SOME REMARKS ON POSTNIKOV COMPLEXES AND FIBRE SPACES

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In § 1 a concise exposition of the Postnikov system of a CW-complex is presented to prepare for the rest of the paper. In § 2 fibrations due to Cartan, Serre, and G. Whitehead, are discussed. In § 3 it is shown that the suspension operation $\widetilde{S}$ (not exactly the usual one) and the loop operation Ω are "dual" in a sense that for certain Postnikov complexes Y of a CW-complex with vanishing lower dimensional homotopy groups, $\Omega\,\widetilde{S}\,(Y)$ is of the same homotopy type as Y. This enables us to construct a fibration such that for pairs of such Postnikov complexes the injection map is homotopically equivallent to the injection map of a fibre into the total fibre space. In § 4 an exact sequence is discussed to study a role of Postnikov complexes and invariants in determining homology and homotopy.

If $f\colon X \rightarrow Y$ is a map, the following notations are adopted unless otherwise stated: 1) f_ρ denotes the homomorphism induced by f between homotopy groups in dimension ρ, 2) $f_\#$ and $f^\#$ denote the homomorphisms induced by f between homology and cohomology groups respectively, and 3) the superscript ρ on the shoulder of a map denotes the ρ-th map of a sequence of maps. The numbers in square brackets refer to the papers of the bibliography at the end of the paper.

1. Postnikov complexes and self-obstruction cocycles. Let X be an arcwise-connected CW-complex and let x be a base point in X. Attaching cells of dimension $(n+1)$ to X by maps representing a set of generators of $\pi_n(X, x)$, we may embed X in a CW-complex X' such that $\pi_n(X', x) = 0$, and for each $\rho < n$, the injection homomorphism $\pi_\rho(X, x) \rightarrow \pi_\rho(X', x)$ is an isomorphism onto. By iterated use of the process of killing homotopy of X in dimensions greater than n, a CW-complex B_n is obtained with the following properties:

1) X is a closed subcomplex of B_n, and $B_n^n = X^n$, where B_n^n and X^n denote the n-skeleton of B_n and X respectvely.

2) $\pi_\rho(B_n, x) = 0$ for each $\rho \geqq n$.

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3) $j: X \to B_n$ is the injection map, then $j_p: \pi_p(X, x) \to \pi_p(B_n, x)$ is an isomorphism onto for each $p < n$.

Thus we have a sequence of spaces $\{B_n : n = 2, 3, \cdots\}$ and B_n will be called a Postnikov complex of type $(\pi_1, \pi_2, \cdots, \pi_{n-1})$, or simply a Postnikov complex if there is no ambiguity.

By using Postnikov complexes we shall define the notion of obstruction cocycles of a map so that we may discuss them in terms of self-obstruction cocycles of a space. Unless otherwise stated, all spaces considered in this work are assumed to be arcwise connected and n-simple for each integer $n \geqq 1$. Let X and $\bar{X}$ be CW-complexes, and let $f: X^n \to \bar{X}$ be a given map. Let $\bar{B}_n$ be a Postnikov complex of type $(\bar{\pi}_1, \cdots, \bar{\pi}_{n-1})$ for $\bar{X}$, then f has an extension $\bar{f}: (X^{n+1}, X^n) \to (\bar{B}_n, \bar{X})$ which is unique up to homotopy. Hence the induced homomorphism $\bar{f}_{n+1}: \pi_{n+1}(X^{n+1}, X^n) \to \pi_{n+1}(\bar{B}_n, \bar{X})$ is uniquely determined by f. Consider the following diagram

$$\begin{array}{ccc} \pi_{n+1}(X^{n+1}, X^n) & \xrightarrow{\bar{f}_{n+1}} & \pi_{n+1}(\bar{B}_n, \bar{X}) \\ \rho \downarrow & & \downarrow \bar{\partial} \\ C_{n+1}(X) = H_{n+1}(X^{n+1}, X^n) & & \pi_n(\bar{X}) \end{array}$$

where ρ is the natural homomorphism of homotopy into homology, $\bar{\partial}$ is the homotopy boundary operator, and $C_{n+1}(X)$ is the integral group of $(n+1)$ chains of X. Define the obstruction cocycle $c^{n+1}(f)$ to be $\bar{\partial}\, \bar{f}_{n+1}\, \rho^{-1}$. Although ρ is not necessarily one-to-one, $c^{n+1}(f)$ is well defined. For X is n-simple.

Let X be a CW-complex and let B_n be a Postnikov complex of type $(\pi_1, \cdots, \pi_{n-1})$. Let $i: X^n \to X$ be the injection map. Since $B_n^n = X^n$ by the construction of B_n, i may be considered as a map $i: B_n^n \to X$. Consider the obstruction cocycle $c^{n+1}(i)$ of extending i over B_n^{n+1}. Then $c^{n+1}(i) = \partial \bar{i}_{n+1}\, \rho^{-1}$, where

$$C_{n+1}(B_n) = H_{n+1}(B_n^{n+1}, X^n) \xleftarrow{\rho} \pi_{n+1}(B_n^{n+1}, X^n) \xrightarrow{\bar{i}_{n+1}} \pi_{n+1}(B_n, X) \xrightarrow{\partial} \pi_n(X).$$

This obstruction cocycle $c^{n+1}(i)$ will be often denoted by c^{n+1} for the sake of brevity, and will be appropriately called a self-obstruction cocycle of X after Adams [1].

PROPOSITION 1.1. *Let $B_n, \bar{B}_n$ be Postnikov complexes of types $(\pi_1, \cdots, \pi_{n-1})$ and $(\bar{\pi}_1, \cdots, \bar{\pi}_{n-1})$ for given CW-complexes $X, \bar{X}$ respectively. Let $f: X \to \bar{X}$ be a given map and let $i: X \to B_n$ and $\bar{i}: \bar{X} \to \bar{B}_n$ be the injection maps. Then there exists a map $F: B_n \to \bar{B}_n$ such that the diagram*

$$\begin{array}{ccc} B_n & \xrightarrow{F} & \bar{B}_n \\ i \uparrow & & \uparrow \bar{i} \\ X & \xrightarrow{f} & \bar{X} \end{array}$$

is commutative. Moreover F is uniquely determined by f up to homotopy relative to X.

