ON CONTACT STRUCTURE OF HYPERSURFACES IN COMPLEX MANIFOLDS, I

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J.W.Gray [3], W. M. Boothby and H.C.Wang [1] introduced the notion of contact and almost contact structures and investigated it from the global viewpoint. An almost contact structure is one of an odd-dimensional manifold such that the structural group of its tangent bundle is reducible to the product of a unitary group with the one-dimensional identity group. It is comparable to almost complex structure of even-dimensional manifolds. S.Sasaki and Y.Hatakeyama [8, 9] proved that an almost contact structure can be represented as a totality of a tensor field and two vector fields satisfying certain conditions. It enables us to research properties of almost contact structures by use of tensor calculus.

In this paper we shall always assume that treated hypersurfaces are orientable. We shall show that a hypersurface in an almost complex manifold has an almost contact structure and that a hypersurface in an almost Hermitian manifold has an almost contact metric structure. Next we shall seek for a condition in order that a hypersurface in a Kählerian manifold has a contact structure. As a consequence we shall be able to obtain an extensive class of contact manifolds, which includes odd-dimensional spheres known as the simplest examples of contact manifolds. Finally we shall investigate the converse problem of imbedding of an almost contact or contact manifolds into an almost complex or complex manifold.

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1. Almost complex structure and almost contact structure. Let M be a $2p$-dimensional differentiable manifold covered with local coordinate systems (x^κ)[1]. An almost complex structure in M is by definition a (1, 1)-tensor field $F=(F_\lambda^{\ \kappa})$ satisfying the equation

$$FF=-E : F_\mu^{\ \lambda}F_\lambda^{\ \kappa}=-\delta_\mu^\kappa, \tag{1. 1}$$

where $E=(\delta_\mu^\kappa)$ is the unit tensor field in M. A manifold M with such a structure F is called an almost complex manifold. Improving the operators of J.A.Schouten and K.Yano [10], M. Obata [6] defined the following operators,

1) In this paper, Geeek indices run on the range $1,\cdots,2p$, and small Latin indices on the range $1,\cdots,2p-1$. Capital Latin indices run on the range $1,\cdots,2p-1$ of small ones and an additional symbol ∞.

illustrated here for a quantity $P=(P_{\mu\lambda}{}^{\kappa})$:

$$
(1.2)\qquad
\begin{cases}
\Phi_1(F)P_{\mu\lambda}{}^{\kappa} = \frac{1}{2}(P_{\mu\lambda}{}^{\kappa} - F_\lambda{}^\beta P_{\mu\beta}{}^\alpha F_\alpha{}^\kappa),\\
\Phi_2(F)P_{\mu\lambda}{}^{\kappa} = \frac{1}{2}(P_{\mu\lambda}{}^{\kappa} + F_\lambda{}^\beta P_{\mu\beta}{}^\alpha F_\alpha{}^\kappa),\\
\Phi_1^*(F)P_{\mu\lambda}{}^{\kappa} = \frac{1}{2}(P_{\mu\lambda}{}^{\kappa} - F_\mu{}^\gamma P_{\gamma\lambda}{}^\alpha F_\alpha{}^\kappa),\\
\Phi_2^*(F)P_{\mu\lambda}{}^{\kappa} = \frac{1}{2}(P_{\mu\lambda}{}^{\kappa} + F_\mu{}^\gamma P_{\gamma\lambda}{}^\alpha F_\alpha{}^\kappa),\\
\Phi_3(F)P_{\mu\lambda}{}^{\kappa} = \frac{1}{2}(P_{\mu\lambda}{}^{\kappa} - F_\mu{}^\gamma F_\lambda{}^\beta P_{\gamma\beta}{}^\kappa),\\
\Phi_4(F)P_{\mu\lambda}{}^{\kappa} = \frac{1}{2}(P_{\mu\lambda}{}^{\kappa} + F_\mu{}^\gamma F_\lambda{}^\beta P_{\gamma\beta}{}^\kappa).
\end{cases}
$$

These operators will be used later.

In an almost complex manifold, there always exists an affine connection, called an F-connection, transposing the structure F parallelly [2, 6]. Further, in an almost complex manifold, there exists a Riemannian metric $G=(G_{\mu\lambda})$, called an almost Hermitian metric, satisfying the condition

$$(1.3)\qquad G=FGF^t:\ G_{\mu\lambda}=F_\mu{}^\gamma F_\lambda{}^\beta G_{\gamma\beta},$$

F^t denoting the transpose of F. A manifold with such a metric is called an almost Hermitian manifold, and there exists a connection, called a metric F-connection, which transposes both F and G parallelly [4, 6].

The covariant tensor field $F_*=(F_{\mu\lambda})$ given by

$$(1.4)\qquad F_*=FG:\ F_{\mu\lambda}=F_\mu{}^\kappa G_{\kappa\lambda}$$

is skew symmetric. We put

$$(1.5)\qquad \Theta=F_{\mu\lambda}dx^\mu\wedge dx^\lambda$$

and call it the fundamental 2-form of the almost Hermitian manifold. If Θ is closed, then the manifold is said to be almost Kählerian.

The (1, 2)-tensor field $N=(N_{\mu\lambda}{}^\kappa)$ defined by

$$(1.6)\qquad N_{\mu\lambda}{}^\kappa=F_\mu{}^\omega(\partial_\omega F_\lambda{}^\kappa-\partial_\lambda F_\omega{}^\kappa)-F_\lambda{}^\omega(\partial_\omega F_\mu{}^\kappa-\partial_\mu F_\omega{}^\kappa)$$

is called the Nijenhuis tensor or torsion tensor of an almost complex structure F. It possesses the properties

$$(1.7)\qquad \Phi_1(F)N=\Phi_1^*(F)N=\Phi_4(F)N=0$$

or the equivalent ones

$$(1.8)\qquad \Phi_2(F)N=\Phi_2^*(F)N=\Phi_3(F)N=N$$

[6]. An almost complex manifold M is complex analytic if and only if its Nijenhuis tensor vanishes [5], or if and only if there is a symmetric F-connection [6, 7]. Moreover, if the manifold is analytic, an almost Hermitian or almost Kählerian manifold is reduced to a Hermitian or Kählerian one respectively. An almost Hermitian manifold is Kählerian if and only if its Riemannian connection is an F-connection [7].

