ON THE SATURATION AND BEST APPROXIMATION

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Let $f(x)$ be an integrable function with period 2π and let its Fourier series be

$$a_0/2+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)=\sum_{k=0}^{\infty}A_k(x).$$

Denote the method of typical means of this series by

$$R_n^{\lambda}(f)=\sum_{k=0}^{n-1}\left(1-\frac{k^{\lambda}}{n^{\lambda}}\right)A_k(x).$$

Then this method saturates with the order $n^{-\lambda}$, that is, we have

THEOREM A. *For the typical means,*

(1°) $f-R_n^{\lambda}(f)=o(n^{-\lambda})\Longleftrightarrow f=a$ constant,

(2°) $f-R_n^{\lambda}(f)=O(n^{-\lambda})\Longleftrightarrow f\in W^{\lambda}$,

where W^{λ} means the class of functions for which

$$\sum_{k=1}^{\infty}k^{\lambda}A_k(x)\sim f^{\lambda}\in L^{\infty}(0,2\pi).$$

See Aljančić [1], Sunouchi [3] Sunouchi-Watari [4]. Recently Aljančić [2] proved the following the orm.

THEOREM B. *Let $k=0,1,\ldots\ldots$ and $0<\alpha\leqq 1$. Then*

$$f^{(k)}(x)\in {}^2\Lambda_\alpha(k+\alpha<\lambda)\Longleftrightarrow f-R_n^{\lambda}(f)=O(n^{-k-\alpha}),$$

where $f^{(k)}(x)\in {}^2\Lambda_\alpha$ means

$$f^{(k)}(x+h)+f^{(k)}(x-h)-2f^{(k)}(x)=O(|h|^{\alpha}).$$

However this fact is not confined to only the typical means, but also is valid for more general approximation processes. Indeed we can deduce Theorem B from Theorem A by method of the moving average.

THEOREM. *Let $k=0,1,2,\ldots\ldots$, and $0<\alpha\leqq 1$. Suppose that for linear approximation processes $T_n(f)$*

(1°) $|f(x)|\leqq M_1$ *implies* $|T_n(f)(x)|\leqq k_1M_1$,

and

(2°) $|f^{\lambda}(x)| \leqq M_2$ *implies* $|f(x) - T_n(f)(x)| \leqq k_2 M_2\ n^{-\lambda}$, *where* $n^{-\lambda}$ *is the best approximation of the class of functions*

$$f^{(k)}(x) \in {}^2\Lambda_\alpha;\ k + \alpha = \lambda,\ k \text{ is an integer},\ 0 < \alpha \leqq 1.$$

Then

$$f^{(k)}(x) \in {}^2\Lambda_\alpha,\ k + \alpha < \lambda \Longleftrightarrow f(x) - T_n(f)(x) = O(n^{-k-\alpha}).$$

Roughly speaking, this method yields the best approximation, whenever the order of the Lipschitz class is smaller than the order of saturation.

PROOF. It is sufficient to prove that $f^{(k)}(x) \in \Lambda_\alpha$, $(k + \alpha < \lambda)$ implies $f - T_n(f) = O(n^{-k-\alpha})$, because the converse part is evident from the best approximation (Zygmund [5], I, p. 119) and the first difference theorem can be transfered to the second difference theorem (Aljančić [2]).

We set $I_1(f)(x)$ the moving average of $f(x)$, that is

$$I_1(f)(x) = \frac{1}{2\delta}\int_{-\delta}^{\delta} f(x + t)dt$$

and

$$I_k(f)(x) = \frac{1}{(2\delta)^k}\int_{-\delta}^{\delta} I_{k-1}(f)(x + t)dt,\ k = 2, 3, \ldots\ldots .$$

At the beginning we suppose that λ is an integer. For simplicity we consider $\lambda = 3$. The proof for $\lambda = 1, 2, \ldots\ldots$ is principally the same.

Case 1. $k = 0$, $0 < \alpha \leqq 1$ and $f \in \Lambda_\alpha$.