PROOF. Since $\pi_\rho(\bar{B}_n)=0$ for $\rho \geqq n$, $\bar{i}\circ f\colon X \to \bar{B}_n$ has an extension $F\colon B_n \to \bar{B}_n$ and the commutativity of the diagram is immediate. Let $G\colon B_n \to \bar{B}_n$ be another extension. Decompose $B_n \times I$ into a CW-complex in the usual manner, and define $H\colon (B_n \times 0) \cup (X \times I) \cup (B_n \times 1) \to \bar{B}_n$ by

$$\begin{aligned} H(\xi, 0) &= F(\xi) && \text{for all } \xi \in B_n, \\ H(\xi, 1) &= G(\xi) && \text{for all } \xi \in B_n, \\ H(\xi, t) &= \mathrm{f}(\xi) && \text{for all } \xi \in X \text{ and for all } t \in I. \end{aligned}$$

Since $\pi_\rho(\bar{B}_n)=0$ for $\rho \geqq n$, H has an extension to $B_n \times I$. Hence F and G are homotopic relative to X.

PROPOSITION 1.2. *Let X and $\bar{X}$ be CW-complexes of the same homotopy type, and let B_n and $\bar{B}_n$ be Postnikov complexes for X and $\bar{X}$ respectively. Then B_n and $\bar{B}_n$ are of the same homotopy type.*

PROOF. Let $f\colon X \to \bar{X}$ be a homotopy equivalence and let $F\colon B_n \to \bar{B}_n$ be an extension as is shown in the previous proposition. Consider the commutative homotopy diagram

$$\begin{tikzcd} \pi_\rho(B_n) \arrow[r, "F_\rho"] & \pi_\rho(\bar{B}_n) \\ \pi_\rho(X) \arrow[u, "i_\rho"] \arrow[r, "f_\rho"] & \pi_\rho(\bar{X}). \arrow[u, "\bar{i}_\rho"'] \end{tikzcd}$$

As f_ρ, i_ρ, and $\bar{i}_\rho$ are isomorphism for $\rho < n$, F_ρ is an isomorphism for all ρ. It follows from a theorem due to J. H. C. Whitehead [16] that F is a homotopy equivalence.

COROLLARY 1.3. *The Postnikov complexes of type $(\pi_1, \cdots, \pi_{n-1})$ for a given CW-complex are of the same homotopy type for each integer $n \geqq 2$.*

PROOF. Let B_n, $\bar{B}_n$ be Postnikov complexes for X. Since the identity map $i\colon X \to X$ is a homotopy equivalence, there exists a homotopy equivalence $\varphi\colon B_n \to \bar{B}_n$ by the above proposition.

Let $c^{n+1}(f)$ be an obstruction cocycle of $f\colon X^n \to \bar{X}$ where X and $\bar{X}$ are CW-complexes. In the next proposition $c^{n+1}(f)$ is expressed in terms of the self-obstruction cocycles c^{n+1}, $\bar{c}^{n+1}$ of X and $\bar{X}$.

PROPOSITION 1.4. *Let $F\colon B_n \to \bar{B}_n$ be the extension of $f\colon X^n \to \bar{X}$ as in Proposition 1.1. Let $i\colon X^n \to B_n$ be the injection map. Then* *we have* (1) $c^{n+1}(f) = \bar{c}^{n+1}\, F_{\#}\, i_{\#}$

and (2) $c^{n+1}(f) = (F|X)_n\, c^{n+1}\, i_{\#}$,

where $F_{\#}: H_{n+1}(B_n^{n+1}, X^n) \to H_{n+1}(\bar{B}_n^{n+1}, \bar{X}^n)$

and $i_{\#}: H_{n+1}(X^{n+1}, X^n) \to H_{n+1}(B_n^{n+1}, X^n)$.

PROOF. For the proof of (1), refer to Adams [1]. For (2) consider the commutative diagram

$$\begin{tikzcd}
\pi_{n+1}(X^{n+1}, X^n) \arrow[rrr, "\bar{f}_{n+1}"] \arrow[dr] \arrow[ddd, "\rho"'] & & & \pi_{n+1}(\bar{B}_n, \bar{X}) \arrow[ddd, "\bar{\partial}"] \\
 & \pi_{n+1}(B_n^{n+1}, X^n) \arrow[r, "\bar{i}_{n+1}"] \arrow[d, "\rho"'] & \pi_{n+1}(B_n, X) \arrow[ur] \arrow[d, "\partial"] & \\
 & H_{n+1}(B_n^{n+1}, X^n) & \pi_n(X) \arrow[dr, "(F|X)_n"'] & \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur, "i_{\#}"] & & & \pi_n(\bar{X})
\end{tikzcd}$$

Then we have $c^{n+1}(f) = \bar{\partial}\bar{f}_{n+1}\,\rho^{-1} = (F|X)_n\,\partial\,\bar{i}_{n+1}\,\rho^{-1}\, i_{\#} = (F|X)_n\, c^{n+1}\, i_{\#}$. This completes the proof.

Let X be a CW-complex and let B_n be a Postnikov complex for X. Then the self-obstruction cocycle c^{n+1} represents an element $l^{n+1}(X)$ of $H^{n+1}(B_n; \pi_n(X))$. By the $\mathcal{L}$-system of a CW-complex X we mean $\mathcal{L}(X) = \{l^{n+1}(X): n = 2, 3, \cdots\}$. The defintion of $\mathcal{L}(X)$ depends upon the choice of the sequence $\{B_n: n = 2, 3, \cdots\}$ of Postnikov complexes for X, but it will be shown in a proposition that the sequence of complexes plays an auxiliary role in the definition.

Let $\mathcal{L}(X)$ and $\mathcal{L}(\bar{X})$ be $\mathcal{L}$-system of CW-complexes X and $\bar{X}$, which are defined by the aid of the sequences of Postnikov complexes $\{B_n\}$, $\{\bar{B}_n\}$ for X and $\bar{X}$ respectively. For the sake of simplicity let us denote $\pi_n(X)$, $\pi_n(\bar{X})$ by π_n, $\bar{\pi}_n$ respectively.

