ON DIFFERENTIABLE MANIFOLDS WITH CERTAIN STRUCTURES
WHICH ARE CLOSELY RELATED TO ALMOST CONTACT
STRUCTURE. II

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1. Introduction. In the previous paper I [2], one of the authors defined
the notions of manifolds with \( (\phi, \xi, \eta) \)-structure and of manifolds with \( (\phi, \xi, \eta, g) \)-structure and studied some algebraic properties of these manifolds. By
definition, a differentiable manifold \( M^{2n+1} \) with \( (\phi, \xi, \eta) \)-structure is a manifold
with three tensor fields \( \phi_j, \xi^i \) and \( \eta_i \) defined over \( M^{2n+1} \) which satisfy the
relations
\[
\begin{align}
\text{rank} \; |\phi_j| &= 2n, \\
\xi^i \eta_i &= 1, \\
\phi_j \xi^i &= 0, \\
\phi_j \eta_i &= 0, \\
\phi_j \phi_k &= -\delta_j^k + \xi^i \eta_i.
\end{align}
\]
Every differentiable manifold with \( (\phi, \xi, \eta) \)-structure has a positive definite
Riemannian metric \( g \) such that
\[
\begin{align}
g_{ij} \xi^i &= \eta_j, \\
g_{ij} \phi_k \phi_l &= g_{kl} - \eta_k \eta_l, \\
g_{kl} \phi_i &= -g_{jl} \phi_i \quad (\equiv \phi_i).
\end{align}
\]
We call such metric \( g \) an associated Riemannian metric of the \( (\phi, \xi, \eta) \)-
structure. Any manifold with \( (\phi, \xi, \eta) \)-structure and its associated Riemannian
metric is called a manifold with \( (\phi, \xi, \eta, g) \)-structure.

In this paper, we shall study mainly about some tensor fields defined by
\( (\phi, \xi, \eta) \)-structures and connections which leave \( \phi_j, \xi^i \) and \( \eta_i \) covariant constant. Notations
are same as in I.

2. Some tensors on manifolds with \( (\phi, \xi, \eta) \)-structure. Let \( M^{2n+1} \) be
a differentiable manifold with \( (\phi, \xi, \eta) \)-structure and \( R \) be a real line, and
consider the product manifold \( M^{2n+1} \times R \). We take a sufficiently fine open
covering \( U \) of \( M^{2n+1} \) by coordinate neighborhoods. If we denote coordinates of
U in U by \( x^i (i, j, k = 1, 2, \ldots, 2n + 1) \) and a cartesian coordinate of R by \( x^m \), then \( (x^i, x^m) \) can be considered as a set of coordinates of \( U \times R \) and \( \{U \times R \mid U \in \mathcal{U} \} \) constitutes an open covering of \( M^{2n+1} \times R \) by coordinate neighborhoods.

Now, suppose that \( U, U' (U \cap U' \neq \emptyset) \) belong to \( \mathcal{U} \) and \( x', x'' \) are their coordinates and let

\[
x'' = x'' (x^1, \ldots, x^{2n+1}),
\]

be the coordinate transformation in \( U \cap U' \). We define the coordinate transformation between \( U \times R \) and \( U' \times R \) by

\[
\begin{aligned}
(x'' &= x'' (x^1, \ldots, x^{2n+1}), \\
x^m &= x^m.
\end{aligned}
\]

Making use of the product manifold \( M^{2n+1} \times R \) and the pseudo-group of the type (2.2), we shall define four tensors \( N^j_k \), \( N^j_\kappa \), \( N^j_k \) and \( N^j \) over \( M^{2n+1} \).

We begin with the following

**Lemma 1.** If we put

\[
(2.3) \quad F_i^j = \phi_i^j, F_i^m = \xi_i^j, F_j^m = -\eta_j^m, F_m^m = 0
\]

in coordinate neighborhoods \( \{U \times R \mid U \in \mathcal{U} \} \), then \( F_{AB}^C (A, B, C = 1, 2, \ldots, 2n + 1, \infty) \) defines a field of mixed tensor over \( M^{2n+1} \times R \) with respect to the pseudo-group of transformations of the type (2.2), and \( F_{AB}^C \) gives an almost complex structure on \( M^{2n+1} \times R \).