Next let us speak of almost contact structure[2)]. Let $\overline{M}$ be a $(2p-1)$-dimensional manifold covered with local coordinate systems (y^h). An almost contact structure in $\overline{M}$ is defined by the totality $(f_i{}^h, \eta_i, \xi^h)$ of a (1, 1)-tensor field $\bar{f} = (f_i{}^h)$, a covariant vector field $\eta = (\eta_i)$ and a contravariant vector field $\xi = (\xi^h)$, which satisfy the conditions

$$\text{(1. 9)} \qquad \operatorname{rank} \bar{f} = 2p - 2$$

and

$$\text{(1.10)} \qquad \begin{cases} f_j{}^i f_i{}^h = -\delta_j^h + \eta_j \xi^h, & f_i{}^h \eta_h = 0, \\ \xi^i f_i{}^h = 0, & \xi^i \eta_i = 1. \end{cases}$$

Putting

$$\text{(1.11)} \qquad f_i{}^\infty = \eta_i, \qquad f_\infty{}^h = -\xi^h$$

and

$$\text{(1.12)} \qquad f = (f_B{}^A) = \begin{pmatrix} f_i{}^h & f_i{}^\infty \\ f_\infty{}^h & 0 \end{pmatrix},$$

we can put the conditions (1.10) into one equation

$$\text{(1.13)} \qquad ff = -E : f_C{}^B f_B{}^A = -\delta_C^A.$$

Accordingly we may call such a matrix field f in $\overline{M}$ an *almost contact structure* or simply an *f-structure*, and the manifold $\overline{M}$ an *almost contact manifold.*

Quantities in $\overline{M}$ with capital Latin indices such like $f = (f_B^A)$ will be called with prefix "C-", for instance, f is an C-tensor in $\overline{M}$.

In an almost contact manifold, there always exists an affine connection transposing parallelly the almost contact structure, that is, the tensor field $f_i{}^h$ and the vector fields $f_i{}^\infty$ and $f_\infty{}^h$ [9]. It will be called an *f-connection.* Furthermore, in an almost contact manifold, there exists a Riemannian metric $\bar{g} = (g_{ji})$ satisfying the equations

$$\text{(1.14)} \qquad \begin{cases} g_{ji} - \eta_j \eta_i = f_j{}^c f_i{}^b g_{cb}, \\ \qquad \eta_i = \xi^h g_{ih}, \end{cases}$$

by use of S.Sasaki's notations [8]. Putting

2) Concerning almost contact structure, see S. Sasaki and Y. Hatakeyama [8, 9].

$$g = (g_{CB}) = \begin{pmatrix} g_{ji} & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.15}$$

the equations (1.13) are put into

$$g = fgf^t : g_{CB} = f_C{}^E f_B{}^D g_{ED}. \tag{1.16}$$

Such a metric $\bar{g}$ or g is called an *associated* metric tensor or C-tensor with the structure f, respectively. Moreover, the structure (f, g) consisting of an almost complex structure f and its associated metric C-tensor g will be called an *almost Grayan structure*, and a manifold with such a structure an *almost Grayan manifold*.

The covariant almost contact C-tensor f_* defined by

$$f_* = fg = \begin{pmatrix} f_{ji} & f_{j\infty} \\ f_{\infty i} & 0 \end{pmatrix} \tag{1.17}$$

is skew symmetric. Putting

$$\theta_1 = f_{i\infty} dy^i, \ \theta_2 = f_{ji} dy^j \wedge dy^i, \tag{1.18}$$

we call θ_1 and θ_2 the *fundamental* 1-*form* and 2-*form* of the almost Grayan structure respectively.

On the other hand, following J.W.Gray [3], a contact structure in $\bar{M}$ is given by two forms θ_1 and θ_2 with conditions $\theta_2 = d\theta_1$ and $\theta_1 \wedge \theta_2{}^{p-1} \neq 0$. Then there exists an almost Grayan structure (f, g) whose fundamental forms coincide with the given forms. We shall call the structure an *almost Sasakian structure*. The condition $\theta_2 = d\theta_1$ is written in

$$f_{ji} = \frac{1}{2} (\partial_j f_{i\infty} - \partial_i f_{j\infty}). \tag{1.19}$$

Returning to an almost contact structure f, we define Obata's operators $\Phi(f)$ for C-quantities by similar expressions to (1. 2). Then algebraic relations among the operators $\Phi(F)$ carry over among the operators $\Phi(f)$.

Similarly to (1. 6), the Nijenhuis C-tensor $n = (n_{CB}{}^A)$ of an f-structure is defined by

$$n_{CB}{}^A = f_C{}^F(\partial_E f_B{}^A - \partial_B f_E{}^A) - f_B{}^E(\partial_E f_C{}^A - \partial_C f_E{}^A), \tag{1.20}$$

where ∂_∞ is interpreted as a null-operator. The Nijenhuis C-tensor n satisfies the equations (1. 6) and (1. 7) with f in place of F. The sets $(n_{ji}{}^h)$, $(n_{ji}{}^\infty)$, $(n_{j\infty}{}^h)$ and $(n_{j\infty}{}^\infty)$ of the components of n define tensor fields in $\bar{M}$ separately, and it is known that the vanishing of the first tensor $\bar{n} = (n_{ji}{}^h)$ implies those of the other tensors, i.e., the vanishing of the C-tensor n itself.

An almost contact structure f with condition $n = 0$ is said to be *normal*. We shall call an almost Grayan or almost Sasakian structure (f, g) with $n = 0$

a *Grayan* or *Sasakian* structure respectively. An almost Grayan structure is Sasakian if and only if the relations

$$
\left\{\begin{aligned} f_{ji} &= \nabla_j f_{i\infty}, \\ \nabla_j f_{ih} &= f_{i\infty} g_{jh} - f_{h\infty} g_{ji} \end{aligned}\right. \tag{1.21}
$$

are satisfied, where ∇ denotes the covariant differentiation with respect to the Riemannian connection of $\bar{g}$.

