}_{2n-1}\eta^{(\rho)}_{2n} - \eta^{(\rho)}_{2n-1}\eta^{(\rho^*)}_{2n})\eta^{(\Delta)}_{2n+1]}\\
&= \sum_{\mu,\nu,\ldots,\rho=1}^{n} 2^n\eta^{(\mu^*)}_{[1}\eta^{(\mu)}_2\eta^{(\nu^*)}_3\eta^{(\nu)}_4\cdots\cdots\eta^{(\rho^*)}_{2n-1}\eta^{(\rho)}_{2n}\eta^{(\Delta)}_{2n+1]}\\
&= \sum_{\mu,\nu,\ldots,\rho=1}^{n} (-1)^{\frac{n(n+1)}{2}}2^n\eta^{(\mu)}_{[1}\eta^{(\nu)}_2\cdots\cdots\eta^{(\rho)}_n\eta^{(\mu^*)}_{n+1}\cdots\cdots\eta^{(\rho^*)}_{2n}\eta^{(\Delta)}_{2n+1]}.
\end{aligned}$$

Therefore, we get

$$\eta_{[1}\phi_{23}\phi_{45}\cdots\cdots\phi_{2n}\phi_{2n+1]} \tag{6.5}$$
$$= (-1)^{\frac{n(n+1)}{2}} 2^n n!\ \eta_{[1}^{(1)}\eta_2^{(2)}\cdots\cdots\eta_n^{(n)}\eta_{n+1}^{(1*)}\cdots\cdots\eta_{2n}^{(n*)}\eta_{2n+1]}^{(\Delta)}.$$

On the other hand, if we consider $2n+1$ 1-forms $\eta^{(h)} = \eta_i^{(h)}dx^i$ and denote the volume element in U_α of our Riemannian manifold M^{2n+1} by dV, then over U_α we see that

$$dV = \eta^{(1)} \wedge \eta^{(2)} \wedge \cdots\cdots \wedge \eta^{(n)} \wedge \eta^{(1*)} \wedge \cdots\cdots \wedge \eta^{(n*)} \wedge \eta^{(\Delta)}$$
$$= (2n+1)!\ \eta_{[1}^{(1)}\eta_2^{(2)}\cdots\cdots\eta_n^{(n)}\eta_{n+1}^{(1*)}\cdots\cdots\eta_{2n}^{(n*)}\eta_{2n+1]}^{(\Delta)}dx^1 \wedge \cdots\cdots \wedge dx^{2n+1}.$$

Threfore we get

$$dV = (-1)^{\frac{n(n+1)}{2}} \frac{(2n+1)!}{2^n\ n!}\ \eta_{[1}\phi_{23}\phi_{45}\cdots\cdots\phi_{2n\ 2n+1]}dx^1 \wedge \cdots\cdots \wedge dx^{2n+1}. \tag{6.6}$$

Comparing the last equation with

$$\eta \wedge (d\eta)^n \doteq \frac{(2n+1)!}{2^n} \eta_{\,1}\phi_{23}\phi_{45}\cdots\cdots\phi_{2n\ 2n+1]}dx^1 \wedge dx^2 \wedge \cdots\cdots \wedge dx^{2n+1}, \tag{6.7}$$

we get finally

$$\eta \wedge (d\eta)^n = (-1)^{\frac{n(n+1)}{2}} n!\ dV \tag{6.8}$$

Consequently, we get the following

THEOREM 6. *Let M^{2n+1} be a differentiable manifold with contact structure and η be the 1-form which defines the structure. Then $\eta \wedge (d\eta)^n$ coincides, within a numerical factor, with the volume element of the Riemannian metric g, where g is the metric of the associated (ϕ, ξ, η, g)-structure to the given contact structure of M^{2n+1}.*

N. B. As $\eta \wedge (d\eta)^n \neq 0$ everwhyere over M^{2n+1}, we see from (6.5) that $\eta_{[1}^{(1)}\eta_2^{(2)}\cdots\cdots\eta_n^{(n)}\eta_{n+1}^{(1*)}\cdots\cdots\eta_{2n}^{(n*)}\eta_{2n+1]}^{(\Delta)}$ has the same sign everywhere over M^{2n+1}. This fact gives a proof that every contact manifold is orientable.

BIBLIOGRAPHY

[1] C. EHRESMANN, Sur les variétés presque complexes, Proc. Int. Congr. Math. Harvard, 412-419.
[2] B. ECKMANN AND A. FRÖLICHER, Sur l'intégrabilité des structures presque complexes, C. R. Acad. Sci., 232(1951), 2248-2286.
[3] W. M. BOOTHBY AND H. C. WANG, On contact manifolds, Ann. of Math., 68(1958), 721-734.
[4] J. W. GRAY, Some global properties of contact structures, Ann. of Math., 69(1959), 421-450.

MATHEMATICAL INSTITUTE,
TÔHOKU UNIVERSITY.