**Proof.** As the Jacobian matrix of the coordinate transformation (2.2) is given by

\[
\begin{pmatrix}
\frac{\partial x''}{\partial x^j} & 0 \\
0 & 1
\end{pmatrix}
\]

we get

\[
F_j^i = \phi_j^i, F_j^m = \xi_j^i, F_i^m = -\eta_i^m, F_m^m = 0
\]

\[
F_i^j = \phi_i^j \frac{\partial x'}{\partial x^i} \frac{\partial x'}{\partial x^j} = F_i^j \frac{\partial x'}{\partial x^i} \frac{\partial x'}{\partial x^j},
\]

\[
F_j^m = \xi_j^m \frac{\partial x'}{\partial x^j} = F_j^m \frac{\partial x'}{\partial x^j},
\]

\[
F^m_j = \xi^m_j \frac{\partial x'}{\partial x^j},
\]

\[
F_i^m = -\eta_i^m \frac{\partial x'}{\partial x^m} = F_i^m \frac{\partial x'}{\partial x^m}
\]
which shows that \( F^\alpha_a \) defines a tensor field on the product manifold \( M^{2n+1} \times \mathbb{R} \).

Making use of the properties (1.1)~(1.5), we can easily see that the tensor \( F^\alpha_a \) satisfies

\[
(2.4) \quad F^\alpha_a F^a_\beta = - \delta^\alpha_\beta.
\]

Now the Nijenhuis tensor of this almost complex structure is given by

\[
N^\alpha_{\beta\gamma} = F^\gamma_\delta(\partial_\beta F^\delta_\alpha - \partial_\alpha F^\delta_\beta) - F^\delta_\alpha(\partial_\beta F^\gamma_\delta - \partial_\gamma F^\delta_\beta).
\]

If we calculate the components of this tensor by grouping their indices, in two groups \((1,2,\ldots,2n+1)\) and \(\infty\), we get

\[
\begin{align*}
N^\alpha_{ij} &= \phi^i_\delta(\partial_\delta \phi^j_\gamma - \partial_\gamma \phi^j_\delta) - \phi^j_\delta(\partial_\delta \phi^i_\gamma - \partial_\gamma \phi^i_\delta) \\
&- \eta^i_{\delta} \partial^j_\delta + \eta^j_{\delta} \partial^i_\delta, \\
N^\alpha_{ik} &= \phi^i_\delta(\partial_\delta \eta^k - \partial^k \eta_\delta) - \phi^k_\delta(\partial_\delta \eta^i - \partial^i \eta_\delta), \\
N^\alpha_{ik} &= \xi^i(\partial_\delta \phi^j_\gamma - \partial_\gamma \phi^j_\delta) - \phi^j_\delta \partial^i_\delta, \\
N^\alpha_{ij} &= \xi^j(\partial_\delta \eta^i - \partial^i \eta_\delta) - \eta^i_{\delta} \partial^j_\delta + \eta^j_{\delta} \partial^i_\delta.
\end{align*}
\]

Now we suppose that an affine connection \( \Gamma^i_{jk} \) is given on the manifold \( M^{2n+1} \). We denote the torsion tensor of the connection by

\[
(2.7) \quad S^i_{jk} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}),
\]

and denote the covariant differentiation by a comma, then we can easily see that the four sets of components \( N^i_{jk}, N^i_j, N^i_k \) and \( N_j \) can be written as follows:

\[
\begin{align*}
N^i_{jk} &= \phi^i_\delta(\phi^j_{\delta, k} - \phi^j_{\delta, h}) - \phi^j_\delta(\phi^i_{\delta, h} - \phi^i_{\delta, k}) + \xi^j_\delta \eta^k - \xi^i_\delta \eta^k \\
&- 2S_{i\delta}^m \phi^j_\delta \phi^m_\delta + 2\phi^i_\delta(S^m_{\delta k} \phi^j_\delta - S^m_{\delta h} \phi^j_\delta) + 2S^i_{jk}, \\
N^i_j &= \xi^j(\phi^i_{\delta, k} - \phi^i_{\delta, h}) - \phi^i_\delta \xi^j_\delta \\
&+ 2S_{i\delta}^m \phi^i_\delta \phi^m_\delta - 2S_{i\delta}^m \phi^j_\delta \phi^m_\delta, \\
N^i_{jk} &= \phi^i_\delta(\eta^k_{\delta, j} - \eta^k_{\delta, h}) - \phi^j_\delta(\eta^i_{\delta, k} - \eta^i_{\delta, h}) \\
&+ 2\eta^i_\delta(S^m_{\delta k} \phi^j_\delta - S^m_{\delta h} \phi^j_\delta), \\
N_j &= \xi^j(\eta^i_{\delta, j} - \eta^i_{\delta, h}) - 2S^i_{ij} \xi_\delta \eta^j.
\end{align*}
\]

Hence, we get the following

**Theorem 1.** If \( M^{2n+1} \) is a manifold with \((\phi, \xi, \eta)\)-structure, then the four sets of components \( N^i_{jk}, N^i_j, N^i_k \) and \( N_j \) of the Nijenhuis tensor of the almost complex structure on \( M^{2n+1} \times \mathbb{R} \) give four tensors on the manifold
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$M^{n+1}$ which are uniquely determined by the $(\phi, \xi, \eta)$-structure.

