ON NORMAL ALMOST CONTACT STRUCTURES WITH A REGULARITY

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Introduction. This paper is a continuation of the previous paper [4] in which we have proved, among others, that the bundle space of a principal circle bundle over a complex manifold, which has a connection satisfying certain conditions, admits a normal almost contact structure (cf. Theorem 6 [4]). In this paper we first consider the converse of the above theorem, and we shall call such a bundle, for the sake of simplicity, a contact bundle over a complex manifold (§1. Theorem 1).

In §2 we consider the period function of a regular closed vector field (Def. 3) and we prove Theorem 4 which says that the period function of a regular closed analytic vector field X on a complex manifold M is the real part of a holomorphic function on M if JX is also a closed vector field on M, J being the complex structure tensor of M. Using this theorem we shall prove that if the vector field ξ of a normal almost contact structure (ϕ, ξ, η) is a regular closed vector field, the period function of ξ is necessarily constant. From this we shall see that there is no other example of normal almost contact structures than the examples constructed in Theorem 6 [4], at least, when the vector field ξ is a closed vector field.

In §3 we consider the family of contact bundles over a complex manifold M_0 and we shall finally show that two contact bundles are isomorphic if and only if there exists a diffeomorphism f_0 of M_0 onto itself such that $f_0^*\overline{\Omega} = \Omega$, where Ω and $\overline{\Omega}$ are associated 2-forms on M_0 to each contact bundle, when M_0 is simply connected (cf. Def. 1).

1. Contact bundles over complex manifolds. Let $M(M_0, S^1, \pi)$ be a principal circle bundle over a (always C^∞-) differentiable manifold M_0, S^1 being the 1-dimensional torus and π being the projection of M onto M_0. Let $\Sigma = (\phi, \xi, \eta)$ be a normal almost contact structure (cf. Def. 2 [4]) on M. The Lie algebra $\mathfrak{r}$ of S^1 being identified with the real number field R, we shall now suppose that η is a connection form on M and that ξ is a vertical fundamental vector field A^* corresponding to the unit vector A of $\mathfrak{r}$. As in [4] we shall denote by $\mathfrak{V}(M)$ the Lie algebra of vector fields on M.

In the sequel we shall often denote the differential of a differentiable map f by the same letter f. We shall now prove the following theorem[1] which

1) Y. Hatakeyama obtained similar results in Tôhoku Math. Journ., 15(1963), pp. 176–181.

may be considered as a converse to Theorem 6 [4].

THEOREM 1. *Notations and assumptions being as above, we can find an unique complex structure J on M_0 such that $\phi(X^*) = (JX)^*$ for $X \in \mathfrak{V}(M_0)$, where X^* denotes the lift of X with respect to the connection η. Moreover, the 2-form Ω on M_0 such that $d\eta = \pi^*\Omega$ satisfies the following condition*

$$\Omega(JX, JY) = \Omega(X, Y)$$

for $X, Y \in \mathfrak{V}(M_0)$, i.e. Ω is a 2-form of type (1. 1) *with respect to J.*

PROOF. Take a tangent vector X of M_0 at $p_0 \in M_0$. Define

$$JX = \pi\phi X_p^*, \tag{1. 1}$$

where $p \in M$, $\pi(p) = p_0$, and X_p^* is the lift of X at p with respect to the connection η. By (1. 1), J is well defined. In fact, take $p' \in M$ such that $\pi(p') = p_0$. Then $p' = R_a \cdot p$ for some element $a \in S^1$, R_a being the right translation corresponding to a. Then $X_{p'}^* = R_a X_p^*$. It must be shown that $\pi\phi X_{p'}^* = \pi\phi X_p^*$. For this, it is sufficient to prove that $\phi \circ R_a = R_a \circ \phi$. Since ξ generates a one parameter group of right translations of M, it is now sufficient to prove that the Lie derivative of ϕ with respect to ξ vanishes identically, i.e.

