ALMOST ANALYTIC VECTORS IN ALMOST COMPLEX SPACES

KENTARO YANO AND MITSUE AKO

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0. Introduction. In a complex space covered by a system of complex coordinate neighborhoods \((z^\alpha, z^\beta)\), the fact that the components \((v^\kappa, v^\bar{\kappa})\) of a self-conjugate contravariant vector field \(v^h\) are analytic functions of complex coordinates of the form

\[(0.1) \quad v^\kappa = v^\kappa(z^\lambda), \quad \bar{v}^\bar{\kappa} = \bar{v}^\bar{\kappa}(z^\bar{\lambda})\]

and the fact that the components \((w_\lambda, w_{\bar{\lambda}})\) of a self-conjugate covariant vector field \(w_i\) are analytic functions of complex coordinates of the form

\[(0.2) \quad w_\lambda = w_\lambda(z^\kappa), \quad \bar{w}_{\bar{\lambda}} = \bar{w}_{\bar{\lambda}}(z^{\bar{\kappa}})\]

have both a meaning which is independent of the choice of the local complex coordinates. We call such vector fields a contravariant analytic vector and a covariant analytic vector respectively.

In the case where the complex space admits a Kähler metric \(ds^2 = 2g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^{\beta}\), equations \((0.1)\) and \((0.2)\) can be written in the form

\[(0.3) \quad \nabla_\mu v^\kappa = 0, \quad \nabla_\mu \bar{v}^{\bar{\kappa}} = 0\]

and

\[(0.4) \quad \nabla_\mu w_\lambda = 0, \quad \nabla_\mu \bar{w}_{\bar{\lambda}} = 0\]

respectively, where \(\nabla_i\) denotes the covariant differentiation with respect to the Riemannian connection \(\{,^i\} \) defined by the Kähler metric \(g_{\mu\bar{\nu}}\).

Using the tensor

\[(0.5) \quad F_i^a = \begin{pmatrix} \sqrt{-1} \delta^a_\lambda & 0 \\ 0 & -\sqrt{-1} \delta^a_{\bar{\lambda}} \end{pmatrix},\]

we can write \((0.3)\) and \((0.4)\) in the form

\[(0.6) \quad F_i^a \nabla_a v^\lambda - F_i^a \nabla_a \bar{v}^a = 0\]

and

\[(0.7) \quad F_i^a \nabla_a w_i - F_i^a \nabla_i \bar{w}^a = 0\]

1) In the sequel, the Latin indices \(h, i, j, \ldots\) run over the range \(1, 2, \ldots, n\); \(\bar{1}, \bar{2}, \ldots, \bar{n}\), and the Greek indices \(\kappa, \lambda, \mu, \ldots\) over the range \(1, 2, \ldots, n\).
respectively.

In the case where the Kähler space is compact, one of the present authors has proved (K. Yano [6], [7]): A necessary and sufficient condition for a vector \( v^h \) in a compact Kähler space to be contravariant analytic is

\[
0^n \nabla_i \nabla^i v^h + K^i_i v^i = 0,
\]

where \( K^i_i \) is the Ricci tensor of the space.

Since a necessary and sufficient condition for a vector field \( v^h \) in a compact orientable Riemannian space to be a Killing vector is that

\[
g^{ih} \nabla_i v^h + K^i_i v^i = 0, \quad \nabla_i v^i = 0,
\]

(K. Yano [7], [8]), we can see from (0.8) and (0.9) that a contravariant analytic vector \( v^h \) satisfying \( \nabla_i v^i = 0 \) in a compact Kähler space is a Killing vector and that a Killing vector in a compact Kähler space is contravariant analytic.

One of the present authors has also proved (K. Yano [6]): A necessary and sufficient condition for a vector \( w_i \) in a compact Kähler space to be covariant analytic is

\[
g^{ih} \nabla_i w^h - K^h_i w_i = 0.
\]

Since a necessary and sufficient condition for a vector \( w_i \) in a compact orientable Riemannian space to be harmonic is given just by (0.10) (K. Yano, [7], [8]), we see that a necessary and sufficient condition for a vector \( w_i \) in a compact Kähler space to be covariant analytic is that \( w_i \) be harmonic.

The purpose of the present paper is to generalize these results to the case of the most general almost Hermitian space, that is, to the case of spaces in which a mixed tensor \( F^h_i \) satisfying

\[
F^h_i F^i_j = - A^h_j,
\]

\( A^h_j \) being the unit tensor, and a Riemannian metric \( g_{\mu} \) satisfying

\[
g_{\mu} = F^j_i F^i_k g_{\nu}
\]

are given.

The Kähler space is characterized by the equation

\[
\nabla_{\mu} F^h_i = 0.
\]

An almost Hermitian space in which we have

\[
\nabla_{\mu} F_{\mu} + \nabla_i F_{hj} + \nabla_{h} F_{\mu} = 0,
\]

where \( F_{\mu} = F^a_j g_{at} \) is called an almost Kähler space (K. Yano [7]). The results

\[2\) See the Bibliography at the end of the paper.\]
mentioned above have been already generalized to the case of almost Kähler spaces by S. Tachibana [4]. S. Tachibana [5] has also generalized these results to the case where

\[(0.15) \quad \nabla_i F^a_i + \nabla_i F^a_i = 0\]

is satisfied. We shall call such an almost Hermitian space an almost Tachibana space.

In § 1 we give some important formulas in the differential geometry of almost complex spaces and of almost Hermitian spaces. In § 2 we give several formulas which are valid in almost Kähler and almost Tachibana spaces. The § 3 is devoted to the discussions of curvature tensors of these spaces. In § 4 and § 5 we discuss contravariant and covariant almost analytic vectors in the most general almost Hermitian space and deduce as corollaries the theorems on these vectors in almost Kähler and almost Tachibana spaces.

1. Preliminaries. We consider a $2n$-dimensional real differentiable manifold $M$ of class $C^\infty$ covered by a system of coordinate neighborhoods $(\xi^a)$. We can introduce in this manifold $M$ a system of complex coordinate neighborhoods $(z^a, \bar{z}^a)$ defined by

\[(1.1) \quad z^a = \xi^a + \sqrt{-1} \bar{\xi}^a, \quad \bar{z}^a = \xi^a - \sqrt{-1} \bar{\xi}^a.\]

If we can cover the manifold $M$ by a system of complex coordinate neighborhoods $(z^a, \bar{z}^a)$ in such a way that in the intersection of two complex coordinate neighborhoods $(z^a, \bar{z}^a)$ and $(z'^a, \bar{z}'^a)$ we have

\[(1.2) \quad z^a = f^a(z^b), \quad \bar{z}^a = f^a(\bar{z}^b)\]

\[(1.3) \quad \left| \frac{\partial f^a}{\partial z^b} \right| = 0,\]

where $f^a$ are complex conjugate functions of $f^b$, we say that the manifold $M$ admits a complex structure and call $M$ a complex manifold.