By a homomorphism $\varphi: \mathcal{L}(X) \to \mathcal{L}(\bar{X})$ we mean two sequences of homomorphisms θ_n for each $n \geqq 1$ and ω_{n+1} for each $n \geqq 2$, such that

$$\theta_n: \pi_n \to \bar{\pi}_n$$

and $$\omega_{n+1}: H^{n+1}(\bar{B}_n; \bar{\pi}_n) \to H^{n+1}(B_n; \bar{\pi}_n),$$

satisfying the condition $\omega_{n+1}(l^{n+1}) = \theta_n^*(l^{n+1})$, where $\theta_n^*: H^{n+1}(B_n; \pi_n) \to H^{n+1}(B_n; \bar{\pi}_n)$ is induced by θ_n. A homomorphism φ between two $\mathcal{L}$-systems is called an isomorphism if and only if all θ_n, ω_{n+1} are isomorphisms.

PROPOSITION 1.5. *If X and $\bar{X}$ are CW-complexes of the same homotopy type, then* $\mathfrak{L}(X)$ and $\mathfrak{L}(\bar{X})$ *are isomorphic.*

PROOF. Let B_n and $\bar{B}_n$ be Postnikov complexes for X and $\bar{X}$, by the aid of which $\mathfrak{L}(X)$ and $\mathfrak{L}(\bar{X})$ are defined respectively. Let $f: X \to \bar{X}$ be a homotopy equivalence. By virtue of Proposition 1.2, f determines $F^n: B_n \to \bar{B}_n$, which is a homotopy equivalence for each integer $n \geqq 2$. As f and F^n are homotopy equivalences, $f_n: \pi_n \to \bar{\pi}_n$ and $F^{n\#}: H^{n+1}(B_n; \bar{\pi}_n) \to H^{n+1}(B_n; \bar{\pi}_n)$ are all isomorphisms onto. Define $\varphi = \{\theta_n, \omega_{n+1}\}$ by $\theta_n = f_n$ and $\omega_{n+1} = F^{n\#}$. In order to show that φ is an isomorphism, it is sufficient to prove $F^{n\#}(\bar{l}^{n+1}) = f_n^*(l^{n+1})$, where $f_n^*: H^{n+1}(B_n; \pi_n) \to H^{n+1}(B_n; \bar{\pi}_n)$. Consider the diagram

$$\begin{tikzcd}
\pi_{n+1}(B_n^{n+1}, X^n) \arrow[rrr, "\bar{i}_{n+1}"] \arrow[dr] \arrow[ddd, "\rho"'] & & & \pi_{n+1}(B_n, X) \arrow[dl] \arrow[ddd, "\partial"] \\
& \pi_{n+1}(\bar{B}_n^{n+1}, \bar{X}^n) \arrow[r, "\bar{\bar{i}}_{n+1}"] \arrow[d, "\rho"] & \pi_{n+1}(\bar{B}_n, \bar{X}) \arrow[d, "\bar{\partial}"] & \\
& H_{n+1}(\bar{B}_n^{n+1}, \bar{X}^n) & \pi_n(\bar{X}) & \\
H_{n+1}(B_n^{n+1}, X^n) \arrow[ur, "F^n_\#"] & & & \pi_n(X) \arrow[ul, "f_n"']
\end{tikzcd}$$

By definition $c^{n+1} = \partial\, \bar{i}_{n+1}\, \rho^{-1}$ and $\bar{c}^{n+1} = \bar{\partial}\ \bar{\bar{i}}_{n+1}\ \bar{\rho}^{-1}$. It follows from the commutativity of the diagram that $\bar{c}^{n+1}\ F^n_\# = f_n\, c^{n+1}$. Hence $F^{n\#}(\bar{l}^{n+1})$ represented by $\bar{c}^{n+1}\ F^n_\#$ is equal to $f_n^*(l^{n+1})$ whose representative is $f_n\, c^{n+1}$. This completes the proof.

COROLLARY 1.6. *The* $\mathfrak{L}$*-system of a CW-complex is uniquely determined up to isomorphism.*

PROOF. In the proof of the previous proposition let $\bar{X} = X$, $f =$ identity map, then we have the corollary.

Let B_n, B_{n+1}, $\bar{B}_n$ and $\bar{B}_{n+1}$ be Postnikov complexes for X and $\bar{X}$ respectively. Given a homomorphism $\theta_n: \pi_n \to \bar{\pi}_n$ and a pair of maps $F^n: B_n \to \bar{B}_n$, $F^{n+1}: B_{n+1} \to \bar{B}_{n+1}$ satisfying

1) the commutative diagram

$$\begin{tikzcd}
\pi_n(B_{n+1}) \arrow[r, "F_n^{n+1}"] & \pi_n(\bar{B}_{n+1}) \\
\pi_n \arrow[u] \arrow[r, "\theta_n"] & \bar{\pi}_n \arrow[u]
\end{tikzcd}$$

2) $F^n | X^{n-1} = F^{n+1} | X^{n-1}$.

Then the pair (F^n, F^{n+1}) is called to be regularly associated with θ_n.

An isomorphism $\varphi: \mathfrak{L}(X) \to \mathfrak{L}(\bar{X})$, where $\varphi = \{\theta_n, \omega_{n+1}\}$ is called to have a canonical realization if and only if

1) There exists a sequence of maps $F^n: B_n \to \bar{B}_n$ for each $n \geqq 2$ such that $\omega_{n+1} = F^{n\#}$, and the diagram

$$\begin{tikzcd} \pi_{n-1}(B_n) \arrow[r, "F^n_{n-1}"] & \pi_{n-1}(\bar{B}_n) \\ \pi_{n-1} \arrow[u] \arrow[r, "\theta_{n-1}"] & \bar{\pi}_{n-1} \arrow[u] \end{tikzcd}$$

is commutative, where the vertical arrows are injection isomorphisms,

2) For any integer $n \geqq 2$ each pair (F^n, F^{n+1}) is regularly associated with θ_n.