Our terminologies are compared with those for complex structure as follows:

Complex structures

$$
\begin{tikzcd}
(\exists\, F) \arrow[r] & \text{almost complex} \arrow[r, "(N=0)"] \arrow[d, "(\exists\, G)"] & \text{complex} \arrow[d] \\
\begin{pmatrix} \exists\, \Theta: \\ d\Theta = 0 \end{pmatrix} \arrow[d] & \text{almost Hermitian} \arrow[r] \arrow[d, "(d\Theta = 0)"] & \text{Hermitian} \arrow[d] \\
\text{symplectic} \arrow[r, "(\exists\, G)"] & \text{almost Kählerian} \arrow[r] & \text{Kählerian}
\end{tikzcd}
$$

Contact structures

$$
\begin{tikzcd}
(\exists\, f) \arrow[r] & \text{almost contact} \arrow[r, "(n=0)"] \arrow[d, "(\exists\, g)"] & \text{normal almost contact} \arrow[d] \\
\begin{pmatrix} \exists\theta_1, \theta_2: \\ \theta_2 = d\theta_1 \end{pmatrix} \arrow[d] & \text{almost Grayan} \arrow[r] \arrow[d, "(\theta_2 = d\theta_1)"] & \text{Grayan} \arrow[d] \\
\text{contact} \arrow[r, "(\exists\, G)"] & \text{almost Sasakian} \arrow[r] & \text{Sasakian}
\end{tikzcd}
$$

2. Almost contact structure of a hypersurface in an almost complex manifold. We consider a $2p$-dimensional almost complex manifold M with structure F and an orientable differentiable hypersurface $\bar{M}$. Let $\bar{M}$ be represented by $x^\kappa = x^\kappa(y^h)$ by using local coordinate systems in M and in $\bar{M}$. We put

$$
B_i^\kappa = \partial_i x^\kappa, \tag{2. 1}
$$

which span the tangent hyperplane of $\bar{M}$ at each point, and choose a vector field C^κ which is complementary to the tangent hyperplane of $\bar{M}$ at each point. The vector field C^κ is called a pseudo-normal to $\bar{M}$ and will be sometimes denoted by B_∞^κ. The matrix $B = (B_B{}^\kappa) = \begin{pmatrix} B_i^\kappa \\ C^\kappa \end{pmatrix}$ is of rank $2p$, and its inverse will be denoted by $B^{-1} = (B_\lambda{}^A) = (B_\lambda{}^h, B_\lambda{}^\infty) = (B_\lambda{}^h, C_\lambda)$. Then we have the equations

$$(2.\ 2)\qquad B_B{}^\lambda B_\lambda{}^A = \delta_B^A, \quad \begin{array}{ll} B_i{}^\lambda B_\lambda{}^h = \delta_i^h, & B_i{}^\lambda C_\lambda = 0, \\ C^\lambda B_\lambda{}^h = 0, & C^\lambda C_\lambda = 1, \end{array}$$

and the equivalent equation

$$(2.\ 3)\qquad B_\lambda{}^A B_A{}^\kappa = B_\lambda{}^i B_i{}^\kappa + C_\lambda C^\kappa = \delta_\lambda^\kappa.$$

Now we put

$$(2.\ 4)\qquad f = BFB^{-1} : f_B{}^A = B_B{}^\lambda F_\lambda{}^\kappa B_\kappa{}^A,$$

which is a C-tensor in $\overline{M}$. The sets $(f_i{}^h)$, $(f_i{}^\infty)$, $(f_\infty{}^h)$ and $(f_\infty{}^\infty)$ of the four kinds of the components

$$(2.\ 5)\qquad \begin{cases} f_i{}^h = B_i{}^\lambda F_\lambda{}^\kappa B_\kappa{}^h, & f_i{}^\infty = B_i{}^\lambda F_\lambda{}^\kappa C_\kappa, \\ f_\infty{}^h = C^\lambda F_\lambda{}^\kappa B_\kappa{}^h, & f_\infty{}^\infty = C^\lambda F_\lambda{}^\kappa C_\kappa \end{cases}$$

define a (1, 1)-tensor field, a covariant vector field, a contravariant vector field and a scalar field in $\overline{M}$ respectively. The C-tensor f is obviously of rank $2p$ and satisfies the equations

$$(2.\ 6)\qquad fB = BF$$

and

$$(2.\ 7)\qquad ff = -E.$$

A pseudo-normal vector field C^κ can be chosen such as

$$(2.\ 8)\qquad f_\infty{}^\infty = C^\lambda F_\lambda{}^\kappa C_\kappa = 0.$$

Indeed, since there is an almost Hermitian metric in M and the covariant almost complex structure F_* is skew symmetric, the unit normal vector field of the hypersurface $\overline{M}$ with respect to the almost Hermitian metric always satisfies (2. 8). A vector field complementary to the tangent plane and lying in the hyperplane represented by $F_\lambda{}^\kappa C_\kappa$ at each point of $\overline{M}$ may be chosen as a pseudo-normal vector field satisfying (2. 8).

Once such a choice is fixed, the equation (2. 7) is written separately as

$$(2.\ 9)\qquad \begin{cases} f_j{}^i f_i{}^h + f_j f^h = -\delta_j^h, & f_j{}^i f_i = 0, \\ f^i f_i{}^h = 0, & f^i f_i = -1. \end{cases}$$

Here and hereafter we drop the index symbol ∞ from $f_i{}^\infty$ and $f_\infty{}^h$ unless confusions give arise. Since f is of rank $2p$, the vectors f_i and f^h do not vanish. The second and third equations mean that rank of the matrix $(f_i{}^h)$ is less than $2p-1$. We can see that the rank is in fact equal to $2p-2$. For, if there is a vector v_h satisfying

$$f_i{}^h v_h = 0, \tag{2.10}$$

then, by contracting (2.10) with $f_j{}^i$, we have $v_j = -f_j f^h v_h$. This means that the equation (2.10) admits solutions proportional to f_i only. Thus we have the following

THEOREM 1. *A hypersurface $\overline{M}$ in an almost complex manifold M has an almost contact structure.*

We shall call the structure the *induced almost contact structure* of $\overline{M}$ by a pseudo-normal vector field C^κ.