$$[\xi, \phi Y] = \phi[\xi, Y] \tag{1. 2}$$

for all $Y \in \mathfrak{V}(M)$. However, (1. 2) was proved as (2. 13) in [4]. Hence by (1. 1), J is well defined. First, we prove that $J^2 = -1, 1$ being the identity map. In fact, for $X \in T_{p_0}(M_0)$, $J^2(X) = \pi\phi(JX)_p^* = \pi\phi((\pi\phi X_p^*)_p^*) = \pi\phi(\phi X_p^*) = -\pi X_p^* = -X$, where we have used the fact that ϕX_p^* is horizontal in the third equality. Hence J is an almost complex structure on M_0. To prove that J is integrable, we first remark that

$$(JX)^* = \phi(X^*)$$

for $X \in \mathfrak{V}(M_0)$. Next we shall prove that

$$J[X, Y] = [JX, Y] + [X, JY] + J[JX, JY]$$

for $X, Y \in \mathfrak{V}(M_0)$. For this purpose, X^* being the lift of X with respect to η, we calculate the lift $(J[X, Y])^*$ of $J[X, Y]$, using (2. 3) [4], as follows:

$$\begin{aligned}(J[X, Y])^* &= \phi[X, Y]^* = \phi(h[X^*, Y^*]) = \phi[X^*, Y^*] \\ &= [\phi X^*, Y^*] + [X^*, \phi Y^*] + \phi[\phi X^*, \phi Y^*] - \{\phi X^* \cdot \eta(Y^*) - \phi Y^* \cdot \eta(X^*)\}\xi \\ &= [(JX)^*, Y^*] + [X^*, (JY)^*] + \phi[(JX)^*, (JY)^*] \\ &= \eta([(JX)^*, Y^*])\xi + [JX, Y]^* + \eta([X^*, (JY)^*])\xi + [X, JY]^* \\ &\qquad + \phi([JX, JY]^*) \\ &= [JX, Y]^* + [X, JY]^* + (J[JX, JY])^* + \{\eta([(JX)^*, Y^*]) \\ &\qquad + \eta([X^*, (JY)^*])\}\xi.\end{aligned}$$

It is now sufficient to prove that

$$\eta([\phi X^*, Y^*]) + \eta([X^*, \phi Y^*]) = 0$$

for $X, Y \in \mathfrak{B}(M_0)$. By (2.7) [4] we see

$$\begin{aligned}\eta([\phi X^*, Y^*]) &= \phi X^* \cdot \eta(Y^*) - Y^* \cdot \eta(\phi X^*) + \eta([\phi^2 X^*, \phi Y^*]) \\ &= \eta([-X^* + \eta(X^*)\xi, \phi Y^*]) = -\eta([X^*, \phi Y^*]).\end{aligned}$$

Hence we have proved that J is integrable.

Now it is well known that there exists (uniquely) a 2-form Ω on M_0 such that $d\eta = \pi^*\Omega$. For this Ω we calculate as follows:

$$\begin{aligned}\Omega(JX, JY) &= \Omega(\pi\phi X^*, \pi\phi Y^*) = (\pi^*\Omega)(\phi X^*, \phi Y^*) \\ &= d\eta(\phi X^*, \phi Y^*) = -\frac{1}{2}\eta([\phi X^*, \phi Y^*]) = -\frac{1}{2}\eta([X^*, Y^*]) \\ &= d\eta(X^*, Y^*) = \pi^*\Omega(X^*, Y^*) = \Omega(X, Y),\end{aligned}$$

where we have used (2.7) [4] in the fifth equality. The uniqueness of J is clear from $\phi(X^*) = (JX)^*$. Thus Theorem 1 is proved.

DEFINITION 1. Let $M(M_0, S^1, \pi)$ be a principal circle bundle, and $\Sigma = (\phi, \xi, \eta)$ be a normal almost contact structure on M satisfying the conditions of Theorem 1. For the sake of simplicity we shall call such a bundle $M(M_0, S^1, \pi)$ with Σ *a contact bundle over a complex manifold* M_0. The 2-form Ω on M_0 in Theorem 1 will be called the *associated 2-form to the contact bundle* (or associtead 2-form to η).