When we write above equations in the form

\[(1.4) \quad \xi^a = \xi^a(\xi),\]

$\xi^a(\xi)$ are real analytic functions of $\xi^a$ and

\[(1.5) \quad \left| \frac{\partial \xi^a}{\partial \xi^b} \right| = \left| \frac{\partial z^a}{\partial z^b} \right| \cdot \left| \frac{\partial \bar{z}^a}{\partial \bar{z}^b} \right| > 0.\]

Thus a complex manifold is of class $C^\infty$ and orientable.

The complex structure is also characterized by the existence of a mixed tensor $F^a_i$ which has numerical components

\[(1.6) \quad F^a_i = \begin{pmatrix} \sqrt{-1} \xi^a & 0 \\ 0 & -\sqrt{-1} \xi^a \end{pmatrix} \]
in all complex coordinate neighborhoods \((z^\kappa, z^{\kappa'})\) and consequently satisfies

\[(1.7) \quad F_i^j F_j^h = - A_i^j,\]

where \(A_i^j\) is the unit tensor.

Indeed the components \((1.6)\) of the tensor in the coordinate neighborhoods \((z^\kappa, z^{\kappa'})\) and the components

\[(1.8) \quad F_i^j = \begin{pmatrix} \sqrt{-1} \delta_{i}^{\alpha'} & 0 \\ 0 & -\sqrt{-1} \delta_{i}^{\alpha'} \end{pmatrix}\]

of the tensor in the coordinate neighborhoods \((z^\alpha', z^{\alpha''})\) should be related by equations of the form

\[F_i^j = \frac{\partial z'}{\partial z^\alpha} \frac{\partial z'}{\partial z^\alpha''} F_i^h\]

from which we find

\[\frac{\partial z'}{\partial z^\alpha} = 0, \quad \frac{\partial z'}{\partial z^\alpha''} = 0\]

and consequently we get equations of the form \((1.2)\).

We next consider a \(2n\)-dimensional real differentiable manifold \(M\) of class \(C^\infty\) covered by a system of coordinate neighborhoods \((\xi^\alpha)\). If there exists a mixed tensor \(F_i^h\) of class \(C^\infty\) which satisfies \((1.7)\), we say that the manifold \(M\) admits an almost complex structure and call such a manifold an almost complex manifold. An almost complex manifold is orientable. If there exists a system of complex coordinate neighborhoods \((z^\kappa, z^{\kappa'})\) with respect to which the tensor \(F_i^h\) has always numerical components \((1.6)\), then we say that the almost complex structure induces a complex structure.

It is now a well known fact (A. Newlander and L. Nirenberg [1]) that an almost complex structure \(F_i^h\) induces a complex structure if and only if the so-called Nijenhuis tensor

\[(1.9) \quad N_{ij}^h = F_j^a (\partial_a F_i^h - \partial_i F_j^a) - F_i^a (\partial_a F_j^h - \partial_j F_i^a)\]

vanishes identically, where \(\partial_a\) denotes partial differentiation with respect to the coordinate \(\xi^a\).

The Nijenhuis tensor \(N_{ij}^h\) satisfies the following identities (K. Yano [7]):

\[(1.10) \quad N_{\alpha\alpha}^a = 0,\]

\[(1.11) \quad N_{ij}^h + N_{ij}^h = 0,\]

\[(1.12) \quad N_{ij}^a F_i^a = - N_{ij}^a F_i^a = - N_{ia}^h F_i^a,\]

\[(1.13) \quad N_{ij}^h + F_i^c F_j^c N_{cb}^h = 0, \quad N_{ij}^h - F_i^c F_j^c N_{cb}^h = 0.\]
We now introduce the following tensors (K. Yano [7]):

(1.14) \[ O_{i_1}^{b_1} = \frac{1}{2} (A_i^b A_i^b - F_i^b F_i^b), \]

(1.15) \[ *O_{i_1}^{b_1} = \frac{1}{2} (A_i^b A_i^b + F_i^b F_i^b). \]

For a mixed tensor $T_i^h$ for example we form

(1.16) \[ O_{i_1}^{b_1} T_{b_1}^a \]

and call it the pure part of the tensor $T_i^h$. In the case where the almost complex structure $F_i^h$ induces a complex structure and $F_i^h$ has numerical components (1.6), putting

\[ T_i^h = \begin{pmatrix} T_{\lambda}^\xi & T_{\lambda}^\xi \\ T_{\lambda}^\xi & T_{\lambda}^\xi \end{pmatrix}, \]

we have

\[ O_{i_1}^{b_1} T_{b_1}^a = \begin{pmatrix} T_{\lambda}^\xi & 0 \\ 0 & T_{\lambda}^\xi \end{pmatrix}. \]

For a mixed tensor $T_i^h$, we form also

(1.17) \[ *O_{i_1}^{b_1} T_{b_1}^a \]

and call it the hybrid part of the tensor $T_i^h$. In the complex case, we have

\[ *O_{i_1}^{b_1} T_{b_1}^a = \begin{pmatrix} 0 & T_{\lambda}^\xi \\ T_{\lambda}^\xi & 0 \end{pmatrix}. \]

Similarly $O_{j_1}^{b_1} T_{c_1}^b$ is the pure part of the tensor $T_{j_1}^{b_1}$ and $*O_{j_1}^{b_1} T_{c_1}^b$ is the hybrid part of the tensor $T_{j_1}^{b_1}$.

Take a general tensor $T^{::a_1}$. If we have

(1.18) \[ O_{i_1}^{a_1} T^{::a_1} = 0, \]

that is, if the pure part of the tensor with respect to the indices $h$ and $i$ vanishes, we say that the tensor $T^{::a_1}$ is hybrid in $h$ and $i$. If we have

(1.19) \[ *O_{i_1}^{a_1} T^{::a_1} = 0, \]

that is, if the hybrid part of the tensor with respect to the indices $h$ and $i$ vanishes, we say that the tensor $T^{::a_1}$ is pure in $h$ and $i$.

Similarly if we have

(1.20) \[ O_{j_1}^{c_1} T^{::c_1} = 0, \]

we say that the tensor $T^{::j_1}$ is hybrid in $j$ and $i$ and if we have
we say that the tensor \( T \) is pure in \( j \) and \( i \).

The almost complex structure \( F_i^h \) satisfying (1.7), we have

\[
^*O_{ih}^a F_b^a = \frac{1}{2} (A_i^h A_a^h + F_i^h F_a^h) F_b^a = 0,
\]

which shows that \( F_i^h \) is pure in \( h \) and \( i \).

The equations (1.13) may respectively be written as

\[
^*O_{ih}^a N_{cb}^h = 0, \quad O_{ih}^a N_{cb}^a = 0,
\]

which shows that \( N_{cb}^h \) is pure in \( j \) and \( i \) and is hybrid in \( h \) and \( i \).