Interpreting ∂_∞ as a null-operator[3], we put

$$\Omega_{CB}{}^A = (\partial_C B_B{}^\kappa - \partial_B B_C{}^\kappa) B_\kappa{}^A, \tag{2.11}$$

whose components are given by

$$\begin{cases} \Omega_{ji}{}^A = \Omega_{\infty\infty}{}^A = 0, \\ \Omega_{j\infty}{}^h = -\Omega_{\infty j}{}^h = (\partial_j C^\kappa) B_\kappa{}^h, \\ \Omega_{j\infty}{}^\infty = -\Omega_{\infty j}{}^\infty = (\partial_j C^\kappa) C_\kappa. \end{cases} \tag{2.12}$$

Then, by the substitution of (2.4) into (1.20) and a straightforward and pretty long computation, the Nijenhuis C-tensor n of the induced almost contact structure f is related to the Nijenhuis tensor N of the almost complex structure F of M by

$$\begin{aligned} n_{CB}{}^A = {} & B_C{}^\mu B_B{}^\lambda N_{\mu\lambda}{}^\kappa B_\kappa{}^A + \Omega_{CB}{}^A + f_B{}^E \Omega_{CE}{}^D f_D{}^A \\ & + f_C{}^F \Omega_{FB}{}^D f_D{}^A - f_C{}^F f_B{}^E \Omega_{FE}{}^A \\ & + [-f_C B_B^\lambda + f_B B_C{}^\lambda - \delta_C^\infty f_B{}^E B_E{}^\lambda + \delta_B^\infty f_C{}^E B_E{}^\lambda] C^\mu (\partial_\mu F_\lambda{}^\kappa) B_\kappa{}^A \end{aligned} \tag{2.13}$$

and, in particular,

$$\begin{aligned} n_{ji}{}^h = {} & B_j{}^\mu B_i{}^\lambda N_{\mu\lambda}{}^\kappa B_\kappa{}^h - f_j{}^l f_i \Omega_{l\infty}{}^h - f_j f_i{}^l \Omega_{\infty l}{}^h \\ & + f_j \Omega_{\infty i}{}^A f^h + f_i \Omega_{j\infty}{}^A f_A{}^h \\ & - f_j B_i{}^\lambda C^\mu (\partial_\mu F_\lambda{}^\kappa) B_\kappa{}^h + f_i B_j{}^\lambda C^\mu (\partial_\mu F_\lambda{}^\kappa) B_\kappa{}^h. \end{aligned} \tag{2.14}$$

Let $\Gamma = (\Gamma_{\mu\lambda}^\kappa)$ be an affine connection in M and define $\gamma = (\gamma_{CB}^A)$ by

$$\gamma_{CB}^A = (B_C{}^\mu B_B{}^\lambda \Gamma_{\mu\lambda}^\kappa + \partial_C B_B{}^\kappa) B_\kappa{}^A, \tag{2.15}$$

which will be called the *induced* C-*connection* in $\overline{M}$. The sets (γ_{ji}^∞), $(\gamma_{j\infty}^h)$ and $(\gamma_{j\infty}^\infty)$ of the components define tensor fields in $\overline{M}$, but $\bar\gamma = (\gamma_{ji}^h)$ given by

$$\gamma_{ji}^h = (B_j{}^\mu B_i{}^\lambda \Gamma_{\mu\lambda}^\kappa + \partial_j B_i{}^\kappa) B_\kappa{}^h \tag{2.16}$$

is the so-called induced affine connection in $\overline{M}$ from Γ. We put

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3) We shall use technical calculus of the non-holonomic theory with this understanding. See K. Yano and E. T. Davies [11].

(2.17) $$\gamma_{ji}^{\infty} = h_{ji},\ \gamma_{j\infty}^{h} = l_j^{\ h},\ \gamma_{j\infty}^{\infty} = m_j,$$

and sometimes use the notation $l_j^{\ \infty}$ for m_j. From (2.15), we have

(2.18) $$\partial_C B_B^{\ \kappa} + B_C^{\ \mu} B_B^{\ \lambda} \Gamma_{\mu\lambda}^{\kappa} - \gamma_{CB}^{A} B_A^{\ \kappa} = 0,$$

and in particular

(2.19) $$\begin{cases} \nabla_j B_i^{\ \kappa} = \partial_j B_i^{\ \kappa} + B_j^{\ \mu} B_i^{\ \lambda} \Gamma_{\mu\lambda}^{\kappa} - \gamma_{ji}^{h} B_h^{\ \kappa} = h_{ji} C^{\kappa}, \\ \nabla_j C^{\kappa} = \partial_j C^{\kappa} + B_j^{\ \mu} C^{\lambda} \Gamma_{\mu\lambda}^{\kappa} = l_j^{\ h} B_h^{\ \kappa} + m_j C^{\kappa}. \end{cases}$$

The left hand sides of these equations are the so-called van der Waerden covariant derivatives. Then we have also

(2.20) $$\begin{cases} \nabla_j B_\lambda^{\ h} = -\ l_j^{\ h} C_\lambda, \\ \nabla_j C_\lambda = -\ h_{ji} B_\lambda^{\ i} - m_j C_\lambda. \end{cases}$$

Denoting the torsion tensor of Γ and the torsion C-tensor of γ by $S = (S_{\mu\lambda}^{\ \ \kappa})$ and $s = (s_{CB}^{\ \ A})$ respectively, it follows from (2.11) and (2.15) that

(2.21) $$2s_{CB}^{\ \ A} = 2B_C^{\ \mu} B_B^{\ \lambda} S_{\mu\lambda}^{\ \ \kappa} B_\kappa^{\ A} + \Omega_{CB}^{\ \ A},$$

or, putting $\overline{S} = (B_C^{\ \mu} B_B^{\ \lambda} S_{\mu\lambda}^{\ \ \kappa} B_\kappa^{\ A})$,

(2.22) $$2s = 2\overline{S} + \Omega.$$

Now let Γ be an F-connection in M. Then we have the equation

(2.23) $$\nabla_\mu F_\lambda^{\ \kappa} = \partial_\mu F_\lambda^{\ \kappa} + F_\lambda^{\ \beta} \Gamma_{\mu\beta}^{\kappa} - \Gamma_{\mu\lambda}^{\alpha} F_\alpha^{\ \kappa} = 0$$

and the Nijenhuis tensor N is related to the torsion tensor S of Γ by

(2.24) $$N = 8\Phi_2\Phi_3(F)S$$

[6]. Substituting (2.23) into the last term of (2.13), using the commutativity (2. 6) and putting