THEOREM 2. *Let $M(M_0, S^1, \pi)$ and $\bar{M}(\bar{M}_0, S^1, \bar{\pi})$ be contact bundles with Σ and $\bar{\Sigma}$. Let f be an isomorphism of Σ to $\bar{\Sigma}$* (cf. Def. 4 [4]). *Then there exists a holomorphic homeomorphism f_0 of M_0 onto $\bar{M}_0$ such that $\bar{\pi}\circ f = f_0\circ\pi$ and $f_0^*\bar{\Omega} = \Omega$, where Ω and $\bar{\Omega}$ denote the associated 2-form to the contact bundles M and $\bar{M}$ respectively.*

PROOF. Since f is an isomorphism of Σ to $\bar{\Sigma}$, $f(\xi) = \bar{\xi}$. Hence f is a fibre preserving map of M onto $\bar{M}$. Therefore f induces a diffeomorphism f_0 of M_0 onto $\bar{M}_0$ such that $\bar{\pi}\circ f = f_0\circ\pi$. To prove that f is holomorphic, it is sufficient to prove that

$$\bar{J} f_0 X = f_0 J X$$

for all $X \in \mathfrak{B}(M_0)$, J and $\bar{J}$ being the complex structures of M_0 and $\bar{M}_0$ respectively. Now

$$f_0 JX = f_0 \pi\phi X^* = \bar{\pi} f\phi X^* = \bar{\pi}\bar{\phi} f X^*. \tag{1.3}$$

We shall next prove that

$$f X^* = (f_0 X)^*. \tag{1.4}$$

In fact, $\bar{\eta}(fX^*) = (f^*\bar{\eta})X^* = \eta(X^*) = 0$, whence fX^* is horizontal. On the other hand $\bar{\pi}(fX^*) = f_0\pi X^* = f_0X$, which proves (1.4). Inserting (1.4) into (1.3), we have

$$f_0JX = \bar{\pi}\bar{\phi}(f_0X)^* = \bar{J}f_0X.$$

Thus Theorem 2 is proved.

2. Period functions of regular closed vector fields.

DEFINITION 2. Let M be a differentiable manifold and let X be a vector field on M such that $X_p \neq 0$ for any $p \in M$. Then clearly X defines a 1-dimensional (involutive) distribution on M i.e. X defines a 1-dimensional vector subspace of the tangent space of M at each point of M. Let C_p be the maximal integral curve of this distribution through the point p. X is called *regular* if for each point $p_0 \in M$ there exists a coordinates system $\{x_1, x_2, \cdots, x_n\}$ on a neighborhood $U(p_0)$ of p_0 such that

(i) $\quad x_i(p_0) = 0 \qquad i = 1, 2, \cdots, n,$

(ii) $\quad C_p \cap U(p_0) = \{q \in U(p_0) \mid x_i(q) = x_i(p)\ i = 2, 3, \cdots, n\}$

for all point $p \in U(p_0)$.

DEFINITION 3. Let X be a regular vector field on M. We shall call X a (regular) *closed vector field* on M, if for each $p \in M$, C_p is a closed curve. When X is a closed vector field, $\varphi_t = \exp tX$ denoting the 1-parameter group of transformations on M generated by X, we define a function $\lambda_X(p)$ on M as follows:

$$\lambda_X(p) = \inf\{t \mid t > 0,\ \varphi_t(p) = p\}.$$

We shall call $\lambda_X(p)$ the *period function* of X. We shall denote frequently $t \cdot p = (\exp tX) \cdot p$ for $-\infty < t < \infty$, $p \in M$, whenever there is no confusion.

It is to be noted that $\lambda_X(p) > 0$ for each $p \in M$ by the regularity of X.

For the period function $\lambda(p) = \lambda_X(p)$ of a closed vector field X we have the following lemma (cf. [1] p. 722).