Now we can introduce in the differentiable manifold \( M \) of class \( C^\alpha \) a positive definite Riemannian metric \( a_{ij} \) of class \( C^\alpha \). From this we form

\[
g_{ij} = \frac{1}{2} (a_{ij} + F_i^c F_j^d a_{cd}) = \ ^*O_{ij}^a a_{cb},
\]

then the \( g_{ij} \) thus defined is also positive definite and satisfies

\[
g_{ij} = F_i^c F_j^d g_{cd},
\]

that is

\[
O_{ij}^a g_{cb} = 0,
\]

which shows that the covariant tensor \( g_{ij} \) is hybrid in \( j \) and \( i \).

When an almost complex manifold admits a hybrid positive definite Riemannian metric \( g_{ij} \), we call such a metric a Hermitian metric. We call an almost complex manifold admitting a Hermitian metric an almost Hermitian space. When the almost complex manifold reduces to a complex manifold an almost Hermitian space is called a Hermitian space.

In an almost Hermitian space, we put

\[
F^h_{ij} = F_i^a g_{ah},
\]

then (1.7), (1.25) and (1.27) give

\[
F_{ij} = - F_{ji},
\]

\[
O_{ij}^a F_{cb} = 0.
\]

These equations show that \( F^h_{ij} \) is skew-symmetric and is hybrid.

Raising and lowering indices by use of the fundamental metric tensor \( g_{ij} \), we can define \( F^{*h} \) and \( N_{ih}^a \). These tensors satisfy

\[
O_{ih}^{*a} F^{*a} = 0,
\]

\[
^*O_{ih}^{*a} N_{jh}^a = 0
\]
respectively. Thus $F_{\mu h}^k$ is hybrid and $N_{\mu h}^k$ is pure in $i$ and $h$.

Following two lemmas will be very useful in the sequel.

**Lemma 1.** The operators $O^a_{\mu h}$ and $*O^a_{\mu h}$ satisfy

\[(1.32) \quad O + *O = A, \quad O\cdot O = O, \quad O\cdot *O = 0, \quad *O\cdot O = 0, \quad *O\cdot *O = *O, \]

$A$ being the identity operator.

This will be proved by a straightforward calculation.

**Lemma 2.** Let $R_{\mu}$ be pure in $j$ and $i$ and $S^a_{\mu}$ be hybrid in $j$ and $i$, then we have

\[(1.33) \quad R_{\mu} S^a_{\mu} = 0. \]

Indeed, under the assumption, we have

* $O\cdot R = 0$ and $O\cdot S = 0$.

From the first of the identities in Lemma 1 and the first of these equations we find

\[R = O\cdot R \quad \text{or} \quad R_{\mu} = O^a_{\mu} R_{cb}, \]

thus

\[R_{\mu} S^a_{\mu} = (O^a_{\mu} R_{cb})S^a_{\mu} = R_{cb}(O^a_{\mu} S^a_{\mu}) = 0. \]

Applying these lemmas, we have for example

\[(1.34) \quad F^a_{\mu h} N_{\mu h}^a = 0, \quad F^a_{\mu k} N_{\mu h}^a = 0, \]

$F^a_{\mu h}$ being hybrid in $j$ and $i$ and $N_{\mu h}^a$ being pure in $j$ and $i$ and also in $i$ and $h$.

If we denote by $\nabla _j$ the covariant differentiation with respect to Riemannian connection defined by the almost Hermitian metric $g_{\mu i}$, the Nijenhuis tensor $N_{\mu h}^a$ can be written as

\[(1.35) \quad N_{\mu h}^a = F^a_j (\nabla _a F_i^h - \nabla _i F_a^h) - F^a_i (\nabla _a F_j^h - \nabla _j F_a^h). \]

We now define following tensors:

\[(1.36) \quad F_{\mu h} = \nabla _j F_{\mu h} + \nabla _i F_{h j} + \nabla _h F_{j i}, \]

\[(1.37) \quad F_i = \nabla _a F_{ai}, \]

\[(1.38) \quad G_{\mu h} = \nabla _j F_i^h + \nabla _i F_j^h, \]

where

\[(1.39) \quad \nabla _a = g_{at} \nabla _t. \]
ALMOST ANALYTIC VECTORS

We call an almost Kähler (Tachibana) space an almost Hermitian space in which $F_{jk} = 0$ ($G_{jk} = 0$) is satisfied. We call a Kähler (Tachibana) space an almost Kähler (Tachibana) space in which $N_{jk} = 0$ is satisfied.

2. Almost Kähler and almost Tachibana spaces. The covariant components of the Nijenhuis tensor may be written in the form

$N_{jk} = F_j^a F_{ak} - F_i^a F_{ai} + 2F_j^a (\nabla_k F_{ia}),$ (2.1)

from which we have

THEOREM 2.1. In an almost Kähler space, we have

$N_{jk} = 2F_j^a (\nabla_k F_{ia}),$ (2.2)

and consequently $\nabla_j F_{ik}$ is pure in $j$ and $i$.

THEOREM 2.2. In a Kähler space, we have

$\nabla_j F_{ik} = 0,$ (2.3)

and conversely, if (2.3) is satisfied in a Hermitian space, it is a Kähler space.

On the other hand, the Nijenhuis tensor is also written in the form

$N_{jk} = -4(\nabla_j F_i^a) F_{ak} + 2G_{jk} F_{ik} - F_j^a G_{ai} + F_i^a G_{aj},$ (2.4)

from which we get

THEOREM 2.3. In an almost Tachibana space, we have

$N_{jk} = -4(\nabla_j F_i^a) F_{ak}$ (2.5)

and consequently $\nabla_j F_{ik}$ is pure in $j$ and $i$.

THEOREM 2.4. In a Tachibana space, we have (2.3) and consequently a Tachibana space is a Kähler space.

Contracting $F_{ik}$ to equation (2.1) we find, by virtue of (1.10) and (1.34),

$0 = N_{jk} F_{ik} = F_j^a F_{ak} F_{ik} + 2F_j,$

from which

(2.6) $F_j F_{ik} = 2F_{j} F_{ik}$.

Equation (2.6) proves

THEOREM 2.6. In an almost Kähler space, we have

(2.7) $F_i = 0$.
and consequently $F_{ij}$ is a harmonic tensor.

A skew-symmetric tensor $T_{\ldots ij}$ is called a Killing tensor (K. Yano and S. Bochner [8]) when it satisfies

$$\nabla_i T_{ij\ldots k} + \nabla_k T_{ij\ldots k} = 0$$

and consequently

$$\nabla^a T_{a\ldots ij\ldots k} = 0.$$ 

Thus in an almost Tachibana space, the tensor $F_{ij}$ is a Killing tensor and we have $F_{ij} = 0$.

