ON SOME REPRESENTATIONS OF C^*-ALGEBRAS

MASAMICHI TAKESAKI

(Received November 5, 1962)

Introduction. Let $\pi = \int_\Gamma \pi_\gamma d\mu(\gamma)$ be an irreducible decomposition of a representation π of a C^*-algebra $\boldsymbol{M}$ over a measure space (Γ, μ). Since traditionally we identify the representations within unitary equivalence, it comes into consideration whether the above decomposition can be regarded as a decomposition of the unitary equivaience class of π into the unitary equivalence class of $\pi(\gamma)$. Besides it is desirable from the view point of the duality that the decomposition can be constructed upon the structures which are completely determined by only $\boldsymbol{M}$.

So G.W.Mackey, in [14], introduced the concept of the Borel structure in the dual space of the separable C^*-algebra and that of the separable locally compact group, in order to describe the behavior of the representations, especially the decompositions, on the dual space and this trying succeeded for the so-called C^*-algebras of type I with some well behaved (i.e., smooth) dual spaces. However we can not avoid some measure theoretic pathology for the C^*-algebras having rather badly behaved dual spaces.

Recently, the dual space of a C^*-algebra has been studied successively by several authors: J.Dixmier [2, 3], J.M.G. Fell [6, 7, 8], J.Glimm [9], M.A. Guichardet [10, 11], J.Tomiyama [26] and J.Tomiyama-M. Takesaki [25]. Among them, J.Glimm [9] obtained the extremally fruitful results for the relation between the dual space and the structure of a separable C^*-algebra, which says that a separable C^*-algebra has the well behaved (smooth) dual space if and only if it has only representations of type I, and that a separable C^*-algebra is of type I if and only if it is GCR-algebra.

It seemed to be the second step to see what happens for the representations of C^*-algebras with *badly bahaved* dual spaces. M.A.Guichardet [10] showed that the representations of type I behave well in their irreducible direct integral decompositions. Moreover, in [11], he gave an example of the C^*-algebra having disjoint factor representations of type II_1 with same kernels.

Thus the present paper is devoted to show that *for a separable C^*-algebra with badly behaved dual space there exists a continuum family of disjoint factor representations of type* II *and of type* III *respectively with the same kernel and moreover there are representations θ^i of type i, $i =$* II, III, *with the following properties: there exist irreducible direct integral decomposition of θ^i,*

$$\theta^{\mathrm{II}} = \int_\Gamma \theta^{\mathrm{II}}_\gamma d\mu^{\mathrm{II}}(\gamma) \qquad \textit{and} \quad \theta^{\mathrm{III}} = \int_\Gamma \theta^{\mathrm{III}}_\gamma d\mu^{\mathrm{III}}(\gamma)$$

where $\theta^{\mathrm{II}}_\gamma$ *and* $\theta^{\mathrm{III}}_\gamma$ *are unitarily equivalent for every* $\gamma \in \Gamma$ *and for any null subset* N *of* Γ *there exists some pair of* $\gamma, \gamma' \notin N$ *such as* $\theta^i_\gamma \simeq \theta^i_{\gamma'}$, *and* $\mu^i(\{\gamma \mid \theta^i_\gamma \simeq \theta^i_{\gamma_0}\ \gamma \in \Gamma\}) = 0$ *for each* $\gamma_0 \in \Gamma$.

Before going into discussion the author must express his thanks to Mr. J. Tomiyama for many conversations with him in the presentation of this paper.

Preliminary. Let $\boldsymbol{M}$ be a separable C^*-algebra with the unit element and φ a representation of $\boldsymbol{M}$ on a separable Hilbert space $\mathfrak{H}$. Let ξ_0 be a generating (unit) vector of $\mathfrak{H}$ for $\varphi(\boldsymbol{M})$, that is, $[\varphi(\mathbf{M})\xi_0] = \mathfrak{H}$ where $[E]$ means the closed subspace generated by E for any subset E of $\mathfrak{H}$. Then ξ_0 is a separating vector for $\varphi(\boldsymbol{M})$,' the commutant of $\varphi(\boldsymbol{M})$. Let $\boldsymbol{A}$ be a commutative C^*-subalgebra of $\varphi(\boldsymbol{M})'$ whose spectrum space is Γ. If we define the linear functional $\omega_{\xi,\eta}$ on the full operator algebra $\mathbf{B}(\mathfrak{H})$ on $\mathfrak{H}$ by $\omega_{\xi,\eta}(x) = (x\xi, \eta)$, then $\omega_{\xi,\eta} \mid \boldsymbol{A}$, the restriction of $\omega_{\xi,\eta}$ on $\boldsymbol{A}$, defines a Radon measure $\mu_{\xi,\eta}$ in Γ and for every $\xi, \eta \in \mathfrak{H}$, $\mu_{\xi,\eta}$ is absolutely continuous with respect to μ_{ξ_0,ξ_0}. We fix the measure μ_{ξ_0,ξ_0} and denote it by μ. Suppose that $\boldsymbol{M}_0 = \{x_n\}$ is a countable dense self-adjoint subalgebra of $\boldsymbol{M}$ on the complex rational number field. Putting $\xi_n = \varphi(x_n)\xi_0$, $\{\xi_n\}$ forms a vector subspace $\mathfrak{H}'$ of $\mathfrak{H}$ over the complex rational numbers field. Let $h_{\xi,\eta}$ be the density of the measure $\mu_{\xi,\eta}$ with respect to μ. The countability of $\mathfrak{H}'$ implies the existence of a null set N_1 in Γ such that for $\gamma \notin N_1$ the function: $(\xi, \eta) \to h_{\xi,\eta}(\gamma)$ is a positive hermitian conjugate bilinear functional on $\mathfrak{H}' \times \mathfrak{H}'$. Let $\mathfrak{H}(\gamma)$ be the completion Hilbert space of the space $\mathfrak{H}'$ by the inner product $h_{\xi,\eta}(\gamma)$. Putting $N_2 = \{\gamma \notin N_1 \mid \mathfrak{H}(\gamma) = 0\}$, we can easily verify $\mu(N_2) = 0$. Let $T(\gamma)$ be the canonical mapping of $\mathfrak{H}'$ into $\mathfrak{H}(\gamma)$. Putting $T(\gamma)\xi = \xi(\gamma)$ for $\xi \in \mathfrak{H}'$ and $\gamma \notin N_1 \cup N_2$, $\{\xi_n(\gamma)\}$ becomes a dense subset of $\mathfrak{H}(\gamma)$. $\mathfrak{H}'$ determines uniquely the structure of measurable field on $\mathfrak{H}(\gamma)$ that contains $\mathfrak{H}'$. $\mathfrak{H}$ is represented as the direct integral $\int_\Gamma \mathfrak{H}(\gamma)\, d\mu(\gamma)$. In this direct integral each operator $\varphi(x)$ of $\varphi(\boldsymbol{M})$ is decomposable and $\boldsymbol{A}$ becomes the algebra of all continuously diagonalizable operators. By [1 : Chap. II, §2, Prop. 6, p. 163] the map $\varphi(x) \to \varphi(x)(\gamma)$ becomes a representation of C^*-algebra $\varphi(\boldsymbol{M})$ onto $\mathfrak{H}(\gamma)$ except for γ of some null set N_3 of Γ. Putting $\varphi_\gamma(x) = \varphi(x)(\gamma)$ for $x \in \boldsymbol{M}$ and $\gamma \notin N_3$, we get a direct integral decomposition $\int_\Gamma \varphi_\gamma d\mu(\gamma) = \varphi$. Moreover as in the proof of [1 : Chap.II, §2, Prop.6] φ_γ is given as the unique extension of uniformly continuous representation of $\boldsymbol{M}_0$. Now we shall explain briefly how φ_γ is given. Putting $\varepsilon_\varphi(x) = h_{\varphi(x)\xi_0,\xi_0}$ for each $x \in \boldsymbol{M}$, the function $\varepsilon_\varphi(x)$ belongs to $L^\infty(\Gamma, \mu)$ and the mapping ε_φ of $\boldsymbol{M}$ to $L^\infty(\Gamma, \mu)$ has the following properties: 1^0 $\varepsilon_\varphi(I) = I$: 2^0