LEMMA 1. *The period function $\lambda(p)$ is a differentiable function on M, especially it is continuous.*

THEOREM 3. *Let (ϕ, ξ, η) be a normal almost contact structure on M. Suppose that ξ is a closed vector field such that its period function $\lambda_\xi(p)$ is a constant. Then there exist a complex manifold M_0 and a C^∞-map π of M onto M_0 such that $M(M_0, S^1, \pi)$ is a principal circle bundle over M_0, η is a connection form on M and ξ is a vertical vector field on M.*

PROOF. Let $\lambda_\xi(p) = c_0 =$ const. Then the torus group $S^1 = R/Z \cdot c_0$ of real numbers modulo c_0 operates on M by

$$(t, p) \to \varphi_t(p) \qquad \text{for } t \in R,\ p \in M,$$

where $\varphi_t = \exp t\xi$ is the 1-parameter group of transformations of M generated by ξ. Clearly the only element of S^1 having a fixed point in M is the identity. Hence by a well known theorem [2] and the same argument in [1] p.725, M has a S^1-bundle structure. Let M_0 be the base space of this bundle. To prove that η defines a connection form on M it is sufficient to prove that η is right invariant. For this it suffices to see that the Lie derivative of η with respect to ξ vanishes identically i. e. $\xi \cdot \eta(X) - \eta([\xi, X]) = 0$ for $X \in \mathfrak{V}(M)$. However, this is an immediate consequence of (2. 7) [4] by putting $Y = \xi$. Hence the bundle $M(M_0, S^1, \pi)$ satisfies the conditions of Theorem 1 and thus M_0 has a complex structure, which proves Theorem 3.

Now we want to prove that if the vector field ξ of a normal almost contact structure (ϕ, ξ, η) on M is closed, the period function of ξ is necessarily constant on M. For this purpose we shall prove the following theorem which may be considered as an analogue of Lemma 1 for the complex case.

THEOREM 4. *Let M be a complex manifold and X be an analytic vector field on M,* i. e. *X generates a local 1-parameter group of holomorphic transformations of M. Suppose that X and JX are both closed vector fields, J denoting the complex structure of M. Put $f(p) = \lambda_X(p) + \sqrt{-1}\,\lambda_{JX}(p)$.*
Then f is a holomorphic function on M.

PROOF. Put $\tilde{X} = X - \sqrt{-1}\,JX$. Then $\tilde{X}$ is a holomorphic vector field on M, i.e. $\tilde{X}$ can be expressed locally as follows:

$$\tilde{X} = \sum_{i=1}^{n} h_i(w) \frac{\partial}{\partial w_i}$$

for complex coordinates system $\{w_1, \cdots, w_n\}$, where $h_i = h_i(w_1, \cdots, w_n)$ is a holomorphic function for $i = 1, 2, \cdots, n$. Fix a point $p_0 \in M$. Since $X \neq 0$ on M, by a well known theorem, we can find a complex coordinates system $\{z_1, \cdots, z_n\}$ on a neighborhood $U = U(p_0)$ of p_0 such that

(i) $\quad z_i(p_0) = 0 \qquad i = 1, 2, \cdots, n,$

(ii) $\quad \tilde{X} = \dfrac{\partial}{\partial z_1} \qquad$ on U.

Let C denote the additive group of complex numbers. Then C operates holomorphically on M by $z \cdot p = \exp tX \cdot \exp sJX \cdot p$, for $z = t + \sqrt{-1}\,s$ and $p \in M$. We note here that $\exp tX$ and $\exp sJX$ commutes since $[X, JX] = 0$. Put $f(p_0) = z^0 = x^0 + \sqrt{-1}\,y^0$, x^0 and y^0 being real and put

$$z_1(z \cdot p) = g(z, z_1(p), \cdots, z_n(p))$$

for $|z - z^0| < \varepsilon$ and $p \in U$, ε being sufficiently small. Since $\tilde{X}$ is a holomorphic vector field, the function $g(z, z_1, \cdots, z_n)$ is holomorphic for $|z - z^0| < \varepsilon$, $|z_i| < a$ for some $a > 0$. We want to solve the equation