By a straightforward calculation, we find

$$3\nabla_{[i} F_{j]h} - F_{[i}h = G_{[ih} - G_{j}h],$$

from which

**THEOREM 2.6.** In an almost Tachibana space we have

$$3\nabla_{[i} F_{j]h} = F_{[i}h$$

and consequently $\nabla_{[i} F_{j]h}$ is skew-symmetric in all its indices.

In general the Nijenhuis tensor $N_{ijh}$ satisfies

$$N_{ijh} + N_{jhi} = - F_{[i}h F_{j]a} F^a - F_{[i}a F_{j]a} + 2F_{[i}h G_{j]a}.$$ 

Thus if, in an almost Kähler space, we have $N_{ijh} = 0$ and the space is an almost Tachibana space. Thus from Theorem 2.6 we find $3\nabla_{[i} F_{j]h} = F_{[i}h = 0$ and consequently the space is a Kähler space. It is evident that in a Kähler space we have $N_{ijh} + N_{jhi} = 0$. Thus we have

**THEOREM 2.7.** An almost Kähler space is a Kähler space if and only if

$$N_{ijh} + N_{jih} = 0,$$

that is, if and only if $N_{ijh}$ is skew-symmetric in all its indices (S. Sawaki and S. Kotô [3]).

Also the tensor $G_{ij}^h$ satisfies

$$2\delta^h_{[i} G_{j]c} = - (F_{[i}^c F_{j]e} + F_{[i}^c F_{j]e})F^e_h,$$

from which

**THEOREM 2.8.** In an almost Kähler space, the tensor $G_{ij}^h$ is pure in $j$ and $i$.

From (2.10) and (2.12), we have in general
(2.13) \[ N_{\mu \lambda} + N_{\lambda \mu} = 2F^c_\mu O^a_{\mu c} G_{bce} \]
and we get

**Theorem 2.9.** In an almost Tachibana space, the Nijenhuis tensor \( N_{\mu \lambda} \) is skew-symmetric in all its indices.

For an almost Tachibana space, we have from Theorems 2.3 and 2.6

(2.14) \[ N_{\mu \lambda} = \frac{4}{3} F^a_{\mu a} F^a_{\lambda} , \]
or equivalently

(2.15) \[ F^a_{\mu \lambda} = -\frac{3}{4} N^a_{\mu \lambda} F^a_{\lambda} . \]
Thus we have

**Theorem 2.10.** In an almost Tachibana space, the tensor \( N^a_{\mu \lambda} F^a_{\lambda} \) is skew-symmetric in all its indices.

### 3. Curvature tensors.

We denote the curvature tensor of the Hermitian metric \( g_{ij} \) by

(3.1) \[ K_{kij}^h = \partial_k \left( \frac{h}{j} \right) - \partial_j \left( \frac{h}{k} \right) + \left( \frac{h}{k a} \right) \left( j i \right) - \left( \frac{h}{j a} \right) \left( k i \right) , \]
its covariant components by

(3.2) \[ K_{kij}^h = K^a_{kij} g_{ah} , \]
and the Ricci tensor and the curvature scalar by

(3.3) \[ K_{ji} = K^a_{aji} \quad \text{and} \quad K = g^{ij} K_{ij} \]
respectively. Moreover we put

(3.4) \[ H_{kj} = \frac{1}{2} K_{kij} F^{th} \]
and

(3.5) \[ K_{ji}^* = -H_{ja} F^a_{i} \quad \text{or} \quad K^*_{ja} F^a_{i} = H_{ji} . \]
Thus \( H_{ji} \) and \( K^*_{ja} F^a_{i} \) are both skew-symmetric. From (3.5) we find \( K^*_{ja} F^a_{i} + K^*_{ia} F^a_{j} = 0 \), from which

(3.6) \[ K^*_{ji} = F^c_{i} F^b_{j} K^*_{cb} . \]
Thus if \( K^*_{ja} \) is symmetric then it is hybrid.

Applying the Ricci identity to the tensor \( F^h_{i} \), we find

(3.7) \[ \nabla_k \nabla_j F^h_{i} - \nabla_j \nabla_k F^h_{i} = K^h_{kj} F^a_{i} - K^a_{kj} F^h_{i} . \]
Contraction with respect to $k$ and $h$ in this equation gives

\begin{equation}
\nabla_a \nabla_j F_i^a = K_{ja} F_i^a - H_{ji} - \nabla_j F_i,
\end{equation}

from which

\begin{equation}
\nabla_a G_{ij}^a = K_{ja} F_i^a + K_{ia} F_j^a - (\nabla_j F_i + \nabla_i F_j).
\end{equation}

Thus

**THEOREM 3.1.** In an almost Tachibana space, the tensor $K_{ja} F_i^a$ is skew-symmetric and consequently $K_{ji}$ is hybrid.

Equation (3.8) can be written as

\begin{equation}
\nabla_a \nabla_j F_i^a + \nabla_j F_i = (K_{ja} - K_{ja}^*) F_i^a.
\end{equation}

Thus we have

**THEOREM 3.2.** In order that $K_{ji} = K_{ji}^*$ in an almost Hermitian space, it is necessary and sufficient that

\begin{equation}
\nabla_a \nabla_j F_i^a + \nabla_j F_i = 0.
\end{equation}

**COROLLARY 1.** In order that $K_{ji} = K_{ji}^*$ in an almost Hermitian space with $F_i = 0$, it is necessary and sufficient that

\begin{equation}
\nabla_a \nabla_j F_i^a = 0.
\end{equation}

**COROLLARY 2.** In order that $K_{ji} = K_{ji}^*$ in an almost Tachibana space, it is necessary and sufficient that

\begin{equation}
\nabla^a F_{aih} = 0 \quad \text{or} \quad \nabla^a \nabla_a F_{ih} = 0.
\end{equation}

**COROLLARY 3.** In a Kähler space, we have

\begin{equation}
K_{ji} = K_{ji}^*.
\end{equation}

4. **Contravariant almost analytic vectors.** Let us consider a self-conjugate contravariant vector field $(v^a, v^b)$ in a complex manifold. If the components $v^a(v^b)$ are functions of $z^i(z^j)$ only, then the vector is called a contravariant analytic vector (S. Sasaki and K. Yano [2], K.Yano [7]). The condition for $v^a$ to be a contravariant analytic vector is given by

\begin{equation}
\partial_i v^a = 0, \quad \partial_{\bar{i}} v^a = 0,
\end{equation}

where

\[ \partial_i = \partial / \partial z^i, \quad \partial_{\bar{i}} = \partial / \partial \bar{z}^i. \]

The condition (4.1) is equivalent to

\begin{equation}
*O_{ia} \partial_{\bar{j}} v^a = 0,
\end{equation}

\[ *O_{ia} \partial_{\bar{j}} v^a = 0. \]
that is to the fact that \( \partial_b v^a \) is pure. The condition (4.2) is also written as

\[
(4.3) \quad \xi_v F^h_i = v^a \partial_a F^h_i - F^a_i \partial_a v^h + F^h_a \partial_i v^a = 0,
\]

which is a tensor equation, where \( \xi_v \) denotes the Lie differentiation with respect to the vector field \( v^h \) (K. Yano [7]).