Especially, when the connection $\Gamma_{jk}^i$ is symmetric, (2.8) can be simplified
as follows:

\[
\begin{align*}
N_{jk}^i &= \phi^i_k(\phi^j_{k}, h - \phi^i_{j}, h) - \phi^i_j(\phi^j_{k}, h - \phi^i_{j}, h) + \xi^i, \eta_k - \xi^i, \eta_j, \\
N_j^i &= \xi^i(\phi^i_{j}, h - \phi^i_{j}, h) - \phi^i_j(\xi^i, h), \\
N_{jk} &= \phi^j(\eta_{j,k} - \eta_{j,k}) - \phi^j(\eta_{j,k} - \eta_{k,j}), \\
N_j &= \xi(\eta_{j} - \eta_{j}).
\end{align*}
\]

(2.9)

3. Some properties of the tensor fields $N_{jk}^i, N_j^i, N_{jk}$ and $N_j$. In this
section we study some properties of the tensors defined in §2. If we calculate
the Lie derivatives of $\eta_j$ and $\phi_j$ with respect to the infinitesimal transformation
$\xi^i$, we get

\[
\begin{align*}
(\mathcal{L}(\xi)\eta)_j &= \xi^i\partial_i \eta_j + \eta_i \partial_i \xi^k \\
&= \xi^i \partial_i \eta_j - \xi^i \partial_i \eta_j = N_j
\end{align*}
\]

and

\[
\begin{align*}
(\mathcal{L}(\xi)\phi)_j &= \xi^i \partial_i \phi_j - \phi_i \partial_i \xi^j + \phi_i \partial_i \xi^k \\
&= \xi^i \partial_i \phi_j - \phi_i \partial_i \xi^j - \xi^i \partial_i \phi_j \\
&= N_j
\end{align*}
\]

Therefore we have the following

THEOREM 2. (3.1) $N_j^i = (\mathcal{L}(\xi)\phi)^i_j,$

(3.2) $N_j = (\mathcal{L}(\xi)\eta)_j.$

COROLLARY 1. $N_j = 0$ if and only if $\eta_j$ is invariant under the trans-
formations generated by the infinitesimal transformation $\xi^i$.

COROLLARY 2. $N_j^i = 0$ if and only if $\phi_j$ is invariant under the trans-
formations generated by the infinitesimal transformation $\xi^i$.

We know that the Nijenhuis tensor $N_{jk}^a$ is hybrid with respect to the
indices $A$ and $C$ and pure with respect to the indices $B$ and $C$. The condition
of hybrid is, by definition, given by

\[
N_{jk}^a F_{b}^c = - N_{bk}^a F_{j}^c.
\]

(3.3) $N_{jk}^a F_{b}^c = - N_{bk}^a F_{j}^c.$

If we write down the components of both sides of the last equation by
grouping their indices in two groups $(1, 2, \ldots, 2n + 1)$ and $\infty$, we get the
following eight relations

\[
\begin{align*}
\phi^i_{j,k} N_{jk} + N_{jk} \phi^i_{h} + \xi^i N_{jk} - N_j \eta_k &= 0, \\
N_{jk} \xi^h + \phi^i_{h} N_j - \xi^i N_j &= 0,
\end{align*}
\]
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\[
\begin{align*}
\eta_h N^i_k - N^i_{jk} \phi^k - N^j k = 0, \\
\phi^i N^i_j + N^i_{jk} \phi^k - \xi^i N_j = 0, \\
N^i_k \xi^k = 0, \\
\eta_h N^i_j - N^j_h \xi^k = 0, \\
\eta_h N^i_j + N^i_{jk} \phi^k = 0, \\
N^i_k \xi^k = 0,
\end{align*}
\]

(3.4)

The condition of purity is, by definition, given by

\[
N^i_{hk} F^h_c = N^i_{hc} F^h_c.
\]