PROPOSITION 1.7. *Let X and $\bar{X}$ be CW-complexes. They are of the same homotopy type if and only if there exists an isomorphism between their $\mathfrak{L}$-systems which has a canonical realization.*

PROOF. Suppose X and $\bar{X}$ are of the same homotopy type. By the Proposition 1.5 there exists an isomorphism $\varphi: \mathfrak{L}(X) \to \mathfrak{L}(\bar{X})$. We wish to show that this isomorphism has a canonical realization. Let $f: X \to \bar{X}$ be a homotopy equivalence. Then f defines a homotopy equivalence $F^n: B_n \to \bar{B}_n$ for each integer $n \geqq 2$ as was shown in the Proposition 1.2. By the proof of the proposition we have the commutative diagram

$$\begin{tikzcd} \pi_n(B_{n+1}) \arrow[r, "F^{n+1}_n"] & \pi_n(\bar{B}_{n+1}) \\ \pi_n \arrow[u] \arrow[r, "f_n"] & \bar{\pi}_n \arrow[u] \end{tikzcd}$$

for $n \geqq 1$. Moreover, from the construction of $F^n: B_n \to \bar{B}_n$ we have $F^{n+1} | X^n = F^n | X^n$. This proves the first half of the theorem.

Assume that $\varphi: \mathfrak{L}(X) \to \mathfrak{L}(\bar{X})$ is an isomorphism which has a canonical realization determined by a sequence of maps $F^n: B_n \to \bar{B}_n$ for $n \geqq 2$. Let us define $f: X \to \bar{X}$ by $f | X^n = F^{n+1} | X^n$. Then f is well defined because $F^n | X^{n-1} = F^{n+1} | X^{n-1}$. Consider the diagram

$$\begin{array}{ccc}
\pi_n & \xrightarrow{\quad f_n \quad} & \bar{\pi}_n \\
\downarrow & & \downarrow \\
\pi_n(B_{n+1}) & \xrightarrow{\quad F_n^{n+1} \quad} & \pi_n(\bar{B}_{n+1}) \\
\uparrow & & \uparrow \\
\pi_n & \xrightarrow{\quad \theta_n \quad} & \bar{\pi}_n
\end{array}$$

where the vertical arrows are injection isomorphisms. The upper half of the diagram is commutative by the construction of f, while the lower half is commutative because the isomorphism φ has the canonical realization. Hence we have $f_n = \theta_n$ for all $n \geqq 1$. This completes the proof.

2. Fibrations. Let a triple (T, p, B) be a fibre space with a fibre map $p: T \to B$ in the sense of Serre [10]. Then, as is usual, let us call T the (total) fibre space, B the base space, and $F = p^{-1}(p(x_0))$ a fibre over $b = p(x_0)$ with $x_0 \in T$. We shall often use a notation

$$0 \longrightarrow F \longrightarrow T \xrightarrow{p} B \longrightarrow 0$$

in case when we have a fibre space in the above sense. Notice that "$B \to 0$" does not mean the surjection of a fibre map p.

DEFINITION 2.1. Given spaces X and Y with a map $f: Y \to X$. A triple (Y, f, X) is said to have a fibration of the first category, if and only if there exists a fibre space (T, p, B) and homotopy equivalences $\sigma: T \to Y$, $\tau: B \to X$ such that the diagram

$$\begin{array}{ccc}
T & \xrightarrow{\quad p \quad} & B \\
\sigma \downarrow & & \downarrow \tau \\
Y & \xrightarrow{\quad f \quad} & X
\end{array}$$

is homotopically commutative; namely $f\sigma$ is homotopic to τp.

DEFINITION 2. 2. A triple (Y, f, X) with a map $f: Y \to X$ is said to have a fibration of the second category if and only if there exists a fibre space (T, p, B) such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & T & \longrightarrow & B & \longrightarrow & 0 \\
 & & \eta \downarrow & & \downarrow \sigma & & & & \\
 & & Y & \xrightarrow{f} & X & & & &
\end{array}$$

is homotopically commutative, where η and σ are homotopy equivalences.

DEFINITION 2.3. Let X and Y be pathwise connected spaces and let $f\colon Y \to X$ be a map. f is called an algebraic homotopy equivalences if and only if the induced homomorphism of homotopy groups,

$$f_\rho\colon \pi_\rho(Y) \longrightarrow \pi_\rho(X),$$

is an isomorphism in all dimensions ρ.

It is well known that the fibration of the second category is not always possible for a given triple (Y, f, X) with $f\colon Y \to X$, although any triple can be fibrated in the first sense. For a given triple (Y, f, X), a map $f\colon Y \to X$ may be considered to be homotopically equivalent to an injection map in the following sense. Let M_f be the mapping cylinder of f and $i\colon Y \to M_f$ be the injection map. Then we have the commutative diagram

$$\begin{tikzcd} & M_f \arrow[d, "r"] \\ Y \arrow[ur, "i"] \arrow[r, "f"'] & X \end{tikzcd}$$

where r is a retraction. Hence it is sufficient to consider fibrations for a triple (Y, i, M_f) with the injection map i, so that we have fibrations for the given triple (Y, f, X).

Throughout the paper we shall use the following notations and fibrations due to Cartan and Serre [2]. Let A and B be non-empty subspaces of a pathwise connected space X, and let us denote by $P(X;\, A, B)$ the space of paths in X which start at A and end in B. If B consists of a single point, we abbreviate $P(X;\, A, B)$ to $P(X;\, A)$, and we denote by $\Omega(X)$ the space of all loops in X with a base point $* \in X$.

Let X be a pathwise connected CW-complex and let B_s be the s-th Postnikov complex for X. Then a map $p^s\colon P(B_s;\, X) \to X$, defined by $p^s(\omega) = \omega(0)$ for $\omega \in P(B_s;\, X)$, is a $(s-1)$ connective fibre map. Consider a commutative diagram

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ 0 \arrow[r] & \Omega(B_{s+1}) \arrow[r] \arrow[d] & P(B_{s+1};\, X) \arrow[r, "p^{s+1}"] \arrow[d, "i"] & X \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega(B_s) \arrow[r] \arrow[d] & P(B_s;\, X) \arrow[r, "p^s"] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P(B_s;\, B_{s+1}) \arrow[r] & P(B_s;\, B_{s+1}) \arrow[r] & \{*\} \arrow[r] \arrow[d] & 0 \\ & & & 0 & \end{tikzcd}$$

with fiberings in the rows. It is easy to see that $P(B_s;\ B_{s+1})$ is an Eilenberg-MacLane space $K(\pi_s(X), s)$, and that Cartan and Serre constructed a fibration of the second category (in a weaker sense that σ and η are algebraic homotopy equivalences in the definition 2.3) for a pair $(P(B_s;\ X),\ P(B_{s+1};\ X))$ in case when X is a space.