Thus in an almost complex space, we define a \textit{contravariant almost analytic vector field} as a vector which satisfies (4.3), that is, as a vector \( v^h \), an infinitesimal transformation with respect to which does not change the almost complex structure.

In an almost Hermitian space, the equation (4.3) may be written as

\[
(4.4) \quad \xi_v F^h_i = v^a \nabla_a F^h_i - F^a_i \nabla_a v^h + F^h_a \nabla_i v^a = 0,
\]

from which

\[
v^a \nabla_a F^h_i - F^a_i \nabla_a v^h - F^h_a \nabla_i v^a = 0,
\]

and taking the symmetric part of this with respect to \( i \) and \( h \)

\[
O^h_i (\nabla_i v_h + \nabla_h v_i) = 0 \quad \text{or} \quad O^h_i (\xi_v g_{ij}) = 0,
\]

and also

\[
O^h_i (\nabla^i v^h + \nabla^h v^i) = 0 \quad \text{or} \quad O^h_i (\xi_v g^{ij}) = 0,
\]

where

\[
\xi_v g_{ij} = v^a \nabla_a g_{ij} + g_{ai} \nabla_j v^a + g_{aj} \nabla_i v^a = \nabla_j v_i + \nabla_i v_j
\]

and

\[
\xi_v g^{ih} = v^a \nabla_a g^{ih} - g^{ai} \nabla_a v^j - g^{aj} \nabla_a v^i = - \nabla^i v^j - \nabla^j v^i.
\]

Equation (4.5) and (4.6) show that \( \xi_v g_{ij} \) and \( \xi_v g^{ih} \) are both hybrid for a contravariant almost analytic vector \( v^h \) in an almost Hermitian space.

Now by a straightforward calculation we can prove

\[
\frac{1}{2} (F_j^a F_{ah}^b + F_i^a F_{ah}^b) - G_{ji} g^h_a
\]

\[
+ F_j^c F_i^h \left[ \frac{1}{2} (F_c^a F_{ah}^b + F_b^a F_{ah}^c) - G_{cb} g^h_a \right] = 0,
\]

which shows that the tensor

\[
\frac{1}{2} (F_j^a F_{ah}^b + F_i^a F_{ah}^b) - G_{ji} g^h_a,
\]

symmetric in \( j \) and \( i \), is pure in \( j \) and \( i \). Thus from Lemma 2 and (4.6) which shows that \( \xi_v g^{ih} \) is hybrid, we obtain
\[
\frac{1}{2} (F^a_i F_{al}^j + F^a_i F_{aj}^l) \left( \frac{\partial}{\partial y^i} g^{jl} \right) - G^a_i F_{al}^j \left( \frac{\partial}{\partial y^i} g^{jl} \right) = 0
\]

or

(4.7) \[
\frac{1}{2} F^a_i \left( \frac{\partial}{\partial y^i} F^{ij} \right) = G^a_i F^{ij} \left( \nabla' v^j \right)
\]

by virtue of \( \xi_v F^a_i = 0 \) for a contravariant almost analytic vector field \( v^a \) in an almost Hermitian space.

Now applying the operator \( \nabla' \) to (4.4), we find

\[
F^a_i [\nabla' \nabla_i v^a + K^a_i v^j - F^a_i \xi_v F^j - G^a_i F^{ij} (\nabla' v^j)] = 0,
\]

or

(4.8) \[
\nabla' \nabla_i v^a + K^a_i v^j - F^a_i \xi_v F^j - G^a_i F^{ij} (\nabla' v^j) = 0
\]

or equivalently

(4.9) \[
\nabla' \nabla_i v^a + K^a_i v^j - F^a_i \xi_v F^j - \frac{1}{2} F^a_i \left( \frac{\partial}{\partial y^i} F^{ij} \right) = 0
\]

by virtue of (4.7).

Equations (4.7) and (4.8) or (4.7) and (4.9) are necessary conditions for a vector \( v^a \) in a general almost Hermitian space to be contravariant almost analytic. Thus a necessary condition for a vector \( v^a \) in an almost Kähler space to be contravariant almost analytic is

\[
\nabla' \nabla_i v^a + K^a_i v^j = 0, \quad G^a_i (\nabla' v^j) = 0,
\]

and a necessary condition for a vector \( v^a \) in an almost Tachibana space to be contravariant almost analytic is

\[
\nabla' \nabla_i v^a + K^a_i v^j = 0, \quad F^a_i \left( \frac{\partial}{\partial y^i} F^{ij} \right) = 0.
\]

If we put

(4.10) \[
T^a_i = g^{ij} \left( \frac{\partial}{\partial y^i} F^j a \right)
\]

we have

(4.11) \[
\frac{1}{2} T^a_i T^a_i = \frac{1}{2} v^a \left( (\nabla_i F^{ij})(\nabla_j F^a) - v^j (\nabla_i F^{ij}) F^a_j (\nabla_j v^i) \right)
+ v^b (\nabla_i F^{ij}) F^a_j (\nabla_j v^i) + (\nabla' v^j)(\nabla_j v^i)
- F^{ij} F^a_i (\nabla_j v^i) (\nabla_j v^j)
\]

and
Assuming that the Hermitian space is compact, we have from (4.12)

\( (4.13) \quad \int_{\Sigma} \left[ \nabla^{\prime} \nabla_{i} v^{h} + K_{i}^{h} v^{i} - F_{i}^{h}(\bar{\partial}_{h} F') - \frac{1}{2} F_{\mu}^{h}(\bar{\partial}_{h} F'') \right] v_{h} \)

\( + \frac{1}{2} T^{\mu} T_{\mu} = 0, \)

where \( d\sigma \) denotes the volume element of the space.

From (4.9) and (4.13), we obtain

**THEOREM 4.1.** A necessary and sufficient condition for a vector field \( v^{h} \) in a compact almost Hermitian space to be contravariant almost analytic is

\( \nabla^{\prime} \nabla_{i} v^{h} + K_{i}^{h} v^{i} - F_{i}^{h}(\bar{\partial}_{h} F') - \frac{1}{2} F_{\mu}^{h}(\bar{\partial}_{h} F'') = 0. \)

**COROLLARY 1.** A necessary and sufficient condition for a vector field \( v^{h} \) in a compact almost Kähler space to be contravariant almost analytic is

\( (4.14) \quad \nabla^{\prime} \nabla_{i} v^{h} + K_{i}^{h} v^{i} = 0 \)

(S. Tachibana [4]).