$$g(z, 0, z_2, \cdots, z_n) = 0$$

for z. For this, we show first that

$$\frac{\partial g}{\partial z}(z^0, 0, \cdots, 0) \neq 0.$$

In fact,

$$\begin{aligned}
\frac{\partial g}{\partial z}(z^0, 0, \cdots, 0) &= \lim_{z\to 0} \frac{g(z^0 + z, 0 \cdots 0) - g(z^0, 0 \cdots 0)}{z} \\
&= \lim_{z\to 0} \frac{z_1((z^0 + z) \cdot p_0) - z_1(z^0 \cdot p_0)}{z} \\
&= \lim_{z\to 0} \frac{z_1(z \cdot p_0) - z_1(p_0)}{z} \\
&= \lim_{t\to 0} \frac{z_1(t \cdot p_0) - z_1(p_0)}{t} \\
&= X_{p_0}(z_1) = \left(\frac{\partial}{\partial x_1}\right)_{p_0}(x_1 + \sqrt{-1}\, y_1) = \left(\frac{\partial}{\partial x_1}\right)_{p_0}(x_1) = 1,
\end{aligned}$$

where we have put $z_1 = x_1 + \sqrt{-1}\, y_1$, x_1 and y_1 being real. Using the existence theorem of implicit functions, we can find a holomorphic function $h(z_2, \cdots, z_n)$ in $|z_i| < a$ $(i = 2, \cdots, n)$ such that

$$\begin{cases} h(0, \cdots, 0) = z^0 \\ g(h(z_2, \cdots, z_n), 0, z_2, \cdots, z_n) = 0. \end{cases}$$

Next we shall prove that there exists a neighborhood $U_1 \subset U$ of p_0 such that

$$f(p) = h(z_2(p), \cdots, z_n(p)) \tag{2. 1}$$

for $p \in U_1$. For this purpose we shall prove the following lemma essentially proved in [1].

LEMMA 2. *Let X be a closed vector field on a differentiable manifold M and $\lambda(p) = \lambda_X(p)$ its period function. Let $\{x_1, \cdots, x_m\}$ be a system of coordinates on a neighborhood U of p_0 such that*

(i) $\qquad X_p = \left(\frac{\partial}{\partial x_\alpha}\right)_p \qquad$ *for* $p \in U$

(ii) $\qquad x_i(p_0) = 0 \qquad i = 1, 2, \cdots, m.$

Put $x_\alpha(t \cdot p) = g(t, x_1(p), \cdots, x_m(p))$. Suppose that $m \geqq 3$ and that there exists *a continuous function* $h(x_1, \cdots, \hat{x}_\alpha, \cdots, \hat{x}_\beta, \cdots, x_m)$ *for some* β *such that*

(iii) $$h(0, \cdots, 0) = \lambda(p_0)$$

(iv) $$g(h(x_1, \cdots, \hat{x}_\alpha, \cdots, \hat{x}_\beta, \cdots, x_m), x_1, \cdots, x_{\alpha-1}, 0, x_{\alpha+1}, \cdots, x_{\beta-1}, 0, x_{\beta+1}, \cdots, x_m) = 0$$

for $|x_i| < a$, *a being sufficiently small positive number, where* $\wedge$ *denotes the omission of the letter under the* $\wedge$.

Then there exists a neighborhood U_1 *of* p_0 *contained in* U *such that*

$$\lambda(p) = h(x_1(p), \cdots, \hat{x}_\alpha(p), \cdots, \hat{x}_\beta(p), \cdots, x_m(p))$$

for $p \in U_1$ *satisfying* $x_\beta(p) = 0$.