(2.25) $$T_{CB}^{\ \ A} = (\delta_C^{\infty} B_B^{\ \lambda} - \delta_B^{\infty} B_C^{\ \lambda}) C^\mu \Gamma_{\mu\lambda}^{\kappa} B_\kappa^{\ A},$$

we have the equation

(2.26) $$\begin{aligned} [f_C B_B^{\ \lambda} - f_B B_C^{\ \lambda} + \delta_C^{\infty} f_B^{\ E} B_E^{\ \lambda} - \delta_B^{\infty} f_C^{\ E} B_E^{\ \lambda}] C^\mu (\partial_\mu F_\lambda^{\ \kappa}) B_\kappa^{\ A} \\ = T_{CB}^{\ \ A} + f_C^{\ F} T_{FB}^{\ \ D} f_D^{\ A} + f_B^{\ E} T_{CE}^{\ \ D} f_D^{\ A} - f_C^{\ F} f_B^{\ E} T_{FE}^{\ \ A} \\ = 4\Phi_2\Phi_3(f) T_{CB}^{\ \ A}. \end{aligned}$$

Hence, by (1.8), (2.13), (2.21), (2.24) and (2.26), we have

(2.27) $$\begin{aligned} n &= \Phi_2\Phi_3(f)(\overline{N} + 4\Omega - 4T) \\ &= 4\Phi_2\Phi_3(f)(2\overline{S} + \Omega - T) \\ &= 4\Phi_2\Phi_3(f)(2s - T), \end{aligned}$$

where $\overline{N}$ denotes the C-tensor $(B_C^{\mu} B_B^{\lambda} N_{\mu\lambda}{}^{\kappa} B_{\kappa}{}^{A})$. We notice that

$$
\begin{aligned}
2s_{CB}{}^{A} - T_{CB}^{A} &= \gamma_{CB}^{A} - \gamma_{BC}^{A} - \delta_C^{\infty}\gamma_{\infty B}^{A} + \delta_B^{\infty}\gamma_{\infty C}^{A} \\
&= \delta_C^{j}\gamma_{jB}^{A} - \delta_B^{i}\gamma_{iC}^{A} \\
&= \delta_C^{j}\delta_B^{\infty} l_j{}^{A} - \delta_C^{\infty}\delta_B^{j} l_j{}^{A} + 2\delta_C^{j}\delta_B^{i} s_{ji}{}^{A},
\end{aligned}
\tag{2.28}
$$

and its non-trivial components are only

$$
\begin{cases}
2s_{ji}{}^{A} - T_{ji}{}^{A} = 2s_{ji}{}^{A}, \\
2s_{j\infty}{}^{A} - T_{j\infty}{}^{A} = -(2s_{\infty j}{}^{A} - T_{\infty j}{}^{A}) = l_j{}^{A}.
\end{cases}
\tag{2.29}
$$

From (2. 8) we have also

$$
l_j^{i} f_i = h_{ji} f^{i}. \tag{2.30}
$$

3. Structure of a hypersurface of a complex manifold. Let M be a complex manifold, $N = 0$. Now we seek for a condition that the induced almost contact structure f of a hypersurface $\overline{M}$ is normal, that is, $n = 0$. By means of the notice at the end of §1, we need only to consider the vanishing of $\bar{n} = (n_{ji}{}^{h})$.

In a complex manifold, there exists a symmetric F-connection, and it will be adopted in this paragraph. As is seen from (2.16), the induced connection $\bar{\gamma}$ in $\overline{M}$ is also symmetric and so is $\gamma_{ji}^{\infty} = h_{ji}$. Hence we know that $s_{ji}{}^{A} = 0$ and the non-trivial components of (2.28) are only ones given by the second of (2.29). Substituting the components into the expression of $n_{ji}{}^{h}$ given by (2.27), we have

$$
n_{ji}{}^{h} = f_i l_j{}^{A} f_A{}^{h} - f_j l_i{}^{A} f_A{}^{h} - f_j^{b} f_i l_b{}^{h} + f_j f_i^{b} l_b{}^{h}. \tag{3. 1}
$$

In order that $\bar{n}$ vanishs, we have the equation

$$
f_i(l_j^{A} f_A{}^{h} - f_j^{b} l_b{}^{h}) = f_j(l_i{}^{A} f_A{}^{h} - f_i^{b} l_b{}^{h}). \tag{3. 2}
$$

This equation is equivalent to the fact that the expressions in parentheses for each value of h are proportional to f_j:

$$
l_j{}^{A} f_A{}^{h} - f_j^{b} l_b{}^{h} = f_j \alpha^{h}, \tag{3. 3}
$$

α^{h} being proportional factors. Contracting f^{j}, we see that $\alpha^{h} = -f^{i} l_i{}^{A} f_A{}^{h}$ and the above equation becomes

$$
l_j{}^{A} f_A{}^{h} + f_j f^{i} l_i{}^{A} f_A{}^{h} - f_j^{b} l_b{}^{h} = 0, \tag{3. 4}
$$

or, by the first equation of (2. 9),

$$
f_j^{i}(f_i^{b} l_b{}^{A} f_A{}^{h} + l_i{}^{h}) = 0. \tag{3. 5}
$$

From this equation, we may put

$$
f_i^{b} l_b{}^{A} f_A{}^{h} + l_i{}^{h} = f_i \beta^{h}. \tag{3. 6}
$$

After contracting this equation with f^i and obtaining the expression of β^h, we see that the above equation is equivalent to

$$(3.\ 7)\qquad \begin{aligned} &l_i^h + f_i^b l_b^A f_A^h + f_i f^b l_b^h \\ &\quad = f_i^b(l_b^a f_a^h - f_b^a l_a^h + m_b f^h) = 0. \end{aligned}$$

Further, by contracting f_h, we have

$$(3.\ 8)\qquad f_i^b f_b^a l_a^h f_h + f_i^b m_b = 0$$

and the equation (3. 7) yields

$$f_i^b(l_b^a f_a^h - f_b^a l_a^h - f_b^a l_a^c f_c f^h) = 0$$

or

$$(3.\ 9)\qquad f_i^d(l_d^a + f_d^c l_c^b f_b^a) f_a^h = 0.$$

Thus we have established the following

THEOREM 2. *If we denote by f the induced almost contact structure of a hypersurface in a complex manifold, then, in order that the Nijenhuis C-tensor n of f vanishes, it is necessary and sufficient that the tensors l_i^h and m_i satisfy the equations* (3. 8) *and* (3. 9).