$\mathcal{E}_\varphi(x^*x) \geqq 0$ for every $x \in \boldsymbol{M}$: 3^0 there exists a null set N_4 such that $\omega_\gamma^\varphi(x) = \mathcal{E}_\varphi(x)(\gamma)$ is a state for each $\gamma \notin N_4$: 4^0 if the weak closure of $\boldsymbol{A}$ is maximal abelian in $\varphi(\boldsymbol{M})'$, there exists a null set N_5 of Γ such that ω_γ^φ is pure for every $\gamma \notin N_5$. Since

$$\int a(\gamma)h_{\xi,\varphi(x)\eta}(\gamma)d\mu(\gamma) = \omega_{\xi,\varphi(x)\eta}(a) = (a\xi, \varphi(x)\eta)$$

$$= (\varphi(x)^*a\xi, \eta) = (a\varphi(x^*)\xi, \eta) = \int a(\gamma)h_{\varphi(x^*)\xi,\eta}\, d\mu(\gamma)$$

for every $a \in \boldsymbol{A}$, every $x \in \boldsymbol{M}$ and every pair of ξ, η of $\mathfrak{H}$, we have

$$\begin{aligned}(\xi_m(\gamma), \xi_n(\gamma)) = h_{\xi_m,\xi_n}(\gamma) &= h_{\varphi(x_m)\xi_0,\varphi(x_n)\xi_0}(\gamma) \\ &= h_{\varphi(x_n{}^*x_m)\xi_0\,\xi_0}(\gamma) = \omega_\gamma^\varphi(x_n{}^*x_m).\end{aligned}$$

for every $\gamma \notin \bigcup_{i=1}^{5} N_i$ and every $\xi_m, \xi_n \in \mathfrak{H}'$. Since $\varphi(a)\xi_n$ belongs to $\mathfrak{H}'$ for each $a \in \boldsymbol{M}_0$ and

$$\begin{aligned}((\varphi(a)\xi_n)(\gamma), \xi_m(\gamma)) = h_{\varphi(a)\xi_n,\xi_n}(\gamma) &= h_{\varphi(ax_n)\xi_0,\varphi(x_m)\xi_0}(\gamma) \\ &= \omega_\gamma^\varphi(x_m^*\, ax_n)\end{aligned}$$

for every $\gamma \notin \bigcup_{i=1}^{5} N_i$ and every $\xi_m, \xi_n \in \mathfrak{H}'$, the representation φ_γ of $\boldsymbol{M}$ is the cyclic representation induced by ω_γ^φ for almost every $\gamma \in \Gamma$. Thus we get the direct integral decomposition of $\mathfrak{H}$ and φ:

$$\mathfrak{H} = \int_\Gamma \mathfrak{H}(\gamma)d\mu(\gamma) \qquad \text{and} \qquad \varphi = \int_\Gamma \varphi_\gamma d\mu(\gamma).$$

If in the above discussion φ is the cyclic representation induced by a state σ and if ξ_0 is the cyclic vector corresponding to σ, we write $\mathcal{E}_\sigma$ and ω_γ^σ in place of $\mathcal{E}_\varphi$ and ω_γ^φ.

For each non-zero projection e of $\varphi(\boldsymbol{M})'$, putting $\varphi^e(x) = \varphi(x)e$ for $x \in \boldsymbol{M}$, we get a representation φ^e on $e\mathfrak{H}$ called the (non-trvial) subrepresentation of φ. If two representations φ and ψ have no unitary equivalent subrepresentations, then they are called disjoint representations. On the other hand, if they have no disjoint subrepresentations, we call them quasi-equivalent. By [13: Chap. I], $\infty \cdot \varphi$ and $\infty \cdot \psi$ are unitary equivalent if and only if they are quasi-equivalent, where $\infty \cdot \varphi$ means the product representation on $\mathfrak{H} \otimes \mathfrak{H}_\infty$, the direct product of $\mathfrak{H}$ and an infinite dimensional Hilbert space $\mathfrak{H}_\infty$, defined by φ and the trivial representation of scalar field on $\mathfrak{H}_\infty$. Hence φ and ψ are quasi-equivalent if and only if there is an isomorphism π from the weak closure of $\varphi(\boldsymbol{M})$ onto the one of $\psi(\boldsymbol{M})$ such that $\psi = \pi\circ\varphi$. As we can define the isomorphism from $\varphi(\boldsymbol{M})$ onto $\psi(\boldsymbol{M})$ by $\pi\varphi(x) = \psi(x)$ for $x \in \boldsymbol{M}$ for the representations φ and ψ with the same kernel, φ and ψ are quasi-equivalent if and only if π is σ-weakly continuous. Define the subspace of M^*, the conjugate space of M as Banach space, by

$$V_\varphi = \{{}^t\varphi(\rho),\ \rho \text{ is a } \sigma\text{-weakly continuous linear functional on the weak closure of } \varphi(M)\},$$

then V_φ is an invariant closed subspace of M^* in the sense that the functional $\sigma(a \cdot b)$ also belongs to V_φ for every $\sigma \in V_\varphi$ and $a, b \in M$. Then we can see that φ and ψ are quasi-equivalent if and only if $V_\varphi = V_\psi$. Moreover if φ is the cyclic representation induced by a state σ, then V_φ is the norm closure of the set $\left\{\sum_{i=1}^{n} \sigma(a_i \cdot b_i):\ a_i,\ b_i \in M\right\}$ which is denoted by V_σ.

1. Some examples. Let Γ be a compact space, G a discrete countable group of homeomorphisms of Γ. Let μ be a Borel measure defined on Γ with total measure 1. For each $\alpha \in G$ and Borel set S of Γ, the measure μ_α defined by $\mu_\alpha(S) = \mu(S\alpha)$ is also a Borel measure on Γ. Suppoee that μ is quasi-invariant under the action of G or equivalently μ and μ_α have the same family of null sets for every $\alpha \in G$. Putting $\rho(\gamma, \alpha) \equiv \frac{d\mu_\alpha}{d\mu}(\gamma)$, we have $\rho(\gamma, \alpha) > 0$, $\rho(\gamma, \epsilon) \equiv 1$ and $\rho(\gamma, \alpha\beta) \equiv \rho(\gamma\alpha, \beta) \cdot \rho(\gamma, \alpha)$ for every $\alpha, \beta \in G$, where ϵ means the unit of G. Let $\mathfrak{H}$ be the Hilbert space of all square summable functions over $\Gamma \times G$, that is,

$$\mathfrak{H} = \{\xi\,;\sum_{\alpha\in G}\int_\Gamma |\xi(\gamma, \alpha)|^2 d\mu(\gamma) = \|\xi\|^2 < +\infty\} = L^2(\Gamma, \mu)\otimes l^2(G).$$

Define the operators $l(a)$, $\hat{u}(\alpha)$, $\widehat{w}, r(a)$ and $\hat{v}(\alpha)$ on $\mathfrak{H}$ for $a \in L^\infty(\Gamma, \mu)$ and $\alpha \in G$ as follows:

$$[l(a)\xi](\gamma, \alpha) \equiv a(\gamma)\xi(\gamma, \alpha),\ [\hat{u}(\alpha_0)\xi](\gamma, \alpha) \equiv \rho(\gamma, \alpha_0)^{1/2}\xi(\gamma\alpha_0, \alpha\alpha_0),$$
$$[\widehat{w}\xi](\gamma, \alpha) \equiv \rho(\gamma, \alpha^{-1})^{1/2}\xi(\gamma\alpha^{-1}, \alpha^{-1}),\ [r(a)\xi](\gamma, \alpha) \equiv a(\gamma\alpha^{-1})\xi(\gamma, \alpha),$$
$$[\hat{v}(\alpha_0)\xi](\gamma, \alpha) \equiv \xi(\gamma, \alpha_0^{-1}\alpha).$$

As in [17], $\hat{u}$ and $\hat{v}$ are unitary representations of G on $\mathfrak{H}$ and $\widehat{w}^* = \widehat{w}^{-1} = \widehat{w}$, $\widehat{w}\hat{u}(\alpha)\widehat{w} = \hat{v}(\alpha)$ and $\widehat{w}l(a)\widehat{w} = r(a)$. Moreover we have

$$\hat{u}(\alpha)^{-1}l(a)\hat{u}(\alpha) = l(a^\alpha),\ [l(a)\hat{u}(\alpha)]^* = 1(a^{*\alpha})\hat{u}(\alpha^{-1}),$$
$$\hat{v}(\alpha)^{-1}r(a)\hat{v}(\alpha) = r(a^\alpha),\ [r(a)\hat{v}(\alpha)]^* = r(a^{*\alpha})\hat{v}(\alpha^{-1}),$$
$$\hat{u}(\alpha)r(a) = r(a)\hat{u}(\alpha),\ \hat{v}(\alpha)l(a) = l(a)\hat{v}(\alpha),$$
$$l(a)r(b) = r(b)l(a),\ \hat{u}(\alpha)\hat{v}(\beta) = \hat{u}(\beta)\hat{v}(\alpha)$$

for $a,\ b \in L^\infty(\Gamma,\ \mu)$ and $\alpha, \beta \in G$, where a^α is the function defined by $a^\alpha(\gamma) = a(\gamma\alpha^{-1})$. Now, the operators of the form $\sum_{k=1}^{n} l(a_k)\hat{u}(\alpha_k)$ for $a_k \in L^\infty(\Gamma,\ \mu)$

and $\alpha_k \in G, k = 1,\cdots, n$, constitute a $*$-algebra of bounded operators whose weak closure becomes the crossed product $L^\infty(\Gamma, \mu) \otimes G$ in the sense of [27]. Its commutant is the weak closure of the $*$-algebra consisting of the operators of the form $\sum_{k=1}^{n} r(a_k)\hat{v}(\alpha_k)$. As in [17], every element x of $L^\infty(\Gamma, \mu) \otimes G$ is expressed in the form $x = \sum_{\alpha \in G} l(x_\alpha)\hat{u}(\alpha)$ under the strongest operator topology on $\mathfrak{H}$. This construction of $L^\infty(\Gamma, \mu) \otimes G$ is discussed precisely in [17].