PROOF. If this lemma were false, there would exist a sequence of points $\{p_\nu\}_{\nu=1}^{\infty}$ such that $p_\nu \to p_0 (\nu \to \infty)$, $x_\beta(p_\nu) = 0$ and that

$$\lambda(p_\nu) \neq h(x_1(p_\nu), \cdots, \hat{x}_\alpha(p_\nu), \cdots, \hat{x}_\beta(p_\nu), \cdots, x_m(p_\nu)) > 0.$$

We can suppose that $x_\alpha(p_\nu) = 0$ by virtue of (i). Now

$$\begin{aligned} &x_\alpha(h(x_1(p_\nu), \cdots, \hat{x}_\alpha(p_\nu), \cdots, \hat{x}_\beta(p_\nu), \cdots, x_m(p_\nu)) \cdot p_\nu) \\ &= g(h(x_1(p_\nu), \cdots, \hat{x}_\alpha(p_\nu), \cdots, \hat{x}_\beta(p_\nu), \cdots, x_m(p_\nu)), \cdots, x_1(p_\nu), \\ &\qquad \cdots, x_{\alpha-1}(p_\nu), 0, x_{\alpha+1}(p_\nu), \cdots, x_{\beta-1}(p_\nu), 0, x_{\beta+1}(p_\nu), \\ &\qquad \cdots, x_m(p_\nu)) = 0. \end{aligned}$$

Hence we can find an integer $k_\nu > 1$ such that

$$h(x_1(p_\nu), \cdots, \hat{x}_\alpha(p_\nu), \cdots, \hat{x}_\beta(p_\nu), \cdots, x_m(p_\nu)) = k_\nu \cdot \lambda(p_\nu).$$

By the continuity of h

$$h(x_1(p_\nu), \cdots, \hat{x}_\alpha(p_\nu), \cdots, \hat{x}_\beta(p_\nu), \cdots, x_m(p_\nu)) \to h(0, \cdots, 0) = \lambda(p_0) \quad (\nu \to \infty).$$

Hence $k_\nu \cdot \lambda(p_\nu) \to \lambda(p_0)$ $(\nu \to \infty)$. On the other hand, by virtue of Lemma 1, $\lambda(p)$ is continuous, so $\lambda(p_\nu) \to \lambda(p_0)$ $(\nu \to \infty)$. Therefore we have

$$\lambda(p_0) = \lim k_\nu \cdot \lambda(p_\nu) \geqq \lim 2\lambda(p_\nu) = 2\lambda(p_0),$$

which implies $\lambda(p_0) = 0$. This contradiction completes the proof of Lemma 2.

Return now to the *proof of Theorem* 4. Let $\pi : U \to U$ be the map such that

$$\begin{cases} z_1(\pi p) = 0 \\ z_i(\pi p) = z_i(p) \qquad (i \geqq 2) \end{cases}$$

for $p \in U$. Put

$$g_1(t, x_1(p), y_1(p), z_2(p), \cdots, z_n(p)) = x_1(t \cdot p)$$

and put

$$h(z_2,\cdots,z_n)=h_1(z_2,\cdots,z_n)+\sqrt{-1}\,h_2(z_2,\cdots,z_n),$$

h_1 and h_2 being real. We shall prove that

$$g_1(h_1(z_2(p),\cdots, z_n(p)),\ 0,\ 0,\ z_2(p),\cdots, z_n(p))=0$$

for $p\in U'$, where U' is a neighborhood of p_0 contained in U. In fact,

$$\begin{aligned}
&g_1(h_1(z_2(p),\cdots, z_n(p)),\ 0, 0,\ z_2(p),\cdots, z_n(p))\\
&=\mathrm{Re}\ g(h_1(z_2(p),\cdots, z_n(p)), 0, 0, z_2(p),\cdots, z_n(p))\\
&=\mathrm{Re}\ z_1(h_1(z_2(p),\cdots, z_n(p))\cdot\pi p)\\
&=x_1(h_1(z_2(p),\cdots,z_n(p))\cdot\pi p)\\
&=x_1(\sqrt{-1}\,h_2(z_2(p),\cdots, z_n(p))\cdot h_1(z_2(p),\cdots, z_n(p))\cdot\pi p)\\
&=x_1(h(z_2(p),\cdots, z_n(p))\cdot\pi p)\\
&=\mathrm{Re}\ z_1(h(z_2(p),\cdots, z_n(p))\cdot\pi p)\\
&=\mathrm{Re}\ g(h(z_2(p),\cdots, z_n(p)), 0, z_2(p),\cdots, z_n(p))=0,
\end{aligned}$$