4. Induced f-connection. Returning to considerations of a hypersurface in an almost complex manifold M, let us seek for a condition in order that the induced connection $\bar{\gamma}$ in $\overline{M}$ from an F-connection Γ in M is an f-connection. The covariant derivatives of the tensors of f are given by

$$(4.\ 1)\qquad \begin{cases} \nabla_j f_i^h = h_{ji} f^h - l_j^h f_i = 0, \\ \nabla_j f_i = -h_{jh} f_i^h - m_j f_i = 0, \\ \nabla_j f^h = l_j^i f_i^h + m_j f^h = 0. \end{cases}$$

The first equation implies that h_{ji} and l_j^h should be of the form

$$(4.\ 2)\qquad h_{ji} = \lambda_j f_i,\ l_j^h = \lambda_j f^h,$$

λ_j being a vector field, and the second implies that

$$(4.\ 3)\qquad m_j = 0.$$

Then the third equation is satisfied. Thus we have

THEOREM 3. *The induced connection of a hypersurface from an F-connection is an f-connection if and only if the tensors h_{ji} and l_j^h are of the form* (4. 2) *and the vector m_j vanishes.*

If M is a complex manifold, Γ is a symmetric F-connection in M and the condition of Theorem 3 is satisfied, then h_{ji} is symmetric and hence λ_j in (4. 2) should be proportional to f_j:

$$\lambda_j = \lambda f_j \tag{4. 4}$$

λ being a factor. Hence the tensors h_{ji} and $l_j{}^h$ are given in the forms

$$h_{ji} = \lambda f_j f_i, \quad l_j{}^h = \lambda f_j f^h. \tag{4. 5}$$

Thus we have

THEOREM 4. *In order that the induced connection in $\overline{M}$ from a symmetric F-connection in a complex manifold M is an f-connection, it is necessary and sufficient that the tensors h_{ji} and $l_j{}^h$ are of the forms* (4. 5) *and the vector m_j vanishes.*

5. Metric structure of hypersurfaces. Let M be an almost Hermitian manifold with metric tensor $G = (G_{\mu\lambda})$. The unit normal vector C^κ of a hypersurface $\overline{M}$ satisfies the equation (2. 8) together with its covariant vector C_λ, and hence it induces an almost contact structure in $\overline{M}$, with which we shall confine ourselves in this paragraph.

We put

$$g = (g_{CB}) = \begin{pmatrix} g_{ji} & 0 \\ 0 & 1 \end{pmatrix} = BGB^t : g_{CB} = B_C{}^\mu B_B{}^\lambda G_{\mu\lambda} \tag{5. 1}$$

and

$$g^{-1} = (g^{BA}) = \begin{pmatrix} g^{ih} & 0 \\ 0 & 1 \end{pmatrix}. \tag{5. 2}$$

The minor matrix $\bar{g} = (g_{ji})$ defines the induced Riemannian metric of the hypersurface $\overline{M}$. We see that the inverse matrix of B is given by

$$B^{-1} = GB^t g^{-1} : \; B_\lambda{}^A = G_{\lambda\kappa} B_B{}^\kappa g^{BA}, \tag{5. 3}$$

that is,

$$B_\lambda{}^h = G_{\lambda\kappa} B_i{}^\kappa g^{ih}, \qquad C_\lambda = G_{\lambda\kappa} C^\kappa. \tag{5. 4}$$

By the skew symmetry of the covariant almost complex structure F_*, we have

$$f^h = C^\lambda F_\lambda{}^\kappa B_\kappa{}^h = -f_i g^{ih}. \tag{5. 5}$$

Moreover, substituting the Hermitian condition (1. 3) into (5. 1), we have

$$g = f g f^t. \tag{5. 6}$$

These equations (5. 5) and (5. 6) show that

THEOREM 5. *The induced Riemannian metric of a hypersurface in an* *almost Hermitian manifold is an associated metric with the induced almost contact structure of the hypersurface, that is to say, a hypersurface in an almost Hermitian manifold has an almost Grayan structure.*

If we put

(5. 7) $$f_* = fg = (f_{CB}),$$

then

(5. 8) $$f_* = BF_*B^t = -f_*^t,$$

that is, f_* is skew symmetric and the components are given by

(5. 9) $$\begin{cases} f_{ji} = B_j^\mu F_{\mu\lambda} B_i^\lambda, \\ f_{j\infty} = -f_{\infty j} = B_j^\mu F_{\mu\lambda} C^\lambda = f_j, \\ f_{\infty\infty} = 0. \end{cases}$$

From the definition (1.18) of the fundamental forms and (5. 9), we see that the 2-form θ_2 of the induced almost Grayan structure in $\overline{M}$ is induced from the fundamental form Θ of the almost Hermitian manifold M by the inclusion map of $\overline{M}$ into M. If M is almost Kählerian, $d\Theta = 0$, then we have $d\theta_2 = 0$. Thus we have the following

THEOREM 6. *If M is an almost Kählerian manifold and $\overline{M}$ a hypersurface in M, then the induced almost Grayan structure in $\overline{M}$ has the closed fundamental 2-form.*

Now let us investigate a condition in order that, in an almost Hermitian manifold M, the induced almost Grayan structure in a hypersurface $\overline{M}$ reduces to an almost Sasakian structure. Using a metric F-connection Γ in M and its induced connection $\bar{\gamma}$ in $\overline{M}$, we have $l_j^h = -h_{ji}g^{ih}$ and $m_j = 0$ in (2.19). The equation (1.19) of the almost Sasakian structure is now written in

$$\begin{aligned} 2f_{ji} &= \partial_j(B_i^\lambda F_\lambda^\kappa C_\kappa) - \partial_i(B_j^\lambda F_\lambda^\kappa C_\kappa) \\ &= B_j^\mu B_i^\lambda[\partial_\mu(F_\lambda^\kappa C_\kappa) - \partial_\lambda(F_\mu^\kappa C_\kappa)] \\ &= B_j^\mu B_i^\lambda[\nabla_\mu(F_\lambda^\kappa C_\kappa) - \nabla_\lambda(F_\mu^\kappa C_\kappa) + 2S_{\mu\lambda}{}^\alpha F_\alpha^\kappa C_\kappa] \\ &= B_i^\lambda F_\lambda^\kappa \nabla_j C_\kappa - B_j^\lambda F_\lambda^\kappa \nabla_i C_\kappa + 2B_j^\mu B_i^\lambda S_{\mu\lambda}{}^\alpha B_\alpha^h B_h^\beta F_\beta^\kappa C_\kappa \\ &= -h_{jh}f_i^h + h_{ih}f_j^h + 2\overline{S}_{ji}{}^h f_h. \end{aligned}$$

If we put

(5.10) $$h_{ji} = g_{ji} + k_{ji}$$

and substitute it into the above equation, then we see that k_{ji} has to satisfy the equation

$$k_{jh}f_i^{\ h} - k_{ih}f_j^{\ h} = 2\overline{S}_{ji}^{\ \ h}f_h. \tag{5.11}$$

Therefore we can state that

THEOREM 7. *In order that the induced almost Grayan strucure of a hypersurface $\overline{M}$ in an almost Hermitian manifold M reduces to an almost Sasakian structure, it is necessary and sufficient that the second fundamental tensor h_{ji} of $\overline{M}$ is given by* (5.10) *with a solution k_{ji} of* (5.11).