In the following we shall consider the C^*-subalgebra of $L^\infty(\Gamma, \mu) \otimes G$ which is the uniform closure of the $*$-algebra $C \odot G = \left\{ \sum_{k=1}^{n} l(a_k)\hat{u}(\alpha_k),\ a_k \in C(\Gamma) \text{ and } \alpha_k \in G,\ k = 1,\cdots, n \right\}$ where $C(\Gamma)$ is the algebra of all continuous functions of Γ. We denote this algebra by $C(\Gamma) \widehat{\otimes} G$ or $C \widehat{\otimes} G$ which becomes the uniform crossed product in the sense of [27].

LEMMA 1. *If G is the union of increasing sequence $\{G_n\}$ of finite groups, then the expression $x = \sum_\alpha l(x_\alpha)\hat{u}(\alpha)$ for $x \in C \widehat{\otimes} G$, is uniformly convergent.*

PROOF. Let $\mathfrak{H}_\alpha$ be the subspace of $\mathfrak{H}$ consisting of all elements in $\mathfrak{H}$ vanishing at $\beta \neq \alpha$ and e_α the projection onto $\mathfrak{H}_\alpha$. We have $\hat{u}(\alpha)e_\beta\hat{u}(\alpha)^{-1} = e_{\beta\alpha^{-1}}$. Since $\|a\| = \|l(a)\| = \|l(a)e_\alpha\|$ for every $a \in C$, we have $\|e_\alpha x e_\epsilon\| = \|l(x_\alpha)e_\alpha\hat{u}(\alpha)\| = \|x_\alpha\|$. Hence $\|x_\alpha\| \leqq \|x\|$, which implies $x_\alpha \in C$ for every $x \in C \widehat{\otimes} G$. Put $e(G_n) = \sum_{\alpha \in G_n} e_\alpha$, then $e(G_n)$ commutes with $x = \sum_{\alpha \in G_n} l(x_\alpha)\hat{u}(\alpha)$ and $\|x\| = \|xe(G_n)\|$. In fact, let

$$M_n = \{x \in C \widehat{\otimes} G : x = \sum_{\alpha \in G_n} l(x_\alpha)\hat{u}(\alpha)\}.$$

M_n is a C^*-subalgebra of $C \widehat{\otimes} G$ which is isomorphic to $C \widehat{\otimes} G_n$. Since $\hat{u}(\alpha)\mathfrak{H}_\beta = \mathfrak{H}_{\beta\alpha^{-1}}$, $\hat{u}(\alpha)$ commutes with $e(G_n)$ for $\alpha \in G_n$, so that $x \in M_n$ commutes with $e(G_n)$. Moreover, $e(G_n)x = 0$ implies $x = 0$ for $x \in M_n$, hence the mapping $x \in M_n \to xe(G_n)$ is an isomorphism. Take an arbitrary element x of M. There is an element x^0 of $C \odot G$ such that $\|x - x^0\| < \varepsilon$. Since x^0 is expressed as finite sum $\sum_{i=1}^{m} l(x^0_{\alpha_i})\hat{u}(\alpha_i)$, we can find a subgroup G_n such as $x^0 \in M_n$. Putting $x' = \sum_{\alpha \in G_n} l(x_\alpha)\hat{u}(\alpha)$ and $e = e(G_n)$, we have

$$\begin{aligned}\|x - x'\| &\leqq \|x - x^0\| + \|x^0 - x'\| = \|x - x^0\| + \|e(x^0 - x')e\| \\ &= \|x - x^0\| + \|e(x^0 - x)e\| < 2\varepsilon,\end{aligned}$$

because $exe = ex'e$. This implies the uniform convergence of the summation $\sum_\alpha l(x_\alpha)\hat{u}(\alpha)$.

Thus, we have the following formulas in $C \hat{\otimes} G$:

If $x = \sum_\alpha l(x_\alpha)\hat{u}(\alpha)$ and $y = \sum_\alpha l(y_\alpha)\hat{u}(\alpha)$, then

$$(x + y)_\alpha = x_\alpha + y_\alpha,\ (x^*)_\alpha = (x^*_{\alpha^{-1}})^{\alpha^{-1}} \text{ and } (xy)_\alpha = \sum_\beta x_\beta y_{\beta^{-1}\alpha}{}^{\beta^{-1}},$$

where the last summation is uniformly convergent over Γ by Lemma 1.

Now, put $\mu(a) = \int a(\gamma)\, d\mu(\gamma)$ and $\gamma(a) = a(\gamma)$, then obviously $\mu(a)$ and $\gamma(a)$ are states on C, besides γ is a pure state on C. For each $\sigma \in C^*$ (the dual space of C) and $x = \Sigma_\alpha\, l(x_\alpha)\hat{u}(\alpha) \in C \hat{\otimes} G$, define $\tilde{\sigma}(x) = \sigma(x_\epsilon)$: $\tilde{\sigma}$ is a state if σ is a state.

LEMMA 2. *If $\gamma \neq \gamma\alpha$ for every $\alpha \in G$, $\alpha \neq \epsilon$, then $\tilde{\gamma}$ is a pure state of $C \hat{\otimes} G$.*

PROOF. Let $\tilde{\varphi}_\gamma$ be the cyclic representation of $C \hat{\otimes} G$ defined by $\tilde{\gamma}$ and $\mathfrak{H}_{\tilde{\gamma}}$ the representation space. Let η be the canonical mapping of $C \hat{\otimes} G$ into $\varphi_{\tilde{\gamma}}$, that is,

$$\tilde{\varphi}_\gamma(a)\eta(x) = \eta(ax) \text{ for } x,\ a \in C \hat{\otimes} G,$$

$$\tilde{\gamma}(a) = (\tilde{\varphi}_\gamma(a)\xi_0,\ \xi_0) \text{ for } a \in C \hat{\otimes} G,$$

where $\xi_0 = \eta(I)$. Define the mapping u of $\eta(C \hat{\otimes} G)$ into $l^2(G)$ by

$$[u\eta(x)](\alpha) = x_\alpha(\gamma\alpha^{-1}) \qquad \text{for } x = \sum_\alpha l(x_\alpha)\hat{u}(\alpha).$$

Since $(x^*x)_\epsilon \sum = {}_\alpha(x^*)_\alpha x_{\alpha^{-1}}{}^{\alpha^{-1}} = \sum_\alpha (x^*_{\alpha^{-1}})^{\alpha^{-1}}(x_{\alpha^{-1}})^{\alpha^{-1}} = \sum_\alpha (x^*_\alpha x_\alpha)^\alpha$, we have

$$\tilde{\gamma}(x^*x) = \sum_\alpha (x^*_\alpha x_\alpha)^\alpha(\gamma) = \sum_\alpha (x^*_\alpha x_\alpha)(\gamma\alpha^{-1}) = \sum_\alpha |x_\alpha(\gamma\alpha^{-1})|^2.$$

It follows that u is an isometry of $\eta(C \hat{\otimes} G)$ into $l^2(G)$ which can be extended to the isometry of $\mathfrak{H}_{\tilde{\gamma}}$ into $l^2(G)$. For any function $f(\alpha)$ vanishing at all $\alpha \in G$ except finitely many elements, we define an element x of $C \hat{\otimes} G$ by $x = \Sigma_\alpha f(\alpha)\hat{u}(\alpha)$. Then $x_\alpha = f(\alpha) \cdot I$ and we have

$$[u\eta(x)](\alpha) = x_\alpha(\gamma\alpha^{-1}) = f(\alpha) \cdot I(\gamma\alpha^{-1}) = f(\alpha),$$

that is, $u\eta(x) = f$. Hence u is an isometry of $\mathfrak{H}_{\tilde{\gamma}}$ onto $l^2(G)$ and $u^{-1}f = \sum_\alpha f(\alpha)\eta[\hat{u}(\alpha)]$ for every $f \in l^2(G)$.

Therefore, we have

$$\begin{aligned} u\tilde{\varphi}_\gamma[\hat{u}(\alpha_0)]u^{-1}f &= u\tilde{\varphi}_\gamma[\hat{u}(\alpha_0)]\sum_\alpha f(\alpha)\eta[\hat{u}(\alpha)] \\ &= u\sum_\alpha f(\alpha)\eta[\hat{u}(\alpha_0)\hat{u}(\alpha)] = u\sum_\alpha f(\alpha)\eta[\hat{u}(\alpha_0\alpha)] \\ &= u\sum_\alpha f(\alpha_0^{-1}\alpha)\eta[\hat{u}(\alpha)] \qquad \text{for } f \in l^2(G), \end{aligned}$$

which implies

$$\{u\tilde{\varphi}_\gamma[\hat{u}(\alpha_0)]u^{-1}f\}(\alpha)=f(\alpha_0^{-1}\alpha).$$

Hence the representation $u\tilde{\varphi}_\gamma[\hat{u}(\cdot)]u^{-1}$ is the left regular representation of G.

