SURFACES OF GAUSSIAN CURVATURE ZERO IN EUCLIDEAN 3-SPACE.

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1. Introduction. Books on the classical differential geometry of surfaces in 3-space usually prove a theorem to the effect that a surface of Gaussian curvature 0 is a developable surface or torse. To be more precise, the following two statements are proved: (a) If every point on a surface of Gaussian curvature 0 is a flat point[1] then the surface is a piece of a plane. (b) If no point on a surface of Gaussian curvature 0 is a flat point, then through every point there is a unique asymptotic line, and the tangent plane is constant along this line.[2]

Apparently all such books neglect completely the case of a surface of curvature zero which has both flat and non-flat points. This seems rather strange in view of the fact that many obvious examples illustrate this case. Perhaps most classical differential geometers felt that this case was too complicated and the possibilities were too numerous to obtain interesting results. The following quotation from the footnote to a paper[3] written by the late Professor A. Wintner in 1955 illustrates this attitude:

"Certain difficulties inherent to Euler's definition of a torse are known since Lebesgue's thesis (1902). But it may not be necessary to go such extremes as Lebesgue went (continous but not one-to-one parametrizations) in order to show that the theory of torses is not as simple as it appears from the texts of differential geometry, including the rigor-conscious books. For is it true that if [the Gaussian curvature] $K \equiv 0$ on a surface S of class C^2, then a neighborhood of every point of S can be "ruled" so as to be a torse in Euler's sense also? I can neither prove nor believe this, not even under the assumption that S is of class C^∞, which in view of the possibility of clustering zeros of [the second fundamental form] H (i. e. of flat points where $H^2=0=K$) is hardly stronger than S being of class C^2.…… A counter example, with S of class C^∞, would be the first such instance in the differential geometry of surfaces as to require the full force of (function-theoretical) analyticity —— rather than just C^∞-character."

It is the purpose of this paper to show that interesting and significant results *can* be obtained on surfaces of curvature 0 in the case where there are both flat

1) That is, a point where all the coefficients of the second fundamental form vanish. Such an umbilic point is also called a planar point.

2) For a precise statement and proof of this result under minimum assumptions of differentiability, see theorem (V) of [2].

3) This footnote occurs on p. 355 of [5].

and non-flat points. Strangely enough, in this situation the most striking result and the one that is easiest to state is of a global nature:

THEOREM I. *A complete surface of Gaussian curvature* 0 *in Euclidean* 3-*space is a cylinder.*

As usual, by a "cylinder" is meant the surface generated by the set of lines parallel to the z-axis through a curve in the xy-plane; since the surface is complete, the curve must either be closed or of infinite length. This theorem was announced by A. V. Pogorelov[4] without proof in 1956 under very general hypotheses. A paper by Hartman and Nirenberg [1] contains among other things a proof of this theorem for n-dimensional hypersurfaces of class C^2 in Euclidean $(n+1)$-space. Below we give an elementary proof in the spirit of classical differential geometry for surfaces of class C^4.

If the hypothesis of completeness is omitted, the situation is more complicated. Let S be a surface of Gaussian curvature 0 in 3-space (S need not be complete), let A denote the set of all flat points of S, and $U = S - A$ the set of all non-flat points. It is clear that A is precisely the set of all points where the mean curvature vanishes, hence it is a closed subset of S; similarly, U is an open subset of S. The classical theory says nothing about the subsets A and U, their relationship to one another, etc. The following theorems show that the sets A and U must satisfy rather stringent requirements.

Recall that through each point of U there passes a unique asymptotic line according to the classical theory. These lines do *not* exist in A.

THEOREM II. *The asymptotic lines in U are "maximal" in the sense that they extend to infinity or to the "boundary" of S in each direction.*

Obviously, this imposes restrictions on U, since it is simply covered by this family of asymptotic lines, each of which is maximal.

For the statement of the next theorem, let U' denote the boundary of U in the topology of S.

THEOREM III. *Through each point of U' there passes a straight line on S, and this straight line is entirely contained in U'. Thus U' is the union of straight lines.*

It should be added that the tangent plane to S is constant along each component of U'. (To prove this, note that $U' \subset A$, hence the second funda-

4) See [3]; a review of this note is in *Math. Review.* vol. 19, p. 309. To the best of my knowledge, Pogorelov has not as yet published a proof. [NOTE ADDED IN PROOF. Since this paper was submitted. another proof of Theorem 1 has been given by J, J. Stoker in volume XIV, number 3 of "Communications on Pure and Applied Mathematics."]

mental form vanishes identically at each point of U'; then use the Weingarten equations.)

We give a proof of theorems II and III below for surfaces of class C^4. The proof of theorem II is given first; it depends on the simple observation that $1/M$, the reciprocal of the mean curvature, is a linear function of arc length along any asymptotic curve in U. Theorems I and III then follow by fairly easy arguments.