PROPOSITION 2.4. *If (A, B) is a pair of spaces, then $P(P(A;\ B);\ \Omega(A))$ is of the same homotopy type as $\Omega(B)$.*

PROOF. Let $(E^2;\ S^+,\ S^-)$ be a triad which is defined by $E^2 = \{(x_1,\ x_2) | x_1^2 + x_2^2 \leqq 1\}$, $S^+ = \{(x_1,\ x_2) | x_1^2 + x_2^2 = 1$ and $x_1 \geqq 0\}$, $S^- = \{(x_1,\ x_2) | x_1^2 + x_2^2 = 1$ and $x_1 \leqq 0\}$. Consider a continuous map φ: $(I \times I,\ \partial(I \times I)) \to (E^2,\ \partial E^2)$ such that $\varphi((I \times 1) \cup (1 \times I)) = (0,\ 1)$, $\varphi(0, 0) = (-1,\ 0)$, and φ is topological elsewhere with the property $\varphi(0 \times I) = S^+$, $\varphi(I \times 0) = S^-$. Let Λ be a function space of all maps $f: (E^2;\ S^+,\ S^-) \to (A;\ A, B)$. For any path $\xi \in P(P(A;\ B);\ \Omega(A))$ there exists a map f_ξ: $I \times I \to A$ defined by $f_\xi(t_1,\ t_2) = \xi(t_1)(t_2)$ for any $(t_1,\ t_2) \in I \times I$. Since $f_\xi(0 \times I) \subset A$, $f_\xi(I \times 0) \subset B$, and $f_\xi((I \times 1) \cup (1 \times I)) = *$, there exists a map $\lambda: P(P(A;\ B);\ \Omega(A)) \to \Lambda$ such that $\lambda(\xi) = f_\xi\, \varphi^{-1} \in \Lambda$. Then it is easily seen that λ is a homeomorphism. On the other hand, it is also seen that there exists a homeomorphism μ between Λ and $P(\Omega(A);\ \Omega(A),\ \Omega(B))$. This can be proved in a similar way as before. Since $P(\Omega(A);\ \Omega(A),\ \Omega(B))$ is of the same homotopy type as $\Omega(B)$, the proof is completed.

PROPOSITION 2.5.[1] *Let (A, B) be a pair of spaces. Then a pair $(\Omega(A), \Omega(B))$ of loop spaces has a fibration of the second category.*

PROOF. Consider the following diagram with a fibering in the row

$$\begin{array}{ccccccccc} 0 & \to & P(P(A;B);\Omega(A)) & \to & P(P(A;B);\Omega(A),\ P(A;B)) & \to & P(A;B) & \to & 0 \\ & & \eta \downarrow & & \downarrow \sigma & & & & \\ & & \Omega(B) & \longrightarrow & \Omega(A) & & & & \end{array}$$

Since η and σ are homotopy equivalences, we have a desired fibration.

3. Fibration of a pair $(B_s,\ B_{s+1})$ of Postnikov complexes.

PROPOSITION 3.1. *Given an integer $n > 1$. Let X be an $(n-1)$ connected CW-complex and let B_s be a Postnikov complex for X. If $s \leqq 2n-1$, there exists a CW-complex $\widetilde{B}_s$ such that B_s and the space of loops, $\Omega(\widetilde{B}_s)$, in $\widetilde{B}_s$ are of the same homotopy type.*

1) The authors have been informed that H. Toda has the same proof of the Proposition 2.5.

PROOF. If $s \leqq n$, choose $\widetilde{B}_s$ as a space of a single point. Then the theorem holds true, because B_s is cotractible to a point. Assume $2n-1 \geqq s \geqq n+1$. Let $\widehat{B}_s$ be the suspension of B_s (see [6], or [11]). To kill homotopy groups $\pi_\rho(\widehat{B}_s)$ for $\rho \geqq 2n+1$, we attach to $\widehat{B}_s$ cells of dimensions greater than $2n+1$ so that we may embed $\widehat{B}_s$ in the resultant complex $\widetilde{B}_s$. Then $\pi_\rho(\widetilde{B}_s)=0$ for each $\rho \geqq 2n+1$ and the injection map $k^s: \widehat{B}_s \to \widetilde{B}_s$ induces an isomorphism $k^s_\rho: \pi_\rho(\widehat{B}_s) \to \pi_\rho(\widetilde{B}_s)$ for each $\rho \leqq 2n$. Let $(B_s)_\infty$ be the reduced product space of B_s and let $\Omega(\widehat{B}_s)$ be the space of loops in $\widehat{B}_s$. Then there exist the injection map $j: B_s \to (B_s)_\infty$, a canonical map $\alpha^s: (B_s)_\infty \to \Omega(\widehat{B}_s)$, and the natural map $\Omega(k^s): \Omega(\widehat{B}_s) \to \Omega(\widetilde{B}_s)$ induced by k^s. Let us denote by φ^s the composite map $\Omega(k^s)\ \alpha^s\ j$. Consider the diagram

$$\begin{tikzcd}
\pi_\rho(B_s) \arrow[r, "j_\rho"] \arrow[drr, "E_\rho"'] & \pi_\rho((B_s)_\infty) \arrow[r, "\alpha^s_\rho"] & \pi_\rho(\Omega(\widehat{B}_s)) \arrow[r, "\Omega(k^s)_\rho"] \arrow[d, "\Omega"'] & \pi_\rho(\Omega(\widetilde{B}_s)) \arrow[d, "\widetilde{\Omega}"'] \\
 & & \pi_{\rho+1}(\widehat{B}_s) \arrow[r, "k^s_{\rho+1}"] & \pi_{\rho+1}(\widetilde{B}_s)
\end{tikzcd}$$