Since a necessary and sufficient condition for a vector field \( v^{h} \) in a compact orientable Riemannian space to be a Killing vector field is

\( \nabla^{\prime} \nabla_{i} v^{h} + K_{i}^{h} v^{i} = 0, \quad \nabla_{i} v^{i} = 0 \)

we have

**COROLLARY 2.** A Killing vector field in a compact almost Kähler space is contravariant almost analytic.

This corollary may be proved also in the following way. Since \( v^{h} \) is a Killing vector, we have

\( \frac{\partial}{\partial x^{\mu}} g_{\mu} = 0 \quad \text{and} \quad \frac{\partial}{\partial x^{\mu}} g^{\mu} = 0. \)

On the other hand, it is well known that the Lie derivative of a harmonic tensor with respect to a Killing vector vanishes in a compact orientable Riemannian space (K. Yano [7]). Since \( F_{\mu} \) is harmonic in a compact almost Kähler space, we have
from which
\[ \frac{\partial}{\partial v} F_{ij} = 0 \]
which shows that \( v^h \) is contravariant almost analytic.

**COROLLARY 3.** A contravariant almost analytic vector field \( v^h \) satisfying \( \nabla_i v^j = 0 \) in a compact almost Kähler space is a Killing vector.

From (4.7), (4.8) and (4.13), we have

**COROLLARY 4.** A necessary condition for a vector field \( v^h \) in an almost Tachibana space to be contravariant almost analytic is that
\[ \nabla^i v^h + K_i^h v^j = 0, \quad F_{ij}^h (\frac{\partial}{\partial v} F^{ij}) = 0 \]
and a sufficient condition for \( v^h \) in a compact almost Tachibana space to be contravariant almost analytic is
\[ \nabla^i v^h + K_i^h v^j - \frac{1}{2} F_{ij}^h (\frac{\partial}{\partial v} F^{ij}) = 0. \]

5. **Covariant almost analytic vectors.** In a complex manifold, a self-conjugate covariant vector field \( (w_\lambda, w_\bar{\lambda}) \) is said to be covariant analytic when its components \( w_\lambda (w_\bar{\lambda}) \) are functions of \( z^\lambda (z^{\bar{\lambda}}) \) only. The condition for \( (w_\lambda, w_\bar{\lambda}) \) to be covariant analytic is given by
\[ \begin{align*}
\partial^j w_\lambda &= 0, & \partial_\bar{j} w_\bar{\lambda} &= 0
\end{align*} \]

or
\[ \begin{align*}
*O_{ij} \partial_\bar{j} w_\bar{\lambda} &= 0
\end{align*} \]

or
\[ \begin{align*}
[(\partial_j F_i^a) - (\partial_i F_j^a)] w_a - F_j^a \partial_\bar{j} w_\bar{\lambda} + F_i^a \partial_j w_\lambda &= 0,
\end{align*} \]
which is easily verified to be a tensor equation.

Thus in an almost complex space we define a covariant almost analytic vector as a vector field \( w_i \) which satisfies (5.3).

In an almost Hermitian space, the equation (5.3) may be written as
\[ \begin{align*}
[(\nabla_j F_i^a) - (\nabla_i F_j^a)] w_a - F_j^a \nabla_\bar{j} w_\bar{\lambda} + F_i^a \nabla_j w_\lambda &= 0,
\end{align*} \]
from which, taking the symmetric part with respect to \( j \) and \( i \), we find
\[ \begin{align*}
*O_{ij} (\nabla_\bar{j} w_\bar{\lambda} - \nabla_\lambda w_\lambda) &= 0,
\end{align*} \]
which shows that \( \nabla_j w_\lambda - \nabla_\lambda w_j \) is pure for a covariant almost analytic vector \( w_i \) in an almost Hermitian space.
Contracting $\nabla_k F^\alpha_{\mu}$ to the equation (5.4), we find

\[(\nabla_\alpha F^\mu_{\alpha})(\nabla_j F^a_{\beta})w_\alpha = 0\]

for a covariant almost analytic vector field $w_i$.

Now we define tensors $P_\mu$ and $Q_\mu$ by

\[P_\mu = (\nabla_j F^a_{\alpha} - \nabla_i F^a_{\beta})w_\alpha, \quad Q_\mu = F^a_\alpha \nabla_\alpha w_i - F^a_\beta \nabla_\beta w_a\]

respectively. Then for a covariant almost analytic vector field $w_i$, we have

\[P_\mu = Q_\mu,\]

\[P_\mu P^\mu = 2(F^\alpha_{\mu a} - \nabla_i F^a_{\beta})(\nabla^\beta F^a_{\alpha})w_\alpha w_a\]

or

\[P_\mu P^\mu = 2F^a_\beta (\nabla^\beta F^a_{\alpha})w_\alpha w_a\]

by virtue of (5.6) and

\[P_\mu Q^\mu = F^\beta_\alpha (\nabla_\alpha w_i + \nabla_\beta w_a)(G^\alpha_{\alpha a} - 2\nabla^\alpha F^a_{\alpha})w_a.\]

Thus in an almost Kähler space we have $P_\mu P^\mu = 0$ for a covariant almost analytic vector $w_i$ from which we obtain

\[P_\mu = 0, \quad Q_\mu = 0.\]

But in an almost Kähler space, we have

\[\nabla_j F^a_{\alpha} - \nabla_i F^a_{\beta} = - \nabla^a F^j_{\mu}.\]

Thus $P_\mu = 0$ gives

\[w^a \nabla_a F^j_{\mu} = 0.\]

On the other hand, in an almost Tachibana space, we have, following Theorem 2.6,

\[\nabla_j F^a_{\mu} = \frac{1}{3} F^a_{\mu a}\]

and consequently (5.6) may be written as

\[F^a_{\mu b} F^b_{\alpha a} w_\alpha = 0,\]

from which

\[(F^a_{\mu} w_a)(F^b_{\alpha a} w_a) = 0,\]

and consequently $F^a_{\mu} w_a = 0$, that is,

\[\nabla_j F^a_{\mu} w_a = 0 \quad \text{or} \quad w^a \nabla_a F^j_{\mu} = 0,\]

by virtue of $\nabla_j F^a_{\mu} = \nabla_a F^j_{\mu}$. 