Although, this is an immediate consequence of (3.3), we shall write down the components of both sides as (3.4) for later use, omitting equations which appear in (3.4) too.1

\[
\begin{align*}
N^i_{jk} \phi^h - N^i_{hk} \phi^j - N^j k \eta_j - N^j k \eta_j = 0, \\
N^i_{jk} \xi^h - N^i_{jk} \phi^j = 0, \\
N^i_{jk} \phi^h - N^j h \phi^j + N^j k \eta_k - N^j k \eta_j = 0, \\
N^i_{jk} \xi^h + N^i_{jk} \phi^j = 0.
\end{align*}
\]

(3.6)

THEOREM 3. For any manifold with \((\phi, \xi, \eta)\)-structure, the relations

\[
\begin{align*}
N_j = N^i_{hk} \phi^j_i, \\
N_j = \eta_h N^i_{k} \phi^j, \\
N_j = N^i_{jk} \eta^i, \\
N^i_{jk} = - \eta_h N^i_{k} \phi^j + N^j k \eta_i, \\
N^i_{j} = \phi^i N^i_{ek} \xi^e + \xi^e N^i_{jk} \xi^e
\end{align*}
\]

(3.7)

hold good.

PROOF. We can easily verify that (3.7) follows from (3.6) and (3.4), (3.7) follows from (3.4), (3.7) follows from (3.4), (3.7) follows from (3.4) and (3.4), and (3.7) follows from (3.4) and (3.7).

Now, we put

\[
a^i_{jk} = \partial_j \eta_k - \partial_k \eta_j,
\]

1) We derived several of (3.4) and (3.6) by direct calculations. The usefulness of purity and hybrid is remarked by S. Tachibana.
then \( \frac{1}{2} a_{jk} dx^j \wedge dx^k \) is the exterior derivative of \( \eta = \eta_j dx^j \) and we have the following

**Theorem 5.** \( N_{jk} = 0 \), if and only if

\[
(3.9) \quad a_{jk} \phi^k \phi_m = a_{im},
\]

i.e., if and only if \( d\eta \) is invariant under \( \phi_j \).

**Proof.** Necessity. Since \( N_{jk} = 0 \), we have

\[
\phi_b^a a_{jh} = \phi^a_k a_{bh}.
\]

If we multiply with the last equation by \( \phi^j_l \) and sum for \( j \), we get

\[
a_{jk} \phi^j_l = (- \delta^b_l + \xi^b \eta_l) a_{kh}
\]

\[
= - a_{sl} + a_{sh} \xi^b \eta_s.
\]

On the other hand, by virtue of the definition of \( N_j \) and Theorem 3, we have

\[
a_{kh} \xi_h = - N_k = 0.
\]

Therefore

\[
a_{jk} \phi^j_l = a_{im}.
\]

**Sufficiency.** From

\[
a_{jk} \phi^j_l \phi_m = a_{im},
\]

we get

\[
a_{im} \xi^m = a_{jk} \phi^j_l \phi^k_m \xi^m = 0.
\]

So we have

\[
a_{im} \phi^m = a_{jk} \phi^j_l \phi^k_m a_{pm} = a_{jk} \phi(- \delta^b_l + \xi^b \eta_l)
\]

\[
= - a_{jk} \phi^j_l.
\]

Hence

\[
N_{jk} = \phi^b a_{jh} - \phi^b a_{kh} = 0.
\]

Q.E.D.

**Corollary 1.** \( N_{jk} = 0 \) if and only if

\[
(3.10) \quad a_{jk} \psi^j_l \psi^k_m = a_{im},
\]

where we have put

\[
(3.11) \quad \psi^j_l = \phi^j_l + \xi^j \eta_l.
\]
PROOF. The necessity is easily seen, since
\[ a_{jk}\phi_i^j\phi_m^i = a_{im}, \quad a_{jk}\xi^j = 0. \]
To prove the sufficiency, we multiply (3.10) by \( \xi^m \) and sum for \( m \), then we get
\[ a_{im}\xi^m = a_{jk}\xi^j\psi_i^j. \]
So \( a_{im}\xi^m \) is the characteristic covector of \( \psi_i^j \) corresponding to the characteristic value 1, therefore
\[ a_{im}\xi^m = \lambda\eta_i. \]
And from this we get
\[ \lambda = a_{im}\xi^m\psi_i^j = 0. \]
So we get
\[ a_{jk}\xi^j = 0. \]
Making use of this and the above condition, we have
\[ a_{jk}\phi_i^j\phi_m^i = a_{im}. \]
Therefore by virtue of Theorem 4, we get
\[ N_{jk} = 0. \]

**COROLLARY 2.** If the \((\phi, \xi, \eta)\)-structure is the one induced from a contact structure, then \( N_{jk} \) and \( N_j \) vanish identically.