In retrospect, it seems amazing that these theorems about surfaces of Gaussian curvature 0 were not discovered many years ago. Looked at from a slightly different point of view, theorem I describes all possible isometric immersions of a complete 2-dimensional Riemannian manifold of curvature 0 in Euclidean 3-space. The analogous problem for complete surfaces of constant negative curvature was solved by Hilbert in 1901 with his famous theorem regarding the impossibilty of imbedding the hyperbolic plane in 3-space, while H. Liebmann in 1899 solved this problem for surfaces of constant positive curvature (spheres). Clearly these theorems of Liebmann and Hilbert are deeper than theorem I above, and not so plausible geometrically.

In this paper we shall use the notations of the book of Struik [4]. We will deal *exclusively* with surfaces of differentiability class C^4 and Gaussian curvature 0 in Euclidean 3-space. These assumptions will not be repeated.

2. Two Basic Lemmas. The two lemmas stated below are concerned with the behavior of the asymptotic curves in the open subset U of the surface S. Both are of a local nature. The first is undoubtedly well known; the statement and proof are included for the sake of completeness. The second is the key lemma which asserts that $1/M$ is a linear function of arc length along an asymptotic curve.

LEMMA 1. *Any asymptotic curve in U is a segment of a geodesic.*

LEMMA 2. *On any fixed asymptotic curve in U the mean curvature M satisfies the following differential equation:*

$$\frac{d^2}{ds^2}\left(\frac{1}{M}\right) = 0.$$

Here the derivatives are taken with respect to arc lengths along the asymptotic curve.

PROOFS. Given any point $p \in U$, we can choose a local coordinate system (u, v) in a sufficiently small neighborhood of p in U such that the asymptotic curves and their orthogonal trajectories are the v and u curves respectively. With this coordinate system, the first and second fundamental forms become (in the notation of Struik, [4])

$$I = Edu^2 + Gdv^2,$$
$$II = edu^2, \qquad e \neq 0,$$

i. e. $F = f = g = 0$. The Mainardi-Codazzi equations then become

$$(2.1) \qquad \begin{cases} e_v = \dfrac{eE_v}{2E} \\ 0 = \dfrac{eG_u}{2E} \end{cases}$$

(see Struik [4], p. 113).
From the second of these equations, we conclude that $G_u = 0$, hence G is a function of v alone. If we further suppose (as we may) that the parameter v measures arc length along some one of the curves $u =$ constant, then it is readily seen that $G = 1$, and the first fundamental form becomes

$$I = Edu^2 + dv^2.$$

It follows that v measures arc length along any curve $u =$ constant. The Gaussian curvature is given by

$$K = -\frac{1}{\sqrt{E}} \frac{\partial^2 \sqrt{E}}{\partial v^2}$$

(see Struik [4], p. 113, equation (3-7)), hence

$$(2.2) \qquad \frac{\partial^2 \sqrt{E}}{\partial v^2} = 0.$$

Next, we apply equation (1-10) on p. 130 of Struik [4] to compute the geodesic curvature κ_g of the curves $u =$ constant; the result is that $\kappa_g = 0$, hence the curves $u =$ constant are geodesics. This completes the proof of lemma 1.

It follows from equation (2. 2) that

$$(2.3) \qquad \sqrt{E} = c_1(u) \cdot v + c_2(u)$$

where $c_1(u)$ and $c_1(u)$ are functions of u. From the first equation in (2. 1) it follows that

$$(2.4) \qquad e = c_3(u) \cdot \sqrt{E}, \qquad c_3(u) \neq 0.$$

In our coordinate system,

$$M = \frac{e}{2E}$$

by equation (7-2) on p. 83 of Struik, [4]. Therefore

$$
\begin{aligned}
M &= \frac{c_3(u)\sqrt{E}}{2E} = \frac{c_3(u)}{2\sqrt{E}} \\
&= \frac{c_3(u)}{2[c_1(u)\cdot v + c_2(u)]}.
\end{aligned} \tag{2. 5}
$$

Now if we restrict attention to a single asymptotic curve $u = u_0$ and recall that v measures arc length along this asymptotic curve, the proof of lemma 2 follows easily by differentiation of formula (2. 5).

3. Proof of Theorem II. From lemma 2 it follows that along any fixed asymptotic curve in U the mean curvature M is given by

$$M = \frac{1}{as + b} \tag{3. 1}$$

where a and b are constants and s denotes the arc length measured from a fixed point p_0 on the asymptotic. Assume now that a certain asymptotic curve C is not maximal i. e., it is only a proper segment of a certain geodesic L and not the entire geodesic. This means that the geodesic L meets the set A. Let p be a point on L which is a boundary point of A and U. Then $M(p) = 0$, and hence we must have

$$\lim_{s \to s_0}\left(\frac{1}{as + b}\right) = 0,$$

where s_0 is the distance from p to p_0 along L. But this is impossible; hence C must be the entire geodesic L.