where $E_\rho = \Omega\ \alpha^s_\rho\ j_\rho$ is the suspension homomorphism and $\Omega(k^s)_\rho = \widetilde{\Omega}^{-1} k^s_{\rho+1}\, \Omega$. Hence we have $\varphi^s_\rho = \widetilde{\Omega}^{-1}\ k^s_{\rho+1} E_\rho$. By virtue of the suspension theorem due to James, E_{2n-1} is onto. Since $\pi_{2n-1}(B_s)=0$ for $s \leqq 2n-1$, we have $\pi_{2n}(\widehat{B}_s) \cong \pi_{2n}(\widetilde{B}_s) \cong 0$, so that $\pi_\rho(\Omega(\widetilde{B}_s)) \cong \pi_\rho(B_s) \cong 0$ for each $\rho \geqq 2n-1$. Since E_ρ is isomorphic onto for $\rho \leqq 2n-2$ and $k^s_{\rho+1}$ is isomorphic onto for $\rho \leqq 2n-1$, φ^s_ρ is also isomorphic onto for $\rho \leqq 2n-2$. By a theorem due to Milnor, $\Omega(\widetilde{B}_s)$ may be considered as a CW-complex, so that φ^s is a homotopy equivalence. The proof is completed.

PROPOSITION 3.2. *Let X be an $(n-1)$ connected CW-complex and let B_s be a Postnikov complex for X. Then there exists a sequence of CW-complexes $\{\widetilde{B}_{n+1}, \cdots, \widetilde{B}_s, \widetilde{B}_{s+1}, \cdots, \widetilde{B}_{2n-1}\}$ such that for $2n-1 \geqq s \geqq n+1$, B_s and $\Omega(\widetilde{B}_s)$ are of the same homotopy type and the commutative diagram*

$$\begin{tikzcd}
\widetilde{B}_{n+1} & \cdots \arrow[l] & \widetilde{B}_s \arrow[l] & \widetilde{B}_{s+1} \arrow[l, "\widetilde{i}^{s+1}"'] & \cdots \arrow[l] & \widetilde{B}_{2n-1} \arrow[l] \\
\widehat{B}_{n+1} \arrow[u, "k^{n+1}"'] & \cdots \arrow[l] & \widehat{B}_s \arrow[l] \arrow[u, "k^s"'] & \widehat{B}_{s+1} \arrow[l, "\widehat{i}^{s+1}"'] \arrow[u, "k^{s+1}"'] & \cdots \arrow[l] & \widehat{B}_{2n-1} \arrow[l] \arrow[u, "k^{2n-1}"']
\end{tikzcd}$$

holds true, where all arrows in the diagram denote injection maps.

PROOF. Let $i: B_{s+1} \to B_s$ be the injection map. From the construction of suspension we obtain the injection map $\hat{i}^{s+1}: \widehat{B}_{s+1} \to \widehat{B}_s$ induced by the injection map $i': B_{s+1} \times I \to B_s \times I$ with the commutative diagram

$$\begin{tikzcd} B_{s+1} \times I \arrow[r, "i'"] & B_s \times I \arrow[d, "\lambda^s"] \\ \widehat{B}_{s+1} \arrow[u, "\lambda^{s+1}"] \arrow[r, "\hat{i}^{s+1}"] & \widehat{B}_s, \end{tikzcd}$$

where I denotes the unit interval and λ's are identification maps. By the previous proposition there exists a CW-complex $\widetilde{B}_{2n-1}$ such that B_{2n-1} and $\Omega(\widetilde{B}_{2n-1})$ are of the same homotopy type. Assume that the commutative diagram

$$\begin{tikzcd} \widetilde{B}_{s+1} & \cdots \arrow[l] & \widetilde{B}_{2n-1} \arrow[l] \\ \widehat{B}_{s+1} \arrow[u] & \cdots \arrow[l] & \widehat{B}_{2n-1} \arrow[l] \arrow[u] \end{tikzcd}$$

has been established. It is sufficient to show that there exists a $\widetilde{B}_s$ with the commutative diagram

$$\begin{tikzcd} \widetilde{B}_s \arrow[r, "\tilde{i}^{s+1}"] & \widetilde{B}_{s+1} \\ \widehat{B}_s \arrow[u, "k^s"] \arrow[r, "\hat{i}^{s+1}"] & \widehat{B}_{s+1}. \arrow[u, "k^{s+1}"'] \end{tikzcd}$$

Considering $\widehat{B}_{s+1}$ as its homeomorphic image under k^{s+1}, we may attach $\widetilde{B}_{s+1}$ to $\widehat{B}_s$ by the map $\hat{i}^{s+1}$. Let us denote the resultant complex by $\widehat{B}_s \cup \widetilde{B}_{s+1}$. Then we may consider $\widehat{B}_s$ and $\widetilde{B}_{s+1}$ as subspaces of $\widehat{B}_s \cup \widetilde{B}_{s+1}$. Since we have $(\widehat{B}_s \cup \widetilde{B}_{s+1})^{2n+1} = (\widehat{B}_s)^{2n+1}$ from the construction of $\widetilde{B}_{s+1}$, $\pi_\rho(\widehat{B}_s \cup \widetilde{B}_{s+1}) \cong \pi_\rho(\widehat{B}_s)$ for $\rho \leqq 2n$. Killing homotopy groups $\pi_\rho(\widehat{B}_s \cup \widetilde{B}_{s+1})$ for $\rho \geqq 2n+1$ by attaching to $\widehat{B}_s \cup \widetilde{B}_{s+1}$ cells of dimension greater than $2n+1$, we obtain $\widetilde{B}_s$ with the desired properties. This completes the proof.

PROPOSITION 3.3. *The diagram*

$$\begin{tikzcd} \Omega(\widetilde{B}_{s+1}) \arrow[r, "\Omega(\tilde{i}^{s+1})"] & \Omega(\widetilde{B}_s) \\ B_{s+1} \arrow[u, "\varphi^{s+1}"] \arrow[r, "i^{s+1}"] & B_s \arrow[u, "\varphi^s"'] \end{tikzcd}$$

is commutative.