Now for $a\in C$,

$$u\tilde{\varphi}_\gamma[l(a)]u^{-1}f=u\tilde{\varphi}_\gamma[l(a)]\cdot\sum_\alpha f(\alpha)\eta[\hat{u}(\alpha)]$$
$$=u\sum_\alpha f(\alpha)\eta[l(a)\hat{u}(\alpha)],$$

so that

$$\{u\tilde{\varphi}_\gamma[l(a)]u^{-1}f\}(\alpha)=a(\gamma\alpha^{-1})f(\alpha)\qquad\text{for } f\in l^2(G).$$

Now, for any pair of different elements α_1, α_2 in G, there is an $a\in C$ such as $a(\gamma\alpha_1^{-1})\neq a(\gamma\alpha_2^{-1})$ because $\gamma\alpha_1^{-1}\alpha_2\neq\gamma$. Hence the weak closure of $u\tilde{\varphi}_\gamma[l(C)]u^{-1}$ coincides with the algebra $l^\infty(G)$ consisting of all multiplication operators by bounded functions on G, which is a maximal abelian algebra in the full operator algebra on $l^2(G)$. If $x\in[u\tilde{\varphi}_\prime(C\mathbin{\widehat{\otimes}}G)u^{-1}]'$, x belongs to $l^\infty(G)$ and x is invariant under the left translations of G, hence x must be a scalar. That is, $u\tilde{\varphi}_\gamma u^{-1}$ is an irreducible representation of $C\mathbin{\widehat{\otimes}}G$. This completes the proof.

Put

$$\xi_0(\gamma,\alpha)=\begin{cases}1 & \alpha=\epsilon\\ 0 & \alpha\neq\epsilon.\end{cases}$$

We have, for $x=\sum_\alpha l(x_\alpha)u(\alpha)\in C\mathbin{\widehat{\otimes}}G$,

$$\Big(\sum_\alpha l(x_\alpha)\hat{u}(\alpha)\xi_0,\xi_0)=\sum_\alpha(l(x_\alpha)\hat{u}(\alpha)\xi_0,\xi_0)\Big)$$
$$=\sum_\alpha\sum_\beta\int_\Gamma[l(x_\alpha)\hat{u}(\alpha)\xi_0](\gamma,\beta)\overline{\xi_0(\gamma,\beta)}d\mu(\gamma)$$
$$=\sum_\alpha\int_\Gamma[l(x_\alpha)\hat{u}(\alpha)\xi_0](\gamma,\epsilon)d\mu(\gamma)$$
$$=\sum_\alpha\int_\Gamma x_\alpha(\gamma)[\hat{u}(\alpha)\xi_0](\gamma,\epsilon)d\mu(\gamma)$$
$$=\sum_\alpha\int_\Gamma x_\alpha(\gamma)\xi_0(\gamma\alpha,\alpha)\rho(\gamma,\alpha)^{1/2}d\mu(\gamma)$$
$$=\int_\Gamma x_\epsilon(\gamma)\,d\mu(\gamma)=\mu(x_\epsilon)=\tilde{\mu}(x).$$

That is,

$$\Big(\sum_\alpha l(x_\alpha)\hat{u}(\alpha)\xi_0,\xi_0\Big)=\tilde{\mu}(x).$$

Moreover we can easily show that ξ_0 is a generating vector of $\mathfrak{H}$ for $C \hat{\otimes} G$. Therefore, the identity representation of $C \hat{\otimes} G$ on $\mathfrak{H}$ may be identified with the cyclic representation $\varphi_{\tilde{\mu}}$ induced by $\tilde{\mu}$, so that we shall denote the identity representation of $C \hat{\otimes} G$ by $\tilde{\varphi}_\mu$ and $\mathfrak{H}$ by $\tilde{\mathfrak{H}}_\mu$.

Next, we take $r(C)$ as a commutative C^*-subalgebra of $\tilde{\varphi}_\mu(C \hat{\otimes} G)'$. We shall apply the arguments of the first paraphrase of the preliminary for $r(C)$ and the cyclic representation $\tilde{\varphi}_\mu$ of $C \hat{\otimes} G$. Then the spectrum space of $r(C)$ is Γ and the basic measure $\mu_{\xi_0\,\xi_0}$ on Γ coincides with the original measure μ. Under the notation of the preliminary, we have

LEMMA 3.

$$\varepsilon_{\tilde{\mu}}\Big[\sum_\alpha l(x_\alpha)\hat{u}(\alpha)\Big] = r(x_\epsilon) \qquad \textit{for all } x = \sum_\alpha l(x_\alpha)\hat{u}(\alpha).$$

PROOF.

$$\begin{aligned}\Big(\sum_\alpha l(x_\alpha)\hat{u}(\alpha)r(a)\xi_0,\ \xi_0\Big) &= \sum_\alpha (l(x_\alpha)r(a)\hat{u}(\alpha)\xi_0,\ \xi_0)\\ &= \sum_\alpha \int [l(x_\alpha)r(a)\hat{u}(\alpha)\xi_0](\gamma,\ \epsilon)d\mu(\gamma)\\ &= \sum_\alpha \int x_\alpha(\gamma)a(\gamma)[\hat{u}(\alpha)\xi_0](\gamma, \epsilon)d\mu(\gamma)\\ &= \sum_\alpha \int x_\alpha(\gamma)a(\gamma)\rho(\gamma, \alpha)^{1/2}\xi_0(\gamma\alpha, \alpha)d\mu(\gamma)\\ &= \int x_\epsilon(\gamma)a(\gamma)d\mu(\gamma) = (r(x_\epsilon)r(a)\xi_0, \xi_0),\end{aligned}$$

for all $a \in L^\infty(\Gamma, \mu)$. This implies the assertion.

From this lemma, we have

$$\omega_\gamma^{\tilde{\mu}}(x) = \varepsilon_{\tilde{\mu}}(x)(\gamma) = \gamma(x_\epsilon)(\gamma) = x_\epsilon(\gamma) = \tilde{\gamma}(x)$$

for $x = \sum_\alpha l(x_\alpha)\hat{u}(\alpha)$, that is, $\omega_\gamma^{\tilde{\mu}} = \tilde{\gamma}$ for every $\gamma \in \Gamma$. If we construct the direct integral decomposition of $\tilde{\varphi}_\mu$ with respect to $r(C)$ as follows:

$$\tilde{\mathfrak{H}}_\mu = \int \tilde{\mathfrak{H}}(\gamma)d\mu(\gamma) \qquad \text{and} \qquad \tilde{\varphi}_\mu = \int \tilde{\varphi}_\gamma d\mu(\gamma),$$

then the arguments in the preliminary imply that each component Hilbert space $\tilde{\mathfrak{H}}_{(\gamma)}$ is the cyclic Hilbert space constructed by $C \hat{\otimes} G$ and by the state $\tilde{\gamma}$ and that $\tilde{\varphi}_\gamma$ is the cyclic representation induced by $\tilde{\gamma}$. If $\gamma\alpha \neq \gamma$ for all $\gamma \in \Gamma$ and $\alpha \in G$, $\alpha \neq \epsilon$, then $r[L^\infty(\Gamma, \mu)]$ becomes a maximal abelian subalgebra of $\tilde{\varphi}_\mu(C \hat{\otimes} G)'$ and by Lemma 2, $\tilde{\varphi}_\gamma$ is irreducible for every $\gamma \in \Gamma$. Now we should notice that each irreducible component representation $\tilde{\varphi}_\gamma$ of the above direct integral decomposition of $\tilde{\varphi}_\mu$ does not depend on the choice of the measure μ, while the representation $\tilde{\varphi}_\mu$ depends completely on μ. Indeed, according to the choice of a measure μ, $\tilde{\varphi}_\mu$ becomes of type I, type II or type III whereas the component representations $\tilde{\varphi}_\gamma$'s remain unchanged.

In the following, we shall treat the special compact space Γ. Put $\Gamma_n = \{0, 1\}$, $\Gamma = \prod_{n=1}^\infty \Gamma_n$ equipped with weak topology, $\mu_n(\{0\}) = p$, $\mu_n(\{1\}) = 1 - p$ where $0 < p \leqq 1/2$ and $\mu = \bigotimes_{n=1}^\infty \mu_n (= \mu^p)$. Let G be the subset of Γ consisting of the elements $\alpha = (\alpha_n)$ where $\alpha_n = 0$ except for finitely many n's. We define the action of G on Γ by $(\gamma \cdot \alpha)_n \equiv \gamma_n + \alpha_n \pmod 2$. As in [19], μ becomes a quasi-invariant measure and the representations $\tilde{\varphi}_\mu$ induced by μ becomes of type II_1 and III according to $p = 1/2$ and $0 < p < 1/2$.