Thus in an almost Tachibana space, we have $P^\mu Q^\mu = 0$ and the equations $P^\mu = Q^\mu$ and $P^\mu Q^\mu = 0$ give $P^\mu = 0$, $Q^\mu = 0$, that is,

$$w^a \nabla_a F^\mu = 0, \quad Q^\mu = 0$$

for a covariant almost analytic vector in an almost Tachibana space. Thus we have

**Theorem 5.1.** A necessary and sufficient condition for a vector field $w_i$ in an almost Kähler or an almost Tachibana space to be covariant almost analytic is that

$$w^a \nabla_a F^\mu = 0,$$

$$F_j^a \nabla_a w_i - F_i^a \nabla_j w_a = 0.$$

For a covariant almost analytic vector field $w_i$ in an almost Hermitian space, we have

$$N^a_b w_h = [F_j^a (\nabla_a F_i^h - \nabla_i F_a^h) - F_i^a (\nabla_a F_j^h - \nabla_j F_a^h)] w_h$$

$$= F_j^a (F_i^a \nabla_j w_i - F_i^a \nabla_j w_i) - F_i^a (F_j^a \nabla_i w_j - F_j^a \nabla_i w_j)$$

$$= - (\nabla_j w_i - \nabla_i w_j) - F_j^a F_i^b (\nabla_j w_b - \nabla_i w_b)$$

$$= - 2^a \Omega^a_b (\nabla_c w_b - \nabla_b w_c),$$

that is,

$$N^a_b w_h = 0$$

by virtue of (5.4) and (5.5).

For such a vector, we have also

$$\nabla_j F_i^a - \nabla_i F_j^a (\nabla^j w^i - \nabla^i w^j) F_a^h$$

$$= \frac{1}{2} (\nabla_j F_i^a - \nabla_i F_j^a) (\nabla^j w^i - \nabla^i w^j) F_a^h$$

$$= \frac{1}{2} (\nabla_j F_i^a - \nabla_i F_j^a) \Omega^h_a (\nabla^h w^a - \nabla^a w^h) F_a^h$$

$$= \frac{1}{2} [(\nabla_j F_i^a - \nabla_i F_j^a) - F_j^a F_i^c (\nabla_j F_c^a - \nabla_i F_c^a)] (\nabla^h w^a) F_a^h$$

by virtue of

$$\nabla^j w^i - \nabla^i w^j = \Omega^h_a (\nabla^h w^a - \nabla^a w^h)$$

derived from (5.5). From this we have
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\[(\nabla_j F_i^a - \nabla_i F_j^a)(\nabla^i w^j) F_a^a = -\frac{1}{2} N_{j\ell} (\nabla^i w^j)\]

and consequently

(5.10) \[(\nabla_j F_i^a - \nabla_i F_j^a)(\nabla^i w^j) F_a^a w_h = 0,\]

by virtue of (5.9), for a covariant almost analytic vector field \(w_i\) in an almost Hermitian space.

If we suppose that \(w_i\) is a contravariant and at the same time covariant almost analytic vector field in an almost Hermitian space, then adding

\[\omega^a \nabla_a F^a_j - F^a_j \nabla_a \omega^i + F_a^a \nabla_j \omega^a = 0\]

or

\[\omega^a \nabla_a F^a_i - F^a_i \nabla_a \omega_i - F^a_i \nabla_j \omega_a = 0\]

and

\[(\nabla_j F^a_i - \nabla_i F^a_j) w^a - F^a_j \nabla_a \omega_i + F^a_i \nabla_j \omega_a = 0,\]

we find

(5.11) \[F^a_{ih} w^a - 2F^a_j \nabla_a \omega_i = 0.\]

In an almost Kähler space, equation (5.11) reduces to

\[F^a_j \nabla_a \omega_i = 0.\]

In an almost Tachibana space, (5.11) is also written as

\[3\omega^a \nabla_a F^a_i - 2F^a_j \nabla_a \omega_i = 0\]

or

\[F^a_j \nabla_a \omega_i = 0\]

by virtue of \(\omega^a \nabla_a F^a_j = 0\) in Theorem 5.1. Thus

THEOREM 5.2. If, in an almost Kähler or almost Tachibana space, \(w_i\) is a contravariant and at the same time covariant almost analytic vector field, then it is covariantly constant.

The equation (5.4) is written as

(5.12) \[\nabla_j \tilde{w}_i - \nabla_i \tilde{w}_j = F_j^a (\nabla_a \omega_i - \nabla_i \omega_a),\]

where

(5.13) \[\tilde{w}_i = F_i^a \omega_a.\]

The equation (5.12) may also be written as

(5.14) \[-(\nabla_j \omega_i - \nabla_i \omega_j) = F_j^a (\nabla_a \tilde{w}_i - \nabla_i \tilde{w}_a).\]
The equations (5.12) and (5.14) give

**THEOREM 5.3.** If a vector field \( \bar{v}_i \) in an almost Hermitian space is covariant almost analytic, then the vector field \( \bar{w}_i = F_i^a w_a \) is also covariant almost analytic.

If vectors \( w_i \) and \( \bar{w}_i \) are both closed, then equation (5.12) is satisfied. Thus we have

**THEOREM 5.4.** If vectors \( w_i \) and \( \bar{w}_i = F_i^a w_a \) in an almost Hermitian space are both closed, then they are both covariant almost analytic vectors.

From (5.12) we have by contractions of \( F^i \) and of \( g^i \)

\[
(5.15) \quad F^i \nabla_i \bar{w}_i = 0 \quad \text{and} \quad F^i (\nabla_i w_i - \nabla_i \bar{w}_i) = 0
\]

respectively. Thus applying \( g^i \nabla_j \) to \( \bar{w}_i = F_i^a w_a \), we find

\[
(5.16) \quad g^i \nabla_j \bar{w}_i - F^i w_i = 0.
\]

If a covariant almost analytic vector field \( w_i \) is closed, we have from (5.12)

\[
(5.17) \quad \nabla_i \bar{w}_i - \nabla_i \bar{w}_j = 0.
\]

Thus from (5.16) and (5.17) we have

**THEOREM 5.5.** If, in an almost Hermitian space with \( F^i = 0 \), a covariant almost analytic vector \( w_i \) is closed, then \( \bar{w}_i \) is harmonic (S. Tachibana [4], [5]).

Thus \( \bar{w}_i \) is covariant almost analytic (Theorem 5.3) and is closed and we have

**THEOREM 5.6.** If, in an almost Hermitian space with \( F^i = 0 \), a covariant almost analytic vector field \( w_i \) is closed, then it is harmonic.