**PROOF.** Since the \((\phi, \xi, \eta)\)-structure is given by a contact structure we have
\[ a_{ij} = \phi_{ij}. \]
On the other hand, we may easily show that
\[ \phi_{ij}\phi_k^i\phi_l^j = \phi_{kl}. \]
Therefore, from the last Theorem, we see that
\[ N_{jk} = 0. \]
\( N_j = 0 \) follows from Theorem 4. \( \text{Q.E.D.} \)

Moreover, relative to the tensor \( N'_{jk} \), we have the following

**THEOREM 6.** If the tensor \( N'_{jk} \) vanishes, then other three tensors \( N_j, N_{jk} \) and \( N'_{ij} \) vanish.

**PROOF.** As \( N'_{jk} = 0, N_j = 0 \) by Theorem 4. Hence, we get from (3.4) and
(3.6),

\[ N_{jh} \phi_k = 0, \]
\[ N_{jh} \xi^k = 0. \]

Therefore, we get
\[ N_{jh} = 0. \]

As \( N'_{jk} \) and \( N_{jk} \) vanish, we see by virtue of Theorem 4 that
\[ N'_{j} = 0. \]

\[ \text{Q.E.D.} \]

4. An affine connection which leaves the tensor \( \phi^i_j \) covariant constant.

On an almost complex manifold, we can always find affine connections which leave the fundamental collineation covariant constant. In this section, we shall find such a connection on a manifold with \( (\phi, \xi, \eta) \)-structure and study some properties of this connection. We begin with the following

\textbf{THEOREM 7.} Let \( \Gamma'_{jk} \) be an arbitrary affine connection on a manifold with \( (\phi, \xi, \eta) \)-structure, and put

\begin{equation}
T'_{jk} = - \frac{1}{2} \phi'^{n}_{j,k} \phi^{i}_{m} - \frac{1}{2} \xi'^{i}_{k} \eta_{j} + \xi'^{i} \eta'_{j,k},
\end{equation}

where the comma is the covariant differentiation with respect to the connection \( \Gamma'_{jk} \). Then \( \phi^i_j \) is covariant constant with respect to the connection defined by

\begin{equation}
(4.2) \quad \Gamma'^{i}_{jk} = \Gamma'_{jk} + T'_{jk}.
\end{equation}

\textbf{PROOF.} If we denote the covariant differentiation with respect to the latter connection by \( ; \), then

\[ \phi'^{i}_{jk} = \phi'^{i}_{j,k} + T'^{i}_{jk} \phi^i_j - T'^{i}_{jk} \phi^i_h \]
\[ = \phi'^{i}_{j,k} + \left( - \frac{1}{2} \phi'^{n}_{j,k} \phi'^{i}_{m} - \frac{1}{2} \xi'^{i}_{k} \eta_{j} + \xi'^{i} \eta'_{j,k} \right) \phi^{h}_{j} \]
\[ - \left( - \frac{1}{2} \phi'^{n}_{j,k} \phi^{h}_{m} - \frac{1}{2} \xi'^{h}_{k} \eta_{j} + \xi'^{h} \eta'_{j,k} \right) \phi^{i}_{h} \]
\[ = \phi'^{i}_{j,k} - \frac{1}{2} \phi'^{n}_{j,k} \phi'^{i}_{m} + \xi'^{i} \eta'_{j,k} \phi^{h}_{j} \]
\[ + \frac{1}{2} \phi'^{n}_{j,k} (- \xi'^{i}_{m} + \xi'^{i} \eta_{m}) + \frac{1}{2} \xi'^{i} \eta_{j} \phi^{h}_{k} \]
\[ = \frac{1}{2} \phi'^{i}_{j,k} - \frac{1}{2} \left[ (\phi'^{n}_{m} \phi'^{i}_{j})_{,k} - \phi'^{n}_{m} \phi'^{h}_{j,k} \right] \phi^{i}_{m} \]
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\[ + \xi^i \eta_i \phi^j + \frac{1}{2} \phi^m_{\cdot i} \xi^j \eta_m + \frac{1}{2} \xi^m \eta_i \phi^j \]