In particular, if M is Kählerian and the induced structure in $\overline{M}$ is Sasakian, then the Riemannian connection in M is a metric F-connection, its induced connection is symmetric, so are h_{ji} and k_{ji}, and in addition the equations (3. 8) and (3. 9) should be satisfied by $l_j^{\ h} = -h_{ji}g^{ih}$ and $m_j = 0$. From (5.11), it follows that

$$k_{jh}f_i^{\ h} = k_{ih}f_j^{\ h} \tag{5.12}$$

and we see that (3. 8) is fulfilled. Substituting (5.10) into (3. 9), we have

$$f_i^{\ d}(k_d^{\ a} + f_d^{\ c}k_c^{\ b}f_b^{\ a})f_{ah} = 0,$$

and by use of (5.12) it is verified that this equation reduces to

$$f_i^{\ d}k_d^{\ a}f_{ah} = f_i^{\ d}f_d^{\ a}k_{ah} = 0.$$

Moreover it follows easily that k_{ji} should be of the form $k_{ji} = \mu f_j f_i$, μ being a scalar field in $\overline{M}$. Thus we have the following

THEOREM 8. *In order that the induced almost Grayan structure in a hypersurface M in a Kählerian manifold M is a Sasakian structure, it is necessary and sufficient that the second fundamental tensor h_{ji} of M is of the form*

$$h_{ji} = g_{ji} + \mu f_j f_i. \tag{5.13}$$

As the most special case, we have

COROLLARY. *A totally umbilical hypersurface with positive constant mean curvature in a Kählerian manifold has a Sasakian structure by means of the induced metric.*

This corallary says that an odd-dimensional sphere has a Sasakian structure. Moreover it is to be noticed that, if $\overline{M}$ is a hypersurface with Sasakian structure stated in Theorem 8 and $\overline{M}'$ a hypersurface diffeomorphic to $\overline{M}$ by a map π, then $\overline{M}'$ has also a contact structure given by the induced forms

$\pi^*\theta_1$ and $\pi^*\theta_2$, but its associated metric is not in general the same as the induced metric in $\overline{M}'$ from the metric of $\overline{M}$.

6. Imbedding of Grayan and Sasakian manifolds. Let $\overline{M}$ be an almost Grayan manifold with metric tensor $\bar{g} = (g_{ji})$ and almost contact structure f, and I a straight line parametrized by $t \in (-\infty, +\infty)$. Consider the direct product $M = \overline{M} \times I$ and denote its metic tensor by g. Further we define a metric tensor G in M by

$$G = \rho^2 g = \rho^2 \begin{pmatrix} \bar{g} & 0 \\ 0 & 1 \end{pmatrix}, \tag{6. 1}$$

where ρ is a non-vanishing scalar field in M such that $\rho(y, 0) = 1$ for any point y of $\overline{M}$, and it will be determined later. Since G is conformal to g, the Christoffel symbol Γ of G is related to γ of g by

$$\Gamma_{\mu\lambda}^{\kappa} = \gamma_{\mu\lambda}^{\kappa} + \delta_{\mu}^{\kappa}\rho_{\lambda} + \delta_{\lambda}^{\kappa}\rho_{\mu} - g_{\mu\lambda}\rho^{\kappa}, \tag{6. 2}$$

where we have put

$$\rho_{\lambda} = \partial_{\lambda}(\log \rho), \ \rho^{\kappa} = \rho_{\lambda} g^{\lambda\kappa}. \tag{6. 3}$$

In a local coordinate system (y^h, t) of M, the equation (6. 2) is written separately in the forms

$$\begin{cases} \Gamma_{ji}^{h} = \gamma_{ji}^{h} + \delta_{j}^{h}\rho_{i} + \delta_{i}^{h}\rho_{j} - g_{ji}\rho^{h}, \\ \Gamma_{ji}^{\infty} = -g_{ji}\rho_{\infty}, \\ \Gamma_{j\infty}^{h} = \delta_{j}^{h}\rho_{\infty}, \ \Gamma_{j\infty}^{\infty} = \rho_{j}, \\ \Gamma_{\infty\infty}^{h} = -\rho^{h}, \ \Gamma_{\infty\infty}^{\infty} = \rho_{\infty}, \end{cases} \tag{6. 4}$$

Since $B_i^{\kappa} = \delta_i^{\kappa}$ and $C^{\kappa} = \delta_{\infty}^{\kappa}$ on $\overline{M}$ and we have

$$\nabla_j B_i^{\infty} = B_j^{\mu} B_i^{\lambda} \Gamma_{\mu\lambda}^{\infty} - \gamma_{ji}^{h} B_h^{\infty} = -g_{ji}\rho_{\infty},$$

the second fundamental tensor of $\overline{M}$ as a hypersurface of M is equal to

$$h_{ji} = -g_{ji}\rho_{\infty}, \tag{6. 5}$$

that is, $\overline{M}$ is totally geodesic if $\rho_{\infty} = 0$ identically on $\overline{M}$ or totally umbilical if $\rho_{\infty} \neq 0$ on $\overline{M}$.