PROOF. From propositions 3.1 and 3.2 we have the commutative diagram

$$\begin{array}{ccc}
\Omega(\widetilde{B}_{s+1}) & \xrightarrow{\Omega(\widetilde{i}^{s+1})} & \Omega(\widetilde{B}_s) \\
{\scriptstyle \Omega(k^{s+1})}\uparrow & & \uparrow{\scriptstyle \Omega(k^s)} \\
\Omega(\widehat{B}_{s+1}) & \xrightarrow{\Omega(\widehat{i}^{s+1})} & \Omega(\widehat{B}_s) \\
{\scriptstyle \alpha^{s+1}}\uparrow & & \uparrow{\scriptstyle \alpha^s} \\
(B_{s+1})_\infty & \longrightarrow & (B_s)_\infty \\
{\scriptstyle j^{s+1}}\uparrow & & \uparrow{\scriptstyle j^s} \\
B_{s+1} & \longrightarrow & B_s
\end{array}$$

Hence the proof is completed.

PROPOSITION 3.4. *Given an integer $n > 2$ and an integer s with $2n-2 \geqq s$. Let X be an $(n-1)$ connected CW-complex and let B_s and B_{s+1} be Postnikov complexes for X. Then the pair (B_s, B_{s+1}) has a fibration of the second category.*

PROOF. Let $(\widetilde{B}_s, \widetilde{B}_{s+1})$ be a pair of CW-complexes which we constructed in the Proposition 3.2. By the Propositions 2.5 and 3.1, we have a diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & F & \longrightarrow & T & \longrightarrow & P(\widetilde{B}_s;\ \widetilde{B}_{s+1}) & \longrightarrow 0 \\
 & {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \sigma} & & & \\
 & \Omega(\widetilde{B}_{s+1}) & \xrightarrow{\Omega(\widetilde{i}^{s+1})} & \Omega(\widetilde{B}_s) & & & \\
 & {\scriptstyle \varphi^{s+1}}\uparrow & & \uparrow{\scriptstyle \varphi^s} & & & \\
 & B_{s+1} & \xrightarrow{i^{s+1}} & B_s & & &
\end{array}$$

with a fibering in the first row, where η, σ, φ^{s+1} and φ^s are homotopy equivalences. Since the diagram is homotopically commutative from propositions 2.5 and 3.3, we have a desired fibration.

COROLLARY 3.5. *Let (B_s, B_{s+1}) be a pair of Postnikov complexes for an $(n-1)$ connected CW-complex, where $s \leqq 2n-2$ and $n > 2$. Then we have*

$$H_\rho(B_s, B_{s+1}) \approx H_\rho(\pi_s, s+1) \qquad \textit{for } \rho \leqq s+n.$$

PROOF. Since we have a fibration of the second category for the pair $(B_s,$

B_{s+1}) and since the base space is s connected and the fibre is $(n-1)$ connected, we have the desired result.

4. An exact sequence. Let X be an $(n-1)$ connected CW-complex and let B_s be a Postnikov complex for X. Then we have a decreasing sequence of Postnikov complexes

$$B_{n+1} \supset B_{n+2} \supset \cdots \supset B_s \supset B_{s+1} \supset \cdots.$$

Defining the groups $D_{s,t} = H_{s+t}(B_s)$ and $E_{s,t} = H_{s+t}(B_s, B_{s+1})$, we have the usual exact sequence

$$\cdots \longrightarrow D_{s+1,\, t-1} \xrightarrow{i} D_{s,t} \xrightarrow{j} E_{s,t} \xrightarrow{k} D_{s+1,\, t-2} \longrightarrow \cdots$$

Thus the exact couple $C(X) = < D, E;\ i, j, k >$ is associated with X, where $D = \sum_{s,t} D_{s,t}$ and $E = \sum_{s,t} E_{s,t}$.

Let $l^{s+1}(X) \in H^{s+1}(B_s, \pi_s)$ be a Postnikov invariant represented by a self-obstruction cocycle $c^{s+1} = \partial i \rho^{-1}$, where $C_{s+1}(B_s) = H_{s+1}(B_s^{s+1}, X^s) \xrightarrow{\rho^{-1}} \pi_{s+1}(B_s^{s+1}, X^s) \xrightarrow{i} \pi_{s+1}(B_s, X) \xrightarrow{\partial} \pi_s(X)$. Since c^{s+1} maps the $(s+1)$ bounding cocycles into 0, $c^{s+1} | Z_{s+1}(B_s)$ induces a homomorphism $\tilde{c}^{s+1} : H_{s+1}(B_s) \to \pi_s$. In virtue of the universal coefficient theorem there exists an isomorphism

$$\lambda :\ H^{s+1}(B_s, \pi_s) \to \operatorname{Hom}(H_{s+1}(B_s); \pi_s) + \operatorname{Ext}(H_s(B_s); \pi_s),$$

such that $\lambda(l^{s+1}) = l_1^{s+1} + l_2^{s+1}$, where $l_1^{s+1} \in \operatorname{Hom}$ is the homomorphism $\tilde{c}^{s+1}$ and $l_2^{s+1} \in \operatorname{Ext}$. Then we have

PROPOSITION 4.1. *Given an integer $n > 1$. Then, for any $s > n$,*

1) $D_{s,t} = H_{s+t}(X)$, *if* $t < 0$,
2) $E_{s,t} = 0$, if $t \leqq 0$,
3) *In the diagram*

$$\begin{tikzcd}
E_{s,1} = H_{s+1}(B_s, B_{s+1}) \arrow[ddrr, "\sigma"'] & & \pi_{s+1}(B_s, B_{s+1}) \arrow[ll, "\tilde{\rho}"'] \arrow[d, "\tilde{\partial}"] \\
& & \pi_s(B_{s+1}) \arrow[d, "i_4"] \\
& & \pi_s(X),
\end{tikzcd}$$

all arrows are isomorphisms onto,

4) $\qquad l_1^{s+1} = \sigma \cdot j_{s,1}$, *where* $j_{s,1}\colon D_{s,1} \to E_{s,1}$.