\[ = \frac{1}{2} \phi^j_{\cdot k} - \frac{1}{2} \xi^m \eta_j \phi^m_{\cdot i} - \frac{1}{2} \xi^m \eta_j \phi^m_{\cdot i} \]

\[ + \frac{1}{2} \phi^h_{\cdot j} (\delta^h_k + \xi^j \eta_h) + \xi^j \eta_{\cdot k} \phi^h_{\cdot j} \]

\[ + \frac{1}{2} \phi^m_{\cdot j} \xi^j \eta_m + \frac{1}{2} \xi^m \eta_{\cdot k} \phi^h_{\cdot j} \]

\[ = \phi^h_{\cdot i} \xi^j \eta_h + \xi^h \eta_{\cdot h,k} \phi^h_{\cdot j} = \xi^h (\phi^h_{\cdot i} \eta_h)_{\cdot k} = 0. \quad \text{Q.E.D.} \]

**Theorem 8** If \( \Gamma^i_{\cdot jk} \) is a connection which leaves \( \phi^j_{\cdot i} \) covariant constant, then with respect to this connection

\[(4.3) \quad \xi^i_{\cdot jk} = \lambda^i_k \xi^i, \quad \eta_{\cdot jk} = -\lambda_k \eta_j \]

hold good, where \( \lambda_k \) is a covariant vector.

**Proof.** From the fact that \( \Phi_F^i = 0 \), we get

\[ \phi^i_{\cdot jk} = 0. \]

While the rank \( |\phi^j_{\cdot i}| = 2n \), and \( \xi^i \) is a characteristic vector corresponding to the characteristic value 0, we have

\[ \xi^i_{\cdot jk} = \lambda^i_k \xi^i. \]

Similarly, we get

\[ \eta_{\cdot jk} = \mu^i_k \eta_j. \]

Since \( \xi^i \eta_i = 1 \), we have

\[ \xi^i_{\cdot jk} \eta_j + \xi^i \eta_{\cdot jk} = 0. \]

Therefore

\[ \lambda^i_k + \mu^i_k = 0. \quad \text{Q.E.D.} \]

N.B. We can easily see that

\[(4.4) \quad \lambda_k = \eta \xi^i_{\cdot jk}. \]

In the same way as the above proof, we can prove the following

**Theorem 9.** With respect to the connection which leaves \( \Psi^j_{\cdot i} \) covariant constant, we have
(4.5) \[ \xi^i_{;jk} = \nu_k \xi^i_j, \quad \eta_{,jk} = -\nu_k \eta_j, \]

where \( \nu_k \) is a covariant vector.

From Theorems 8 and 9, we get

**COROLLARY.** \( \phi^i_{,jk} = 0 \), if and only if \( \psi^i_{,jk} = 0 \).

Next if we calculate the covariant derivative of \( \xi^i \) with respect to the connection stated in Theorem 7, we get

\[
\xi^i_{,jk} = \xi^i_{,k} + T^i_{jk} \xi^j
\]

\[ = \xi^i_{,k} - \frac{1}{2} \phi^m_{,k} \phi^i_m \xi^j - \frac{1}{2} \xi^i_{,k} \eta \eta^j + \xi^i \eta_{,jk} \xi^j
\]

\[ = \frac{1}{2} \xi^i_{,k} + \frac{1}{2} \xi^i_{,k} \phi^m \phi^i_m + \xi^i \eta_{,jk} \xi^j
\]

\[ = \frac{1}{2} \xi^i_{,k} + \frac{1}{2} \xi^i_{,k} \left( \eta_i - \xi^i_{,j} \right) - \xi^i \eta_{,jk} \xi^j
\]

i.e.,

(4.6) \[ \xi^i_{,k} = -\frac{1}{2} \xi^i_{,k} \eta \xi^j. \]

And so

(4.7) \[ \eta_{,jk} = \frac{1}{2} \xi^i_{,k} \eta \eta_j. \]

On the other hand, according to a theorem of Ishihara and Obata in [1], we can find a symmetric affine connection which leaves \( \xi^i \) covariant constant.

So if we take this connection as \( \Gamma^i_{jk} \) in Theorem 7, we have

(4.8) \[ \phi^i_{,jk} = 0, \quad \xi^i_{,jk} = 0, \quad \eta_{,jk} = 0. \]

Therefore we get

**THEOREM 10.** On a manifold with a \( (\phi, \xi, \eta) \)-structure we can find an affine connection which leaves \( \phi^i, \xi^i \) and \( \eta \), covariant constant.