Next, we shall consider the infinite C^*-direct product of 2×2 matrix algebra. Let $\boldsymbol{M}_n$ be the 2×2 matrix algebra and $\{e^n{}_{i,j} : i, j = 0, 1\}$ the matrix units of $\boldsymbol{M}_n$. Let $\prod_{n=1}^\infty \hat{\otimes}_\alpha \boldsymbol{M}_n$ be the infinite C^*-direct product of $\boldsymbol{M}_n$ in the sense of [24]. In the following, we shall denote $\prod_{n=1}^\infty \hat{\otimes}_\alpha \boldsymbol{M}_n$ by $\boldsymbol{M}_0$. Putting $u_0^n = I$ and $u_1^n = e^n{}_{1,0} + e^n{}_{0,1}$, we define a unitary element $u(\alpha)$ for $\alpha \in G$ by $u(\alpha) = \prod \bigotimes_{n=1}^\infty u^n_{\alpha_n}$.
For $(\alpha_1, \cdots, \alpha_n)$ and $(\beta_1, \cdots, \beta_n)$ $\alpha_i, \beta_i = 0, 1$, define the element $w(\alpha_1, \cdots, \alpha_n : \beta_1, \cdots, \beta_n)$ by

$$w(\alpha_1, \cdots, \alpha_n : \beta_1, \cdots, \beta_n) = e^1_{\alpha_1\beta_1} \otimes e^2_{\alpha_2\beta_2} \otimes \cdots \otimes e^n_{\alpha_n\beta_n} \otimes I \otimes \cdots .$$

Let $\chi(\alpha_1, \cdots, \alpha_n)$ be the characteristic function of the subset of $\{\gamma \in \Gamma : \gamma_i = \alpha_i, i = 1, 2, \cdots, n\}$. Consider the mapping of $C_0 \odot G$ into $\boldsymbol{M}_0$ defined by

$$\pi[\tilde{u}(\alpha)] = u(\alpha) \text{ and}$$
$$\pi[l(\chi(\alpha_1, \cdots, \alpha_n))] = w(\alpha_1, \cdots, \alpha_n : \beta_1, \cdots, \beta_n),$$

where C_0 means the subalgebra of C that is the totality of finite linear combinations of $\chi(\alpha_1, \cdots, \alpha_n)$'s.

LEMMA 4. *π is an isometric isomorphism of $C_0 \odot G$ onto $\prod_{n=1}^\infty \odot \boldsymbol{M}_n$, the algebraic infinite direct product of $\boldsymbol{M}_n$. So π can be extended to the isomorphism* *of $C\hat{\otimes} G$ onto M_0. Moreover, we have*

$$ {}^t\pi^{-1}(\tilde{\gamma}) = \bigotimes_{n=1}^{\infty} \sigma^n_{\gamma_n} \text{ for } \gamma \in \Gamma \text{ and } {}^t\pi^{-1}(\mu) = \bigotimes_{n=1}^{\infty} [p\sigma^n_0 + (1-p)\sigma^n_1], $$

for the measure μ on Γ defined above, where σ^n_k means the state of M_n defined by $\sigma^n_k(e^n{}_{i,j}) = \delta^i_j\delta^k_i$.

The proof is essentially contained in von Neumann's paper [14 : pp. 71-77], so we omit the detail.

LEMMA 5. *Two irreducible representations φ_γ and $\varphi_{\gamma'}$ of $\mathbf{M}_0$ are unitarily equivalent if and only if $\gamma \equiv \gamma'$ mod G, where φ_γ means the cyclic representation induced by $\sigma_\gamma = \bigotimes_{n=1}^{\infty} \sigma^n_{\gamma_n}$.*

This is a reformulation of the statements of [9 : p.585].

LEMMA 6. *There is a continuum family of disjoint factor representation of M_0 of type i_∞, for $i = I, II, III$.*

PROOF. The existence of disjoint factor representations of type I is the conclusion of Lemma 5. So we shall give the families of representations of type II and III. Put $M^1 = \prod_{n=1}^{\infty} \hat{\otimes}_\alpha M_{2n}$ and $M^2 = \prod_{n=1}^{\infty} \hat{\otimes}_\alpha M_{2n-1}$, then we have naturally $M_0 = M^1 \hat{\otimes}_\alpha M^2$.

Define the state $\tilde{\mu}^1$ of M^1 and σ^2_γ of M^2 for $\gamma \in \Gamma$ by $\tilde{\mu}^1 = \bigotimes_{n=1}^{\infty} [p\sigma^{2n}_0 + (1-p)\sigma^{2n}_1]$ and $\sigma^2_\gamma = \bigotimes_{n=1}^{\infty} \sigma^{2n-1}_{\gamma_n}$ respectively. We denote the cyclic representations of M^1 and M^2 determined by $\tilde{\mu}^1$ and σ^2_γ by φ^1_μ and φ^2_γ respectively. Then φ^1_μ is a factor representation of type II_1 if $p = 1/2$ because $\tilde{\mu}^1$ is the trace of M^1 and φ^1_μ is of type III from [19] if $0 < p < 1/2$. By Lemma 5, $\varphi^2_\gamma \simeq \varphi^2_{\gamma'}$ if and only if $\gamma \equiv \gamma'$ mod G and φ^2_γ is irreducible. Putting $\psi_\gamma = \varphi^1_\mu \otimes \varphi^2_\gamma$, we see that ψ_γ becomes a factor representation by[1 : p. 103, Cor. of Prop. 14]. The type of ψ_γ is the same as the type of φ^1_μ by [1 : p. 111, Ex. 10 and p.250, Ex. 4] and [20].

Suppose ψ_γ and $\psi_{\gamma'}$ are quasi-equivalent. Then $\psi_\gamma | M^2$ and $\psi_{\gamma'} | M^2$ are quasi-equivalent. But $\psi_\gamma | M^2$(resp. $\psi_{\gamma'} | M^2$) is quasi-equivalent to φ^2_γ (resp. $\varphi^2_{\gamma'}$). Hence φ^2_γ and $\varphi^2_{\gamma'}$ are quasi-equivalent, so that they are unitary equivalent, which implies $\gamma \equiv \gamma'$ mod G by Lemma 5. Conversely, it is clear that ψ_γ and $\psi_{\gamma'}$ are unitarily equivalent if $\gamma \equiv \gamma'$ mod G. Therefore, the family $\{\psi_\gamma | \gamma \in \Gamma\}$ contains the required one if we assume that $p = 1/2$ or $0 < p < 1/2$ corresponding to the case of type II or type III. This completes the proof.

2. Consideration of general case and the main results. Now let $\boldsymbol{M}$ be an arbitrary separable $NGCR$-algebra in the sense of [9], that is, $\boldsymbol{M}$ has no non-zero GCR-ideal. Let $\{s_n : n = 1, 2, \cdots\}$ be a sequence of self-adjoint elements of $\boldsymbol{M}$ which is dense in the self-adjoint part of $\mathbf{M}$ and s_0 an arbitrary positive element of $\boldsymbol{M}$ of norm one. Let $\{v(\alpha_1, \cdots, \alpha_n) : \alpha_i = 0, 1\}$ and $(\lambda(\alpha_1, \cdots, \alpha_n : \beta_1, \cdots, \beta_n) : \alpha_i, \beta_i = 0, 1)$ be the system of elements of $\boldsymbol{M}$ and $2^n \times 2^n$-matrix as in [9: Lemma 4]. Put

$$e(n) = \sum_{\alpha_1, \cdots \alpha_n} v(\alpha_1, \cdots, \alpha_n) v(\alpha_1, \cdots, \alpha_n)^*$$

and

$$t_n = \sum_{\alpha_1 \cdots, \alpha_n : \beta_1, \cdots \beta_n,} \lambda(\alpha_1, \cdots, \alpha_n : \beta_1, \cdots, \beta_n)\ v(\alpha_1, \cdots, \alpha_n) v(\beta_1, \cdots, \beta_n)^*.$$

Let $\mathfrak{M}(n)$ be the subspace of $\boldsymbol{M}$ linearly spanned by $v(\alpha_1, \cdots, \alpha_n)v(\beta_1, \cdots, \beta_n)^*$'s and $\mathfrak{M}$ the subspace of $\boldsymbol{M}$ linearly spanned by $\mathfrak{M}(n)$'s. We have

LEMMA 7. *If two states σ and ρ of $\boldsymbol{M}$ have the value one at $e(n)$ for all n and if $\sigma | \mathfrak{M}(n) = \rho | \mathfrak{M}(n)$ for all n, then σ and ρ coincides each other.*

PROOF. By [9: Lemma 6], $\sigma(x) = \sigma(e(n)xe(n))$ and $\rho(x) = \rho(e(n)xe(n))$ for all n. Let a be an arbitrary self-adjoint element of $\boldsymbol{M}$. For each n there exists an integer $j > 2n$ such as $\|a - s_j\| < 1/2n$ and an element $t_j \in \mathfrak{M}(n)$ such as

$$\|e(j+1)(s_j - t_j)e(j+1)\| < 1/(2n+1).$$

Then we have

$$\begin{aligned}
|\sigma(a) - \rho(a)| &\leqq |\sigma(s_j) - \rho(s_j)| + 2\|a - s_j\| \\
&< |\sigma(e(j+1)s_j e(j+1)) - \rho(e(j+1)s_j e(j+1))| + 1/n \\
&\leqq |\sigma(e(j+1)t_j e(j+1)) - \rho(e(j+1)t_j e(j+1))| \\
&\qquad 2\|e(j+1)(s_j - t_j)e(j+1)\| + 1/n \\
&< |\sigma(t_j) - \rho(t_j)| + 2/(2n+1) + 1/n \\
&= 2/(2n+1) + 1/n \to 0 \ as\ n \to \infty.
\end{aligned}$$

Hence $\sigma = \rho$ on $\boldsymbol{M}$.