Since

$$I_3(f)(x) = \{f_3(x + 3\delta) - 3f_3(x + \delta) + 3f_3(x - \delta) - f_3(x - 3\delta)\}/(2\delta)^3,$$

where $f_3(x)$ is the third primitive of $f(x)$, we have

$$\frac{d^3}{dx^3} I_3(f)(x) = \Delta_\delta^3 f(x)/(2\delta)^3$$

and $f(x)$ belonging to the class Λ_α,

$$\left|\frac{d^3}{dx^3} I_3(f)(x)\right| \leqq c_1 \delta^{\alpha-3}.$$

When $0 < \alpha < 1$, $\tilde{f}(x) \in \Lambda_\alpha$ and when $\alpha = 1$, $f(x) \in {}^2\Lambda_1$ which yields $\tilde{f}(x) \in {}^2\Lambda_1$ which yields $f(x) \in {}^2\Lambda_1$ (Zygmund [5], I, p. 121). Hence we get similarly

$$(*) \qquad \left|\frac{d^3}{dx^3} I_3(\tilde{f})(x)\right| \leqq c_2 \delta^{\alpha-3}.$$

On the other hand

$$I_3(f)(x) - f(x) = \frac{1}{(2\delta)^3}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \{f(x + t + u + v) - f(x)\}dtdudv$$

and

$$(^{**})\qquad |I_3(f)(x)-f(x)|\leqq c_3\delta^\alpha.$$

Hence, if we set

$$g(x)=f(x)-I_3(f)(x),$$

then

$$f(x)-T_n(f)(x)=I_3(f)(x)-T_n\{I_3(f)\}(x)+g(x)-T_n(g)(x).$$

From the hypothesis, (*) and (**),

$$|f(x)-T_n(f)(x)|\leqq k_2c_2\delta^{\alpha-3}n^{-3}+k_1c_3\delta^\alpha.$$

We set $\delta=\pi/n$ and

$$|f(x)-T_n(f)(x)|\leqq Cn^{-\alpha}\ (0<\alpha\leqq 1).$$

Case 2. $k=1,\ 0<\alpha\leqq 1,\ f'(x)\in\Lambda_\alpha.$

Applying Taylor's theorem to the fact $f'(x)\in\Lambda_\alpha$,

$$|\Delta_\delta^3 f|=O(\delta^{1+\alpha}).$$

In the same way as Case 1, we have

$$\left|\frac{d^3}{dx^3}I_3(f)(x)\right|\leqq d_1\delta^{\alpha-2},\quad \left|\frac{d^3}{dx^3}I_3(\tilde{f})(x)\right|\leqq d_2\delta^{\alpha-2}.$$

On the other hand

$$\begin{aligned}g'(x)&=I_3(f')(x)-f'(x)\\&=\frac{1}{(2\delta)^3}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\{f'(x+t+u+v)-f'(x)\}dtdudv\end{aligned}$$

and

$$|g'(x)|\leqq d_3\delta^\alpha.$$

Hence from the hypothesis and the result of Case 1, we get

$$\begin{aligned}&|f(x)-T_n(f)(x)|\\&\quad\leqq|I_3(f)(x)-T_n(I_3(f))|+|g(x)-T_n(g)|\\&\quad\leqq k_2d_2\delta^{\alpha-2}n^{-3}+Cd_3\delta^\alpha n^{-1}=Dn^{-(1+\alpha)},\end{aligned}$$

where $\alpha=\pi/n$.

Case 3. $k=2,\ 0<\alpha<1,\ f''(x)\in\Lambda_\alpha.$

In this case, α is fractional and

$$\left|\frac{d^3}{dx^3}I_3(f)(x)\right|\leqq e_1\delta^{\alpha-1},\quad \left|\frac{d^3}{dx^3}I_3(\tilde{f})(x)\right|\leqq e_2\delta^{\alpha-1}.$$

Moreover

$$g''(x)=I_3(f'')(x)-f''(x)$$

$$= \frac{1}{(2\delta)^3}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\{f''(x+t+u+v)-f''(x)\}dtdudv,$$
$$\leqq e_3\delta^{\alpha}.$$

Hence

$$\begin{aligned}&|f(x)-T_n(f)(x)|\\&\qquad\leqq |T_3(f)(x)-T_n(I_3(f))|+|g(x)-T_n(g)|\\&\qquad\leqq k_2\delta^{\alpha-1}n^{-3}+e_3\ D\delta^{\alpha}n^{-2}=En^{-(2+\alpha)}\end{aligned}$$

where $\delta=\pi/n$.