**N. B.** We call the connection stated in the last theorem a \( (\phi, \xi, \eta) \)-connection.

Next, we consider a manifold with \( (\phi, \xi, \eta, \varphi) \)-structure. If we take the Christoffel's symbol \( \{ \frac{i}{j} \} \) with respect to \( g_{ij} \) as \( \Gamma^i_{jk} \), we get

\[ g_{ij;k} = g_{ij;k} - \left( -\frac{1}{2} \phi^m_{,i} \phi^m_{,k} - \frac{1}{2} \xi^i_{,k} \eta + \xi^i \eta_{,ik} \right) g_{ij}. \]
ON DIFFERENTIABLE MANIFOLDS WITH CERTAIN STRUCTURES

\(- \left( - \frac{1}{2} \phi^m \phi^h - \frac{1}{2} \xi^h \eta_j + \xi^h \eta_j \right) g_{ih} \)

\[ = \frac{1}{2} \phi^m \phi^h g_{ih} + \frac{1}{2} \eta_{ijk} \eta_j - \eta_j \eta_{ij} \]

\[ + \frac{1}{2} \phi^m \phi^h g_{ih} + \frac{1}{2} \eta_{ijk} \eta_j - \eta_j \eta_{ij} \]

\[ = \frac{1}{2} \phi^m \phi^h g_{ij} + \frac{1}{2} \phi^m \phi^h g_{ij} - \frac{1}{2} (\eta_j \eta_{ij}) k. \]

By virtue of (1.8), we have

\[ g_{ijk} = - \frac{1}{2} g_{km} \phi^i \phi^j + \frac{1}{2} g_{km} \phi^j \phi^i - \frac{1}{2} (\eta_j \eta_{ij}) k \]

\[ = - \frac{1}{2} (g_{km} \phi^i + \eta_j \eta_{ij}) k = - \frac{1}{2} g_{ij} = 0. \]

Then, as \( \xi^i \) is a unit vector field, we get

\[ 0 = (g_{ij} \xi^i \xi^j)_{ij} = g_{ij} \xi^i \xi^j + g_{ij} \xi^i \xi^j = 2 \lambda. \]

So we have

\[ \xi^i \lambda = 0, \quad \eta_{ij} = 0. \]

Hence, we get the following

**Theorem 11.** On a manifold with \((\phi, \xi, \eta, g)\)-structure we can find an affine connection which leaves \( \Phi^i, \xi^i, \eta \) and \( g_{ij} \) covariant constant.

**5. Symmetric \((\phi, \xi, \eta)\)-connections.** In this section, we study the condition for the existence of symmetric \((\phi, \xi, \eta)\)-connections. We begin with the following lemma.

**Lemma 2.** On the manifold admitting vector fields \( \xi^i, \eta_j \) which satisfy the condition

\[ (5.1) \quad \xi^i \eta_j = 1, \]

there exists a symmetric affine connection which leaves \( \xi^i \) and \( \eta_j \) covariant constant, if and only if \( \eta_j \) is a gradient (i.e., \( \eta \) is closed).

**Proof.** The necessity is trivial. So we prove the sufficiency. By virtue of Ishihara and Obata's theorem, we can find a symmetric affine connection which leaves \( \xi^i \) covariant constant. We denote the coefficients of this connection by \( \Gamma^i_{jk} \) and the operation of covariant differentiation by a vertical line \( | \) respectively. If we set
then \( \Gamma'_{jk} \) defines a symmetric affine connection, as \( \eta_j \) is a gradient by assumption. If we denote the covariant differentiation with respect to the latter connection by a comma, we get
\[
\xi_j'_{jk} = \xi_j'_{jk} + \xi_j \eta_{j|k},
\]
\[
\eta_{j|k} = \eta_{j|k} - \xi_j \eta_{j|k} \eta_i = 0.
\]
So the connection defined by \( \Gamma'_{jk} \) satisfies the condition stated above. Q.E.D.

By virtue of Lemma 2, we have the following

**THEOREM 12.** Let \( M^{2n+1} \) be a manifold with \((\phi, \xi, \eta)\)-structure. If \( \eta_j \) is a gradient and \( N_j = 0 \), then we can find a \((\Phi, \zeta, \eta)\)-connection whose torsion tensor is equal to \( \frac{1}{8} N_{ijk} \).