LEMMA 8. *Let σ be a state of $\boldsymbol{M}$ with $\sigma(e(n)) = 1$ for all n and θ_σ the cyclic representation on $\mathfrak{H}_\sigma$ induced by σ. Let $f_\sigma(n)$ be the projection onto [range of $\theta_\sigma(e(n))$] and put $f_\sigma = \inf_n f_\sigma(n)$. Define the mapping π_σ of $\prod_{n=1}^{\infty} \bigodot \boldsymbol{M}_n$ into $\theta_\sigma(\mathfrak{M})f_\sigma$ by*

$$\pi_\sigma\Big[\sum_{\alpha_1,\dots\alpha_n;\beta_1,\dots\beta_n}\lambda(\alpha_1,\cdots,\alpha_n:\beta_1,\cdots,\beta_n)w(\alpha_1,\cdots,\alpha_n:\beta_1,\cdots,\beta_n)]$$

$$=\sum_{\alpha_1,\dots\alpha_n;\beta_1,\dots\beta_n}\lambda(\alpha_1,\cdots,\alpha_n:\beta_1,\cdots,\beta_n)\theta_\sigma(v(\alpha_1,\cdots,\alpha_n)v(\beta_1,\cdots,\beta_n)^*)f_\sigma.$$

Then π_σ is an isometric isomorphism of $\prod_{n=1}^{\infty}\odot \boldsymbol{M}_n$ onto $\theta_\sigma(\mathfrak{M})f_\sigma$, and π_σ can be extended to the isomorphism of $\boldsymbol{M}_0$ onto the uniform closure of $\theta_\sigma(\mathfrak{M})f_\sigma$. Moreover if ξ_σ is the cyclic vector of $\mathfrak{H}_\sigma$, then π_σ is the cyclic representation induced by ${}^t\pi_\sigma(\omega_{\xi_\sigma,\xi_\sigma})=\mu_\sigma$.

PROOF. Since $\theta_\sigma(\mathfrak{M}(n))$ leaves $f_\sigma(k)\,\mathfrak{H}_\sigma$ invariant for $k\geqq n+1$ by [9: Lemma 5]. $\theta_\sigma(\mathfrak{M}(n))$ is reduced by f_σ. Moreover [9 : Lemma 5] shows us that $\theta_\sigma(\mathfrak{M}(n))f_\sigma$ is $2^n\times 2^n$-matrix algebra with matrix units $\{\theta_\sigma[v(\alpha_1,\cdots,\alpha_n)v(\beta_1,\cdots,\beta_n)^*]f_\sigma$: $\alpha_i,\beta_i=0$ or $1\}$. Hence the norm of

$$\sum_{\alpha_1,\dots,\alpha_n;\beta_1\dots\beta_n}\lambda(\alpha_1,\cdots,\alpha_n:\beta_1,\cdots,\beta_n)\theta_\sigma[v(\alpha_1,\cdots,\alpha_n)v(\beta_1,\cdots,\beta_n)^*]f_\sigma$$

is the same as the operator norm of the matrix

$$\lambda(\alpha_1,\cdots,\alpha_n:\beta_1,\cdots,\beta_n)$$

which is the norm in $\boldsymbol{M}_1\,\widehat{\otimes}_\alpha\cdots\widehat{\otimes}_\alpha\boldsymbol{M}_n$. Since the canonical imbedding of $\boldsymbol{M}_1\,\widehat{\otimes}_\alpha\cdots\widehat{\otimes}_\alpha\boldsymbol{M}_n$ into $\prod_{n=1}^{\infty}\widehat{\otimes}_\alpha\boldsymbol{M}_n=\boldsymbol{M}_0$ is isometric, we have

$$\Big\|\sum_{\substack{\alpha_1,\dots,\alpha_n\\ \beta_1,\dots,\beta_n}}\lambda(\alpha_1,\cdots,\alpha_n:\beta_1,\cdots,\beta_n)\theta_\sigma[v(\alpha_1,\cdots,\alpha_n)\ v(\beta_1,\cdots,\beta_n)^*]f_\sigma\Big\|$$

$$=\Big\|\sum_{\substack{\alpha_1,\dots\alpha_n\\ \beta_1,\dots\beta_n}}\lambda(\alpha_1,\cdots,\alpha_n:\beta_1,\cdots,\beta_n)w(\alpha_1,\cdots,\alpha_n:\beta_1,\cdots,\beta_n)\Big\|.$$

On the other hand, it is clear that π_σ preserves the algebraic operations, hence π_σ is an isometric isomorphism. Moreover $[\theta_\sigma(\boldsymbol{M})\xi_\sigma]=\mathfrak{H}_\sigma$ implies

$$[f_\sigma\theta_\sigma(\boldsymbol{M})f_\sigma\xi_\sigma]=f_\sigma[\theta_\sigma(\boldsymbol{M})\xi_\sigma]=f_\sigma\mathfrak{H}_\sigma.$$

Thus ξ_σ is a generating vector for $\pi_\sigma(\boldsymbol{M}_0)$ since $\theta_\sigma(\mathfrak{M})f_\sigma$ is uniformly dense in $f_\sigma\theta_\sigma(\boldsymbol{M})f_\sigma$, which implies the last half assertion.

THEOREM 1. *Let $\boldsymbol{M}$ be a separable NGCR-algebra, then there exists family $\{\theta^s_\gamma:\gamma\in\Gamma$ and $s\in S\}$ of factor representations of type i_∞, $i=$ II, III, such that $\sum_{s\in S}\oplus\,\theta^s_\gamma$ is faithful and the kernel of $\theta^s_\gamma=$ the kernel of $\theta^s_{\gamma'}$ for every $s\in S$ and every pair of $\gamma,\gamma'\in\Gamma$, while θ^s_γ and $\theta^s_{\gamma'}$ are quasi-equivalent if* *and only if* $\gamma \equiv \gamma'$ *mod* G.

PROOF. Let s_0 be an arbitrary positive element of $\boldsymbol{M}$ of norm one. Define the linear functional ρ_γ on $\mathfrak{M}$ by

$$\rho_\gamma(\mathrm{I}) = 1 \text{ and } \rho_\gamma(v(\alpha_1, \cdots, \alpha_n)v(\beta_1, \cdots, \beta_n)^*)$$
$$= \delta^{\alpha_1, \cdots, \alpha_n}_{\beta_1, \cdots, \beta_n} \prod_{k=1}^{[\frac{n}{2}]} p^{1-\alpha_{2k}}(1-p)^{\alpha_{2k}} \prod_{k=1}^{[\frac{n+1}{2}]} \delta^{\gamma_k}_{\alpha_{2k-1}}.$$

ρ_γ is uniquely extended to the state of $\boldsymbol{M}$ by Lemma 7, which is denoted by ρ_γ too. Let θ_γ be the cyclic representation of $\boldsymbol{M}$ induced by ρ_γ. Following the arguments in [9 : p. 585] we can easily show that the kernel of $\theta^\gamma = \{a \in \mathbf{M} : \lim_{n\to\infty} \|e(n)xaye(n)\| = 0 \text{ for all } x, y \in \boldsymbol{M}\}$, which implies that $\theta_\gamma^{-1}(0) = \theta_{\gamma'}^{-1}(0)$ for every pair of $\gamma, \gamma' \in \Gamma$.