When λ is fractional, the proof may be done in the same idea. For simplicity we suppose $1<\lambda<2$. Then it is sufficient to prove $\alpha=1$ and $1<\alpha<\lambda$. If we can prove these cases, another cases will be proved by method of the moving average (Zygmund [5], I, p. 117).

Case 1. $\alpha=1,\ f\in\Lambda_1,\ \ 1<\lambda<2.$

Since

$$I_2(f)(x)=\{f_2(x+2\delta)-2f_2(x)+f_2(x-2\delta)\}/(2\delta)^2,$$

we have

$$\frac{d^{\lambda}}{dx^{\lambda}}I_2(f)(x)=\frac{1}{(2\delta)^2}\{f_{2-\lambda}(x+2\delta)-2f_{2-\lambda}(x)+f_{2-\lambda}(x-2\delta)\}.$$

$|f'(x)|\leqq M$ implies $f'_{2-\lambda}(x)\in\Lambda_{2-\lambda}$ (Zygmund [5], II, p. 136), and

$$\left|\frac{d^{\lambda}}{dx^{\lambda}}I_2(f)(x)\right|\leqq l_1\delta^{1-\lambda}.$$

Since $2-\lambda$ is fractional, $\tilde{f}'_{2-\lambda}(x)\in\Lambda_{2-\lambda}$ and

$$\left|\frac{d^{\lambda}}{dx^{\lambda}}I_2(\tilde{f})(x)\right|\leqq l_2\delta^{1-\lambda}.$$

$I_2(f)(x)\in W^{\lambda}$ with the constant $l_3\delta^{1-\lambda}$.

On the other hand

$$|g(x)|=|f(x)-I_2(f)(x)|\leqq l_4\delta.$$

Hence

$$\begin{aligned}&|f(x)-T_n(f)(x)|\\&\qquad\leqq |I_2(f)-T_n(I_2(f))|+|g-T_n(g)|\\&\qquad\leqq k_2l_3\delta^{1-\lambda}n^{-\lambda}+k_1l_4\delta\leqq Ln^{-1},\end{aligned}$$

where $\delta=\pi/n$.

Case 2. $k=1,\ 1<1+\alpha<\lambda<2,\ f'(x)\in\Lambda_{\alpha}.$

In this case $f'_{2-\lambda}(x)\in\Lambda_{2-\lambda+\alpha}$, because $0<2-\lambda+\alpha<1$ and $f'(x)\in\Lambda_{\alpha}$ (Zygmund [5], II, p. 136).

Hence

$$\left|\frac{d^\lambda}{dx^\lambda} I_2(f)(x)\right| \leqq m_1\delta^{1-\lambda+\alpha}$$

$$\left|\frac{d^\lambda}{dx^\lambda} I_2(\tilde{f})(x)\right| \leqq m_2\delta^{1-\lambda+\alpha}$$

and $I_2(f) \in W^\lambda$ with the constant $m_3\delta^{1-\lambda+\alpha}$.
Moreover

$$|g'(x)| \leqq m_4\delta^\alpha.$$

Hence we have

$$\begin{aligned} &|f(x) - T_n(f)(x)| \\ &\quad \leqq k_2 m_3 \delta^{1-\lambda+\alpha} n^{-\lambda} + L n^{-1} m_4 \delta^\alpha \\ &\quad = M n^{-(1+\alpha)} \end{aligned}$$

where $\delta = \pi/n$.
Thus we proves the theorem completely.

Applying this, we may deduce Theorem B from Theorem A. An easy corollary is the following.

COROLLARY. *Denote $\sigma_n^r(f)$ the n-th Cesàro means of the r-th order $(0 < r < \infty)$, then*

(1) $f - \sigma_n^r(f) = o(n^{-1}) \Longleftrightarrow f =$ *a constant,*

(2) $f - \sigma_n^r(f) = O(n^{-1}) \Longleftrightarrow \widetilde{f'} \in L^\infty(0, 2\pi)$

(3) $f - \sigma_n^r(f) = O(n^{-\alpha}) \Longleftrightarrow f \in \Lambda_\alpha (0 < \alpha < 1)$.

(1) and (2) is the saturation theorem (Sunouchi-Watari [4]).

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