Now applying \( F^i \nabla_a \) to (5.4) and changing indices, we obtain

\[
(5.18) \quad \nabla^a \nabla_a^i w_i - (2K^a_{ji} - K^a_j) \omega^i + F^i \nabla^a (F_{cba} w^a)
\]

\[
+ (\nabla^a \omega^i) G_{cba} F_i^a + F_i^a (\omega^i \nabla_b F_a + F_b \nabla^i \omega_a) = 0.
\]

For the tensor \( T_{ji} \) defined by

\[
(5.19) \quad T_{ji} = (\nabla_j F_i^a - \nabla_i F_j^a) w_a - F_j^a \nabla_a w_i + F_i^a \nabla_j w_a,
\]

we have the identity

\[
(5.20) \quad \nabla^i (T_{ji} F_a^j \omega^a) + [\nabla^a \nabla_a w_i - (2K^a_{ji} - K^a_j) \omega^i
\]

\[
+ F^i \nabla^a (F_{cba} w^a) + (\nabla^a \omega^i) G_{cba} F_i^a
\]
Thus, in a compact almost Hermitian space, we have

\begin{equation}
(5.21) \int_{\sigma} \left[ \nabla^a \nabla_a \omega_i - (2K^*_{\mu} - K_{\mu}) \omega_j^\prime + F_i^a \nabla^b (F_{cba} \omega^a) \\
+ (\nabla^b \omega^a) G_{cba} F_i^a + F_i^a (\omega^b \nabla_a F_{ba} + F_b \nabla^b \omega_a) \\
- (\nabla_c F_{ba} - \nabla_b F_{ca} (\nabla^c \omega^b) F_i^a \right] \omega^j + \frac{1}{2} T^\prime T_{\mu} \right] d\sigma = 0,
\end{equation}

and consequently

**Theorem 5.7.** A necessary condition for a vector field \( \omega_i \) in an almost Hermitian space to be covariant almost analytic is that (5.10) and (5.18) are satisfied and a sufficient condition for \( \omega_i \) in a compact almost Hermitian space to be covariant almost analytic is

\begin{equation}
(5.22) \nabla^a \nabla_a \omega_i - (2K^*_{\mu} - K_{\mu}) \omega_j^\prime + F_i^a \nabla^b (F_{cba} \omega^a) \\
+ (\nabla^b \omega^a) G_{cba} F_i^a + F_i^a (\omega^b \nabla_a F_{ba} + F_b \nabla^b \omega_a) \\
- (\nabla_c F_{ba} - \nabla_b F_{ca} (\nabla^c \omega^b) F_i^a = 0.
\end{equation}

**Corollary 1.** A necessary condition for a vector field \( \omega_i \) in an almost Kähler space to be covariant almost analytic is that

\begin{equation}
(5.23) (\nabla_a F_{\mu})(\nabla^b \omega^a) F_i^a \omega^b = 0
\end{equation}

and

\begin{equation}
(5.24) \nabla^a \nabla_a \omega_i - (2K^*_{\mu} - K_{\mu}) \omega_j^\prime + (\nabla^b \omega^a) G_{cba} F_i^a = 0
\end{equation}

are satisfied and a sufficient condition for \( \omega_i \) in a compact almost Kähler space to be covariant almost analytic is

\begin{equation}
(5.25) \nabla^a \nabla_a \omega_i - (2K^*_{\mu} - K_{\mu}) \omega_j^\prime \\
+ (\nabla^b \omega^a) G_{cba} F_i^a + (\nabla_a F_{cb}) (\nabla^c \omega^b) F_i^a = 0.
\end{equation}

The equation (5.24) can be written as

\( \nabla^a \nabla_a \omega_i - K_{\mu} \omega_j^\prime - 2(K^*_{\mu} - K_{\mu}) \omega_j^\prime + (\nabla^b \omega^a) G_{cba} F_i^a = 0. \)

On the other hand, taking account of (3.10) and (5.7), we have
and consequently
\[ (\nabla^a \nabla_a \omega_i - K_i \omega^j) \omega^j = - F_i^j (\nabla^a \omega^j) (\nabla_a F_{i} - \nabla_j F_{i}) \omega^j = 0 \]
by virtue of (5.10). Thus the integral formula (K. Yano [7])
\[
\int_{\sigma} \left( (\nabla^a \nabla_a \omega_i - K_i \omega^j) \omega^j + \frac{1}{2} (\nabla^j \omega^j - \nabla^j \omega^j) (\nabla_j \omega_i - \nabla_i \omega_j) + (\nabla_j \omega^j) (\nabla_i \omega^j) \right) d\sigma = 0
\]
shows that
\[ \nabla_j \omega_i - \nabla_i \omega_j = 0, \quad \nabla_i \omega^j = 0, \]
that is, \( \omega_i \) is harmonic. Thus we have

**COROLLARY 2.** A covariant almost analytic vector in a compact almost Kähler space is harmonic.

For a covariant almost analytic vector field \( \omega_i \) in an almost Tachibana space, taking account of (3.10) and of (5.7), we have
\[
- (K_i^j - K_j^i) \omega^j = (\nabla^a F^i_{,a}) (\nabla_i F_{ba}) \omega^j = 0,
\]
that is, \( K_i^j \omega^j = K_j^i \omega^j \). And consequently, from Theorem 5.7, we have

**COROLLARY 3.** A necessary condition for a vector field \( \omega_i \) in an almost Tachibana space to be covariant almost analytic is that
\[
(5.26) \quad (\nabla_i F^i_{,a}) (\nabla^j \omega^j) F_{a}^{,b} \omega_{b} = 0
\]
and
\[
(5.27) \quad \nabla^a \nabla_a \omega_i - K_i \omega^j = 0
\]
are satisfied and a sufficient condition for \( \omega_i \) in a compact almost Tachibana space to be covariant analytic is
\[
(5.28) \quad \nabla^a \nabla_a \omega_i - (2K_i^j - K_j^i) \omega^j + F_i^j \nabla^a (F_{c,b}^a \omega^b)
- 2(\nabla_c F_{ba}) (\nabla^c \omega^b) F_i^a = 0.
\]
From (5.27) we have

**COROLLARY 4.** A covariant almost analytic vector in a compact almost
Tachibana space is harmonic.

By a similar computation, we can prove the following integral formula which is valid in a compact almost Hermitian space:

\[\int_M \left[ \nabla^a \nabla_a w^i - (2K^i_{\mu} - K_{ij}) w^j + F^a_{i}(F_{cda} \omega^a) + (\nabla^a \omega^c) F_{cda} F^a_i + F^a_i (\omega^c \nabla_a F_c + F_b \nabla^b \omega_a) \right] w^j - \frac{1}{2} F^i_i \left( \omega_b (\nabla_c F^a) + F_b \nabla_c w_e + F_{cda}(\nabla^a \omega^a) \right) w^j + \frac{1}{2} T^a_{ij} T_{ji} \]

\[+ \frac{1}{2} \left( \frac{1}{2} (\nabla^a \omega^b - \nabla^b \omega^a) (\nabla_c \omega^c_a - \nabla_a \omega^c) + (\nabla_a \omega^a)^2 \right) \right] d\sigma = 0,\]

from which we have

**Theorem 5.8.** A necessary and sufficient condition for a vector \( w_i \) in a compact almost Kähler space to be covariant almost analytic is

\[\nabla^a \nabla_a w_i - (2K^i_{ij} - K_{ij}) w^j + (\nabla^c \omega^b) F^a_{i}(F_{cda} \omega^a) = 0.\]

**Bibliography**