PROOF. 1) is from $X^s = B_s^s$. Since $\pi_k(B_s, B_{s+1}) = 0$ if $k \leqq s$, we have Hurewicz isomorphism $\tilde{\rho}\colon \pi_{s+1}(B_s, B_{s+1}) \to H_{s+1}(B_s, B_{s+1})$ and $E_{s,t} = 0$ if $t \leqq 0$. It is obvious that $\tilde{\partial}$ and i_4 are isomorphisms. Hence 2) and 3) are proved. Consider the commutative diagram

$$\begin{tikzcd}
Z_{s+1}(B_s) \arrow[r, "j"] \arrow[ddd, "p"'] & H_{s+1}(B_s^{s+1}, B_{s+1}^s) \arrow[ddd, "i_1"'] & \pi_{s+1}(B_s^{s+1}, B_{s+1}^s) \arrow[l, "\rho"'] \arrow[r, "i"] \arrow[d, "i_2"] & \pi_{s+1}(B_s, X) \arrow[dl, "i_3"'] \arrow[ddd, "\partial"] \\
 & & \pi_{s+1}(B_s, B_{s+1}) \arrow[d, "\tilde{\partial}"] \arrow[ddl, "\tilde{\rho}"'] & \\
 & & \pi_s(B_{s+1}) \arrow[dr, "i_4"] & \\
H_{s+1}(B_s) \arrow[r, "j_{s,1}"] & H_{s+1}(B_s, B_{s+1}) \arrow[rr, "\sigma"] & & \pi_s(X)
\end{tikzcd}$$

Then, for any $x \in H_{s+1}(B_s)$ and for any representative x' of x, we have

$$\begin{aligned}
l_1^{s+1}(x) &= \tilde{c}^{s+1}(x) = c^{s+1}(x') = \partial\, i\, \rho^{-1}\, j\,(x') \\
&= i_4 \tilde{\partial}\, i_3\, i\, \rho^{-1}\, j\,(x') \\
&= i_4 \tilde{\partial}\, i_2\, \rho^{-1}\, j\,(x') \\
&= \sigma\, \tilde{\rho}\, i_2\, \rho^{-1} j\,(x') \\
&= \sigma\, i_1\, j\,(x') \\
&= \sigma\, j_{s,1}\, p(x') \\
&= \sigma\, j_{s,1}\,(x).
\end{aligned}$$

Hence the proof is completed.

COROLLARY 4. 2. *Given an integer* $n > 1$ *and an integer* $s > n$. *Let* $\sum_s(X)$ *be the s dimensional spherical homology group of an* $(n-1)$ *connected CW-complex X. Then we have*

$$\pi_s(X) \,/\, l_1^{s+1}(H_{s+1}(B_s)) \cong \textstyle\sum_s(X).$$

PROOF. Consider the commutative diagram

$$\begin{tikzcd}
H_{s+1}(B_s) = D_{s,1} \arrow[r, "j_{s,1}"] \arrow[dr, "l^{s+1}"'] & E_{s,1} \arrow[r, "k_{s,1}"] \arrow[d, "\sigma"] & D_{s+1,-1} = H_s(X) \\
 & \pi_s(X) \arrow[ur, "\tau"'] &
\end{tikzcd}$$

Let $\tau = k_{s,1}\,\sigma^{-1}$, then $\tau(\pi_s) = \sum_s(X)$. Hence the proof is obtained from the exact row.

PROPOSITION 4.3.[2] *If* $s \leqq 2n-2$ *and* $n > 2$, *we have*

$$E_{s,p-s} \cong H_p(\pi_s,\ s+1) \text{ for } 1 \leqq p \leqq s+n.$$

PROOF. It is obvious from the Corollary 3.5.

COROLLARY 4.4. *If* $n > 2$ *and* $s \leqq 2n-2$, *we have*

1) $$E_{s,2} = 0,$$

2) $$E_{s,3} \cong \pi_s/2\pi_s.$$

PROOF. It is obvious from the Prorosition 4. 3.

PROPOSITION 4.5. *If* $n > 2$ *and* $2n-2 \geqq s \geqq n+1$, *we have*

$$H_{s+1}/\textstyle\sum_{s+1} \cong \textit{kernel of } l_1^{s+1},$$

where $\sum_{s+1}$ *is the* $(s+1)$ *dimensional spherical homology group.*

PROOF. Consider the diagram

$$\begin{tikzcd}
\arrow[r] & E_{s+1,1} \arrow[r, "k_{s+1,1}"] & H_{s+1} \arrow[d, "i_{s+2,-1}"] & \\
\arrow[r] & E_{s,2} \arrow[r] \arrow[d, equal] & D_{s+1,0} \arrow[r] \arrow[d, "i_{s+1,0}"] & E_{s+1,0} \arrow[d, equal] \\
& 0 & D_{s,1} \arrow[r, "l_1^{s+1}"] & \pi_s \arrow[r] & {} \\
& & & 0 &
\end{tikzcd}$$

Since $E_{s,2}=0$ from the Corollary 4.4 and $E_{s+1,0} = 0$, $i_{s+2,-1}$ is epimorphic and $i_{s+1,0}$ is monomorphic. It follows from the exactness that $j_{s+1} = i_{s+1,0}\, i_{s+2,-1}$ induces the desired isomorphism.

PROPOSITION 4.6. *Given* $n > 2$. *Let* X *be an* $(n-1)$ *connected complex. Then we have an exact sequence*

$$H_{2n}(B_{2n-1}) \xrightarrow{l_1^{2n}} \pi_{2n-1} \longrightarrow H_{2n-1} \xrightarrow{j_{2n-1}} H_{2n-1}(B_{2n-2}) \longrightarrow \pi_{2n-2} \longrightarrow H_{2n-2} \longrightarrow \cdots \longrightarrow$$

$$H_{n+3}(B_{n+2}) \xrightarrow{l_1^{n+3}} \pi_{n+2} \longrightarrow H_{n+2} \xrightarrow{j_{n+2}} H_{n+2}(\pi_n, n) \xrightarrow{l_1^{n+2}} \pi_{n+1} \longrightarrow H_{n+1} \longrightarrow 0,$$

where $j_{s+1} = i_{s+1,0}\, i_{s+2,-1}$ *and* l_1^{s+1} *is a projection of the Postnikov invariant* l^{s+1} *as in the Proposition* 4.1.

2) K. Shiraiwa informs us of a proof of the Proposition 4. 3 without the restriction $s \leqq 2n-2$.

PROOF. This is the immediate consequence of the Corollary 4. 2, Proposition 4. 5, and the exact couple which we considered.

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