4. Proof of Theorem I. In case the surface is complete, every geodesic, and hence every asymptotic curve, can be extended infinitely far in both directions. Since the mean curvature remains finite on all of S, one must have $a = 0$ in formula (3. 1), i. e., the mean curvature M constant along any geodesic.

Next, recall that a *complete* surface S of Gaussian curvature 0 has the Euclidean plane as its universal covering space. Hence any such surface S is obtained by an isometric immersion of the Euclidean plane in Euclidean 3-space. Moreover, any geodesic on S is the image of a straight line in the plane. Since any asymptotic curve on S is an entire geodesic, and no two asymptotic curves intersect, it follows that the asymptotic curves are all images of lines in the plane which are parallel. Choose a rectangular coordinate system (u, v) in the Euclidean plane such that the lines u = constant (parallel to the v-axis) include all lines which map onto the asymptotic curves. Since the coordinate system is rectangular, the first fundamental form of S becomes with this parametrisation,

$$ds^2 = du^2 + dv^2, \tag{4.1}$$

and the argument used in proving lemmas 1 and 2 shows that the second fundamental form may be written

$$II = edu^2 \tag{4.2}$$

in the entire plane. Since

$$M = \frac{1}{2}e$$

and M is constant along any asymptotic curve, we see that e is a function of u alone.

Now by using the uniqueness part of the classical theorem of Bonnet (see Struik, [4], p. 124), one sees that any surface whose first and second fundamental forms are given by (4.1) and (4.2) with e a function of u alone is congruent to the cylinder generated by lines parallel to the z-axis through a curve in the xy-plane; here the parameter u is the arc length of the given curve and the function $e(u)$ is its curvature[5].

5. Proof of Theorem III. Let $p \in U'$ be any boundary point of the set U, and let $\{p_n\}$ be an infinite sequence of points in U such that $p_n \to p$ as $n \to \infty$. For each point p_n, let $C(p_n)$ denote the unique asymptotic line through p_n. We assert that as $n \to \infty$ the lines $C(p_n)$ approach a limiting line, which we will denote by $C(p)$.

To prove this assertion, draw a line L through p such that L intersects $C(p_n)$ for all sufficiently large n and let θ_n denote the angle between the lines L and $C(p_n)$. It is readily seen from the geometry of the situation that $\{\theta_n\}$ is a Cauchy sequence, because no two of the lines $C(p_n)$ can intersect on S. Hence $\lim \theta_n$ exists as $n \to \infty$, and this determines the direction of the line $C(p)$ through p.

A similar argument shows that $C(p)$ is independent of the choice of the sequence $\{p_n\}$. Hence $C(p)$ is uniquely determined.

Next, we assert that every point of $C(p)$ on S is a boundary point of U. First of all, if q is any point on $C(p)$, then it is possible to choose for each integer n a point $q_n \in C(p_n)$ such that $q_n \to q$ as $n \to \infty$. Thus q belongs to the closure of U. To complete the proof of this assertion, it suffices to show

5) Bonnet's theorem is usually stated as a local theorem. However it applies equally well to the determination of isometric immersions in the large of simply connected 2-dimensional Riemannian manifolds in Euclidean 3-space. The immersion is determined up to a rigid motion by the second fundamental form, which is assumed to satisfy the Gauss-Codazzi equations. The passage from the local to the global formulation is accomplished by a process of "analytic continuation" along paths, entirely analogous to the process used in classical complex function theory.

that q does not belong to U. If, on the contrary, q belonged to U, then it would follow by continuity that $C(p)$ was the unique asymptotic line through q, and hence that $p \in U$, a contradiction.

This completes the proof of theorem III.

BIBLIOGRAPHY

[1] HARTMAN AND L. NIRENBERG, On spherical image maps whose Jacobians do not change signs. Amer. Jour. Math., 81(1959), 901–920.

[2] P. HARTMAN AND A. WINTNER, The fundamental equations of differential geometry, Amer. Jour. Math., 72(1950), 757–774.

[3] A. V. POGORELOV, Extensions of the theorem of Gauss on spherical representation to the case of surfaces of bounded extrinsic curvature, Dokl, Akad, Nauk, SSSR (N. S.), 111(1956), 945–947 (Russian).

[4] D. J. STRUIK, Lectures on classical differential geometry. Addison-Wesley Press, 1950.

[5] A. WINTNER, On Frenet's equations, Amer. Jour. Math., 78(1956), 349–356.

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