**PROOF.** Since \( \eta_j \) is a gradient, we can find a symmetric affine connection such that
\[
\xi_j'_{jk} = 0, \quad \eta_{j|k} = 0.
\]
We denote its coefficients and covariant differentiation by \( \Gamma'_{jk} \) and a comma respectively. Since \( N_j = 0 \) can be written as
\[
\xi_j \phi'_{j,k} - \phi'_j \xi_{j,k} - \xi_k \phi'_{j,k} = 0,
\]
it is transformed to
\[
\xi \phi'_{j,k} = 0.
\]
On the other hand, from (1, 3), (1.4) and (1.5) we get the following relations.
\[
\phi'_{j,k} \xi' = 0, \quad \phi'_{j,k} \eta_i = 0, \quad (\phi' \phi')_{j,k} = 0.
\]
If we set
\[
\Gamma'_{jk} = \Gamma_{jk} + T'_{jk}
\]
where
\[
T'_{jk} = - \frac{1}{4} \left| \phi'_{(j,k)} - \phi'_{(j,k)} + \phi(\phi_{j,k} + \phi_{j,k}) \right|
\]
and denote the covariant differentiation with respect to the connection defined by \( \Gamma'_{jk} \) by \( ; \), then we get
\[ \Phi_{jk} = \Phi_{j,k} - \frac{1}{4} \{ \Phi_{h}(\Phi_{k,i} - \Phi_{i,k}) + \Phi_{i}(\Phi_{k,j} + \Phi_{j,k}) \} \Phi_{i} \]
\[ + \frac{1}{4} \{ \Phi_{i}(\Phi_{k,i} - \Phi_{k,j}) + \Phi_{j}(\Phi_{i,j} + \Phi_{j,k}) \} \Phi_{i} \]
\[ = \Phi_{j,k} - \frac{1}{4} (\Phi_{k,i} - \Phi_{i,k})(- \delta^{i}_{j} + \xi^{i}_{\eta}) \]
\[ + \frac{1}{4} \Phi^{i}_{j,k} + \Phi^{i}_{k,i} \Phi_{i} \]
\[ + \frac{1}{4} \Phi^{i}_{j,k} + \Phi^{i}_{k,i} \Phi_{i} \]
\[ = \frac{1}{2} \Phi_{j,k} + \frac{1}{4} \Phi^{i}_{j,k} \Phi_{i} + \frac{1}{4} \Phi^{i}_{k,i} \Phi_{i} \]
\[ - \frac{1}{4} \Phi^{i}_{j,k} \Phi_{i} + \frac{1}{4} \Phi^{i}_{k,i} \Phi_{i} \]
\[ = \frac{1}{2} \Phi_{j,k} + \frac{1}{4} \Phi^{i}_{j,k} (- \delta^{i}_{j} + \xi^{i}_{\eta}) + \frac{1}{4} \Phi^{i}_{k,i} (- \delta^{i}_{j} + \xi^{i}_{\eta}) \]
\[ = 0, \]
\[ \xi^{i}_{j,k} = \xi^{i}_{j,k} - \frac{1}{4} \{ \Phi_{h}(\Phi_{k,i} - \Phi_{i,k}) + \Phi_{i}(\Phi_{k,j} + \Phi_{j,k}) \} \Phi_{i} \]
\[ = 0, \]

and so by Theorem 8
\[ \eta_{j,k} = 0 \]

The torsion tensor of this connection is easily seen to be given by
\[ S_{jk}^{i} = \frac{1}{2} (T_{jk}^{i} - T_{k,j}^{i}) \]
\[ = \frac{1}{8} \{ \Phi^{i}_{j}(\Phi_{k,i} - \Phi_{i,k}) - \Phi^{i}_{k}(\Phi_{j,i} - \Phi_{i,k}) \} \]

(5.8) \[ S_{jk}^{i} = \frac{1}{8} N_{jk}^{i} \]

Q.E.D.

By virtue of Theorem 12 and Theorem 6, we get
COROLLARY. On a manifold with \((\phi, \xi, \eta)\)-structure, we can find a symmetric \((\phi, \xi, \eta)\)-connection if and only if the following conditions are satisfied:

i) \(\eta_i\) is a gradient,

ii) \(N_{jk} = 0\).

REMARK. If a \((\phi, \xi, \eta)\)-structure is the one defined by a contact structure, then \(\eta_i\) is not a gradient. So, in this case, there exists no symmetric \((\phi, \xi, \eta)\)-connection.

BIBLIOGRAPHY