Let π_γ be the representation of $\boldsymbol{M}_0$ defined by ρ_γ in Lemma 8. Let ξ_γ be the cyclic vector for the representation θ_γ, that is, $\rho_\gamma(a) = (\theta_\gamma(a)\xi_\gamma, \xi_\gamma)$ for $a \in \boldsymbol{M}$. Then we have

$$(\pi_\gamma\Big[\sum_{\alpha_1,\cdots,\alpha_n;\beta_1,\cdots,\beta_n} \lambda(\alpha_1, \cdots, \alpha_n : \beta_1, \cdots, \beta_n)w(\alpha_1, \cdots, \alpha_n : \beta_1, \cdots, \beta_n)\Big]\xi_\gamma, \xi_\gamma)$$
$$= (\theta_\gamma\Big[\sum_{\alpha_1,\cdots,\alpha_n;\beta_1,\cdots,\beta_n} \lambda(\alpha_1, \cdots, \alpha_n : \beta_1, \cdots, \beta_n)v(\alpha_1, \cdots, \alpha_n)v(\beta_1, \cdots, \beta_n)^*\Big]\xi_\gamma, \xi_\gamma)$$
$$= \sum_{\alpha_1,\cdots\alpha_n;\beta_1\cdots\beta_n} \lambda(\alpha_1, \cdots, \alpha_n : \beta_1, \cdots, \beta_n)\rho_\gamma[v(\alpha_1, \cdots, \alpha_n)v(\beta_1, \cdots, \beta_n)^*]$$
$$= \sum_{\alpha_1,\cdots\alpha_n} \lambda(\alpha_1, \cdots, \alpha_n : \alpha_1, \cdots, \alpha_n) \prod_{k=1}^{[\frac{n}{2}]} p^{1-\alpha_{2k}}(1-p)^{\alpha_{2k}} \prod_{k=1}^{[\frac{n+1}{2}]} \delta^{\gamma_k}_{\alpha_{2k-1}}$$
$$= \bigotimes_{n=1}^{\infty} [p\sigma_0^{2n} + (1-p)\sigma_1^{2n}] \otimes \Big[\bigotimes_{n=1}^{\infty} \sigma_{\gamma_n}^{2n-1}\Big]\Big(\sum_{\alpha_1,\cdots\alpha_n;\beta_1\cdots,\beta_n} \lambda(\alpha_1, \cdots, \alpha_n : \beta_1, \cdots, \beta_n)\cdot$$
$$\cdot w(\alpha_1, \cdots, \alpha_n : \beta_1, \cdots, \beta_n)\Big).$$

Thus π_γ is unitary equivalent to ψ_γ defined in the proof of Lemma 6, and π_γ is a factor representation type II or of type III according to $p = 1/2$ or $0 < p < 1/2$. It follows that $\pi_\gamma(\boldsymbol{M}_0)'$ is a factor of type II or type III. Since $\pi_\gamma(\mathbf{M}_0)$ is the uniform closure of $\theta_\gamma(\mathfrak{M})f_\gamma$, $\pi_\gamma(\boldsymbol{M}_0)'$ is considered as the commutant of the weak closure of $f_\gamma\theta_\gamma(\boldsymbol{M})f_\gamma$ on the Hilbert space $f_\gamma\mathfrak{H}_\gamma$, where $\mathfrak{H}_\gamma$ is the representation space of θ_γ and f_γ is the projection defined in Lemma 8. Since ξ_γ is a generating vector for $\theta_\gamma(\boldsymbol{M})$, ξ_γ is separating for $\theta_\gamma(\boldsymbol{M})'$ and we get $\theta_\gamma(\boldsymbol{M})' \cong \pi_\gamma(\boldsymbol{M}_0)'$. Therefore θ_γ is a factor representation of type II or type III according to $p = 1/2$ or $0 < p < 1/2$.

Suppose that θ_γ and $\theta_{\gamma'}$ are quasi-equivalent, which is implemented by an isomorphism π of the weak closure of $\theta_\gamma(\boldsymbol{M})$ onto the weak closure of $\theta_{\gamma'}(\boldsymbol{M})$. Since $\pi\theta_\gamma(e(n)) = \theta_{\gamma'}(e(n))$, $\pi(f_\gamma) = f_{\gamma'}$. Hence we have

$$\pi\Big[\sum_{\substack{\alpha_1,\cdots,\alpha_n \\ \beta_1,\cdots,\beta_n}} \lambda(\alpha_1,\cdots,\alpha_n:\beta_1,\cdots,\beta_n)\theta_\gamma(v(\alpha_1,\cdots,\alpha_n)v(\beta_1,\cdots,\beta_n)^*)f_\gamma\Big]$$

$$= \sum_{\substack{\alpha_1,\cdots\alpha_n \\ \beta_1,\cdots,\beta_n}} \lambda(\alpha_1,\cdots,\alpha_n:\beta_1,\cdots,\beta_n)\theta_{\gamma'}(v(\alpha_1,\cdots,\alpha_2)v(\beta_1,\cdots,\beta_n)^*)f_{\gamma'}),$$

which implies

$$\pi\circ\pi_\gamma\Big[\sum_{\substack{\alpha_1,\cdot\alpha_n \\ \beta_1,\cdots\beta_n}} \lambda(\alpha_1,\cdots,\alpha_n:\beta_1,\cdots,\beta_n)w(\alpha_1,\cdots,\alpha_n:\beta_1,\cdots,\beta_n)\Big]$$

$$= \pi_{\gamma'}\Big[\sum_{\substack{\alpha_1,\cdots,\alpha_n \\ \beta_1,\cdots,\beta_n}} \lambda(\alpha_1,\cdots,\alpha_n:\beta_1,\cdots,\beta_n)w(\alpha_1,\cdots,\alpha_n:\beta_1,\cdots,\beta_n)\Big].$$

Thus, π gives the quasi-equivalence of π_γ and $\pi_{\gamma'}$, and by Lemma 6, $\gamma\equiv\gamma'$ mod G. Therefore if $\gamma\not\equiv\gamma'$ mod G, then θ_γ and $\theta_{\gamma'}$ are disjoint factor representations.

Suppose $\gamma\equiv\gamma'$ mod G, that is, $\gamma'=\gamma\cdot\alpha$ for some $\alpha\in G$. Let $u(\hat{\alpha})$ be the unitary operator of $\boldsymbol{M}_0$ defined by the similar manner in p.87, where $\hat{\alpha}$ is the element of G defined by

$$\hat{\alpha}_{2i-1}=\alpha_i,\ i=1,2,\cdots,\hat{\alpha}_{2i}=0,\ i=1,2,\cdots.$$

Then there is an element $v(\alpha)$ of $\mathfrak{M}$ such that $[\theta_\gamma(v(\alpha))]f_\gamma=\pi_\gamma[u(\hat{\alpha})]$. On the other hand, we have

$$\tilde{\mu}^1\otimes\sigma_\gamma^2(u(\hat{\alpha})xu(\hat{\alpha})^*)=\tilde{\mu}^1\otimes\sigma_{\gamma'}^2(x) \qquad \text{for } x\in\boldsymbol{M}_0$$

by the direct calculations, where $\tilde{\mu}^1$, σ_γ^2 and $\sigma_{\gamma'}^2$ are the states as in the proof of Lemma 6. It follows that

$$(\pi_\gamma[u(\hat{\alpha})xu(\hat{\alpha})^*]\xi_\gamma,\xi_\gamma)=(\pi_{\gamma'}(x)\xi_{\gamma'},\xi_{\gamma'})$$

for all $x\in\boldsymbol{M}_0$, so that we have for every $x\in\mathfrak{M}$,

$$\begin{aligned}(\theta_\gamma[v(\alpha)xv(\alpha)^*)]\xi_\gamma,\xi_\gamma) &= (\theta_\gamma[v(\alpha)xv(\alpha)^*]f_\gamma\xi_\gamma,\xi_\gamma)\\ &= (\pi_\gamma[u(\hat{\alpha})x'u(\hat{\alpha})^*]\xi_\gamma,\xi_\gamma)\\ &= (\pi_{\gamma'}(x')\xi_{\gamma'},\xi_{\gamma'})=(\theta_{\gamma'}(x)\xi_{\gamma'},\xi_{\gamma'}),\end{aligned}$$

where x' is the element of $\boldsymbol{M}_0$ such that $\pi_\gamma(x')=\theta_\gamma(x)f_\gamma$, or equivalently $\pi_{\gamma'}(x') = \theta_{\gamma'}(x)f_{\gamma'}$. Hence we have

$$\rho_\gamma[v(\alpha)xv(\alpha)^*]=\rho_{\gamma'}(x) \qquad \text{for all } x\in\boldsymbol{M}$$

by Lemma 7. Therefore $\rho_{\gamma'}$ is contained in the closed invariant subspace of $\boldsymbol{M}^*$

generated by ρ_γ. By the symmetric arguments, ρ_γ is contained in the subspace generated by $\rho_{\gamma'}$, so that we have $V_{\rho\gamma} = V_{\rho\gamma'}$. This implies the quasi-equivalence of θ_γ and $\theta_{\gamma'}$.

Finally, we can see that θ_γ depends only on the choice of s_0. Hence if we denote this dependence on a positive element s with $\|s\| = 1$ by θ_γ^s, then the family of representations $\{\theta_\gamma^s : \gamma \in \Gamma$ and $s \in S$ where S is the set of all positive elements in $\boldsymbol{M}$ with norm one$\}$ is the required one.