Next, if we put

$$F = \begin{pmatrix} f_i^{\ h} & f_i \\ f^h & 0 \end{pmatrix} \tag{6. 6}$$

with respect to a local coordinate system (y^h, t) in M, then F defines a (1,1)-tensor field in M. It is obvious that the tensor field F is an almost complex structure in M and the metric tensor G is almost Hermitian for any choice of ρ with condition $\rho(y, 0) = 1$ for any $y \in \overline{M}$. Therefore we can state that

THEOREM 9. *An almost Grayan manifold can be imbedded in an almost Hermitian manifold as a totally geodesic or umbilical hypersurface.*

From the definitions of the Nijenhuis tensor and C-tensor, it is obvious that

THEOREM 10. *A Grayan manifold can be imbedded in a Hermitian manifold as a totally geodesic or umbilical hypersurface.*

The covariant almost complex structure of M is given by

$$F_{*} = FG = \rho^2 f_{*} = \rho^2 \begin{pmatrix} f_{ji} & f_j \\ -f_i & 0 \end{pmatrix} \tag{6. 7}$$

and the fundamental form Θ by

$$\Theta = F_{\mu\lambda} dx^{\mu} \wedge dx^{\lambda} = \rho^2 f_{CB} dy^C \wedge dy^B. \tag{6. 8}$$

Therefore the 3-tensor $F_{\mu\lambda\kappa}$ of the derived form $d\Theta$ has the following independent components:

$$\begin{aligned} F_{jih} &= \rho^2(\partial_j f_{ih} + \partial_i f_{hj} + \partial_h f_{ji}) + 2\rho^2(\rho_j f_{ih} + \rho_i f_{hj} + \rho_h f_{ji}), \\ F_{ji\infty} &= \rho^2(\partial_j f_i - \partial_i f_j) + 2\rho^2(\rho_j f_i - \rho_i f_j) + 2\rho^2 \rho_{\infty} f_{ji}. \end{aligned} \tag{6. 9}$$

If $\overline{M}$ is almost Sasakian, then we have the equation (1.19) and make the derived form $d\Theta$ vanish by choosing ρ as

$$\rho = e^{-t}, \tag{6.10}$$

and $\overline{M}$ has a positive constant mean curvature $h = h_{ji} g^{ji} = (2p - 1)$ as a hypersurface of M. Thus we have

THEOREM 11. *An almost Sasakian manifold can be imbedded in an almost Kählerian manifold as a totally umbilical hypersurface with positive constant mean curvature.*

Finally let us show that a Sasakian manifold can be imbedded in a Kählerian manifold. Since a Hermitian manifold is Kählerian if and only if the Riemannian connection is an F-connection, it follows from (6. 4) and (6. 6) that the conditions for M to be Kählerian are

$$\begin{aligned} \nabla_j F_{ih} = \rho^2(\overline{\nabla}_j f_{ih} - \rho_i f_{jh} + \rho_h f_{ji} + g_{ji} \rho^a f_{ah} \\ - g_{jh} \rho^a f_{ai} - g_{ji} \rho_{\infty} f_h + g_{jh} \rho_{\infty} f_i) = 0, \end{aligned}$$

$$\nabla_\infty F_{ih} = \rho^2(\rho_i f_h - \rho_h f_i) = 0,$$
$$\nabla_j F_{i\infty} = \rho^2(\overline{\nabla}_j f_i - \rho_i f_j + g_{ji}\rho^h f_h + f_{ji}) = 0,$$
$$\nabla_\infty F_{i\infty} = \rho^2 \rho^h f_{ih} = 0,$$

$\overline{\nabla}$ denoting the covariant differentiation with respect to $\bar{\gamma}$ in $\overline{M}$. The last equation implies that ρ should satisfy

$$\rho_i = \tau f_i, \tag{6.11}$$

τ being a scalar function in M, and then the second equation is fulfilled. The first and the third equations are reduced to

$$\begin{cases} \overline{\nabla}_j f_i = \tau(f_j f_i - g_{ji}) + f_{ji}, \\ \overline{\nabla}_j f_{ih} = \tau(f_i f_{jh} - f_h f_{ji}) - \rho_\infty(f_i g_{jh} - f_h g_{ji}). \end{cases} \tag{6.12}$$

Substituting these equations into the identity $\overline{\nabla}_j(f^i f_{ih}) = 0$, we have $1 + \rho_\infty = 0$ and hence

$$\rho = Ae^{-t}, \tag{6.13}$$

A being a function of y^h. However, since ρ is identically equal to one for $t = 0$, we should choose $A = 1$. Therefore ρ does not depend on the variables y^h and ρ_i vanishes. Then the equations (6.12) are reduced to

$$\begin{cases} \overline{\nabla}_j f_i = f_{ji}, \\ \overline{\nabla}_j f_{ih} = f_i g_{jh} - f_h g_{ji}, \end{cases} \tag{6.14}$$

which are just the same as (1.21). Thus we have established the following

THEOREM 12. *A Sasakian manifold $\overline{M}$ can be imbedded into a Kählerian manifold as a totally umbilical hypersurface.*

REFERENCES

[1] W. M. BOOTHBY AND H. C. WANG, On contact manifolds, Ann. of Math., 68(1958), 721–734.

[2] A. FRÖLICHER, Zur Differentialgeometrie der komplexen Strukturen, Math. Annalen, 129(1955), 50–95.

[3] J. W. GRAY, Some global properties of contact structures, Ann. of Math., vol. 69(1959), 421–450.

[4] A. LICHNEROWICZ, Espaces homogènes kähleriens, Coll. Int. CNRS. Géométrie Différentielle, Strasbourg, 1953, 171–184.

[5] A. NEWLANDER AND L. NIRENBERG, Complex analytic coordinates in almost complex manifolds, Ann. of Math., 65(1957), 391–404.

[6] M. OBATA, Affine connections on manifolds with almost complex, quaternion or Hermitian structure, Japan. Journ. Math., 26(1956), 43–77.

[7] E. M. PATTERSON, A characterization of Kaehler manifolds in terms of parallel fields of planes, Journ. London Math. Soc., 28(1953), 260–269.

[8] S. SASAKI, On differentiable manifolds with certain structures which are closely related to almost contact structure, I, Tôhoku Math. Journ., (2) 12(1960), 456–476.

[9] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with certain structures

which are closely related to almost contact structure, II, Tôhoku Math. Journ., (2) 13(1961), 281–294.

[10] J. A. SCHOUTEN AND K. YANO, On an intrinsic connexion in an X_{2n} with an almost Hermitian structure, Indag. Math., 17(1955), 1–9.

[11] K. YANO AND E. T. DAVIES, Contact tensor calculus, Annali di Mat., 37(1954), 1–36.

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