THEOREM 2. *Let $\boldsymbol{M}$ be as in Theorem 1. There exists a representation θ^i of type i, $i =$ II, III, such that we get a direct integral decomposition of θ^i into irreducible representations over measure space (Γ, μ^i),*

$$\mathfrak{H}^i = \int_\Gamma \mathfrak{H}^i(\gamma)d\mu^i(\gamma),\ \theta^i = \int_\Gamma \theta^i_\gamma d\mu^i(\gamma)$$

with the following property:

1^0 $\theta_\gamma^{\mathrm{II}} \simeq \theta_\gamma^{\mathrm{III}}$ *for all* $\gamma \in \Gamma$, *say* θ_γ,

2^0 $\theta_\gamma \simeq \theta_{\gamma'}$ *if and only if* $\gamma \equiv \gamma'$ mod G.

PROOF. Let ρ be the functional on $\mathfrak{M}$ defined by

$$\rho(I) = 1,\ \rho[v(\alpha_1, \cdots, \alpha_n)v(\beta_1, \cdots, \beta_n)^*] = \delta^{\alpha_1 \cdots \alpha_n}_{\beta_1 \cdots \beta_n} \prod_{k=1}^\infty p^{1-\alpha_k}(1-p)^{\alpha_k},$$

where we assume $p = 1/2$ or $0 < p < 1/2$ according to the case of type II or of type III. Then ρ can be uniquely extended to a state of $\boldsymbol{M}$. Let θ be the cyclic representation of $\boldsymbol{M}$ induced by ρ. Then θ becomes a representation of type II or of type III according to the assumptions $p = 1/2$ or $0 < p < 1/2$. In the following, we shall treat the case of type II and type III simultaneously.

Let μ be the measure on Γ defined in p. 87. Let π be the isomorphism of $C \hat{\otimes} G$ onto $\boldsymbol{M}_0$ defined in Lemma 4 and π_ρ be the representation $\boldsymbol{M}_0$ on the Hilbert space $f_\rho \mathfrak{H}_\rho$ defined in Lemma 8. Then the arguments of [9 : p.588 and p. 589] show that the representation $\pi_\rho \circ \pi$ of $C \hat{\otimes} G$ is unitarily equivalent to the representation $\tilde{\varphi}_\mu$ appeared in p. 86 and this unitary equivalence is implemented by an isometry u of $\tilde{\mathfrak{H}}_\mu$ onto $f_\rho \mathfrak{H}_\rho$ which carries the cyclic vector ξ_0 of $\tilde{\mathfrak{H}}_\mu$ to the cyclic vector ξ_ρ of $\mathfrak{H}_\rho$. Let $\boldsymbol{A}$ be the abelian subalgebra of $\tilde{\varphi}_\mu(C \hat{\otimes} G)'$ defined in p. 86 as $r(C)$. Then $u\boldsymbol{A}u^*$ is the abelian subalgebra of $\pi_\rho \circ \pi(C \hat{\otimes} G)' = [\theta_\rho(\mathfrak{M})f_\rho]'$. Since the mapping: $x \to xf$ is an isomorphism of $\theta_\rho(\boldsymbol{M})'$, we get an abelian subalgebra $\boldsymbol{A}_\rho$ of $\theta(\boldsymbol{M})'$ such that $u\boldsymbol{A}u^* = \boldsymbol{A}_\rho f_\rho$. Now we decompose θ into the direct integral of irreducible representations relative to $\boldsymbol{A}_\rho$ whose weak closure $\tilde{\boldsymbol{A}}_\rho$ is isomorphic to $L^\infty(\Gamma, \mu)$. Let ε_ρ be the mapping of $\boldsymbol{M}$ into $\tilde{\boldsymbol{A}}_\rho$ such as

$$(\theta(x)a\xi_\rho, \xi_\rho) = (\varepsilon_\rho(x)a\xi_\rho, \xi_\rho)$$

for all $a \in \widetilde{A}_\rho$ and $x \in \boldsymbol{M}$. Then, for each $a \in \widetilde{A}_\rho$, we have

$$\begin{aligned}
&(\theta_\rho[v(\alpha_1,\cdots,\alpha_n)v(\beta_1,\cdots,\beta_n)^*]a\xi_\rho,\xi_\rho)\\
&=(\theta'[v(\alpha_1,\cdots,\alpha_n)v(\beta_1,\cdots,\beta_n)^*]f_\rho af_\rho\xi_\rho,\xi_\rho)\\
&=(\pi_\rho\circ\pi[l(\chi(\alpha_1,\cdots,\alpha_n))\hat{u}(\beta_1+\alpha_1,\ \beta_2+\alpha_2,\cdots,\ \beta_n+\alpha_n,\ 0,0,\cdots)]af_\rho\xi_\rho,\xi_\rho)\\
&=(\tilde{\varphi}_\mu[l(\chi(\alpha_1,\cdots,\alpha_n))\hat{u}(\beta_1+\alpha_1,\cdots,\beta_n+\alpha_n,0,0,\cdots)]u^*af_\rho u\xi_0,\xi_0).
\end{aligned}$$

Therefore

$$\varepsilon_\rho[v(\alpha_1,\cdots,\alpha_n)v(\beta_1,\cdots,\beta_n)^*]=\varepsilon_{\hat{\mu}}[l(\chi(\alpha_1,\cdots,\alpha_n))\dot{u}(\beta_1+\alpha_1,\cdots,\beta_n+\alpha_n,0,\cdots)]$$

if we regard the both side of equation as the function over Γ, where $\varepsilon_{\tilde{\mu}}$ is the mapping defined in Lemma 3. It follows that

$$\varepsilon_\rho[v(\alpha_1,\cdots,\alpha_n)v(\beta_1,\cdots,\beta_n)^*](\gamma)=\delta^{\alpha_1,\cdots,\alpha_n}_{\beta_1,\cdots,\beta_n}\delta^{\alpha_1,\cdots,\alpha_n}_{\gamma_1\cdots\gamma_n}.$$

Hence if we define the state ρ_γ on $\boldsymbol{M}$ by $\rho_\gamma(x)=\varepsilon_\rho(x)(\gamma)$, then ρ_γ becomes a pure state of $\boldsymbol{M}$ by the arguments of [9 : 584], and ρ_γ does not depend on the choice of p. Moreover if we decompose the representation θ' relative to $\boldsymbol{A}_\rho$ as follows:

$$\theta'=\int\theta_\gamma d\mu(\gamma)\qquad \mathfrak{H}_\rho=\int\mathfrak{H}(\gamma)d\mu(\gamma)$$

then θ_γ is the cyclic representation induced by ρ_γ, which implies 1^0. Besides the arguments in [9 : p.585 and p.594] show that $\theta_\gamma\cong\theta_{\gamma'}$ if and only if $\gamma\equiv\gamma'$ mod G. This completes the proof.

Concluding remarks. 1^0 Theorem 1 does not hold in the case of type II_1. Indeed, let φ be an arbitrary finite factor representation of $\boldsymbol{M}_0$. If τ is a trace on the weak closure of $\varphi(\boldsymbol{M}_0)$, then ${}^t\varphi(\tau)$ is also a trace on $\boldsymbol{M}_0$. Since there is only one trace $\bigotimes_{n=1}^{\infty}[1/2(\sigma_0^n+\sigma_1^n)]$ on $\boldsymbol{M}_0$, we have ${}^t\varphi(\tau)=\bigotimes_{n=1}^{\infty}[1/2(\sigma_0^n+\sigma_1^n)]$. Hence there is only one finite factor representation of $\boldsymbol{M}_0$ within quasi-equivalence.

2^0 In [10], Guichardet shows that the components of irreducible direct integral decomposition of every multiplicity free representation is mutually disjoint except a null set. However in the cases of type II and of type III, this is imposible. In fact, if $\mu(\Gamma_0)=1$ and $\Gamma_0\cap\Gamma_0\alpha=\phi$ then $\mu(\Gamma_0\alpha)=0$ for every subset Γ_0 of Γ. Hence there is no subset Γ_0 such that $\Gamma_0\cap\Gamma_0\alpha=\phi$ and $\mu(\Gamma_0)=1$, which implies that there are some elements γ and γ' in Γ_0 such that $\theta_\gamma\cong\theta_{\gamma'}$ if $\mu(\Gamma_0)=1$. Moreover, $\mu(\{\gamma\in\Gamma\,|\,\theta_\gamma\cong\theta_{\gamma_0}\})=0$ for every $\gamma_0\in\Gamma$.

3^0 If we define the measure in the dual space $\widehat{\boldsymbol{M}}$ of $\boldsymbol{M}$ by $\hat{\mu}(S)=\mu(\{\gamma\in\Gamma\,|\,\hat{\theta}_\gamma\in S)$, where $\hat{\theta}_\gamma$ means the unitary equivalence class of θ_γ, then $\hat{\mu}$ is not standard measure. And the identity mapping in $\widehat{\boldsymbol{M}}$ is not integrable with respect to $\hat{\mu}$ in the sense of [14].

4^0 After writing this paper, the author find the papers [4] and [5] that treat the factor decompositions. Comparing the results of [5] and this paper, it seems the factor decomposition to be more natural object in the case of the representations of type II or type III.

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DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.