where we have used the commutativity $[X, JX]=0$ in the fifth equality. By virtue of Lemma 2 there exists a neighborhood U_1' of p_0 contained in U' such that $\lambda_X(p)=h_1(z_2(p),\cdots, z_n(p))$ for $p\in U_1'$ satisfying $y_1(p)=0$. In the same way, using Lemma 2 again, we can find a neighborhood U_1'' of p_0 contained in U_1' such that

$$\lambda_{JX}(p)=h_2(z_2(p),\cdots, z_n(p))$$

for $p\in U_1''$ satisfying $x_1(p)=0$. From this it follows that

$$f(p)=h(z_2(p),\cdots,z_n(p)) \tag{2.1}$$

for $p\in U_1''$ satisfying $z_1(p)=0$. On the other hand it is easy to see that $f(p)=f(z\cdot p)$ for z with sufficiently small $|z|$. Hence we conclude that there exists a neighborhood U_1 of p_0 contained in U_1'' such that (2.1) holds for $p\in U_1$. Hence f is holomorphic in U_1, which completes the proof of Theorem 4.

In the rest of §2, M is assumed to be connected.

THEOREM 5. *Let (ϕ,ξ,η) be a normal almost contact structure on M such that ξ is a closed vector field on M. Then the period function λ_ξ of ξ is a constant on M.*

PROOF. Put $\widetilde{M}=M\times S^1$, S^1 being the 1-dimensional torus with the natural (normal) almost contact structure. Then $\widetilde{M}$ has a complex structure J induced by (ϕ,ξ,η). Consider ξ as a vector field on $\widetilde{M}$. Then ξ is an analytic vector field on $\widetilde{M}$ with respect to J, since (ϕ,ξ,η) is normal. Let $f(\widetilde{p})=\lambda_\xi(\widetilde{p})+\sqrt{-1}\,\lambda_{J\xi}(\widetilde{p})$ be the function associated to ξ as in Theorem 4. It is clear that $\lambda_{J\xi}(\widetilde{p})$ is constant, so $\lambda_\xi(\tilde{p})$ is also constant since $f(\tilde{p})$ is holomorphic on $\tilde{M}$. On the other hand, since $\lambda_\xi(\tilde{p}) = \lambda_\xi(p)$ for $\tilde{p} = (p, t)$, $t \in S^1$, $p \in M$, $\lambda_\xi(p)$ is also constant on M, which completes the proof.

COROLLARY 1. *Let (ϕ, ξ, η) be a normal almost contact structure on M such that ξ is a closed vector field on M. Then M has a circle bundle structure over a complex manifold M_0.*

In fact, since the period function of ξ is constant we can apply Theorem 3.

COROLLARY 2. *Let (ϕ, ξ, η) be a normal almost contact structure on a compact manifold M such that ξ is a regular vector field on M. Then M has a circle bundle structure as in Corollary* 1.

In fact, every maximal integral curve of a regular vector field on M is a closed set in M, so compact in M, which says that ξ is a closed vector field on M. Hence we can apply Corollary 1.

In the case when ξ is a *proper* vector field, i.e. ξ generates a global 1-parameter group $\exp t\xi(-\infty < t < \infty)$ of transformations on M, we want to show that ξ is a closed vector field if there exists one point $p_0 \in M$ such that the maximal integral curve C_{p_0} through p_0 is a closed curve. For this purpose we prepare the following two lemmas.

For a proper vector field X we define $\lambda_X(p)$ as in Def. 3, while $\lambda_X(p) = \infty$ if $t \cdot p \neq p$ for any $t > 0$.

LEMMA 3. *Let X be a regular, proper vector field on a differentiable manifold M. Let M^0 be the set of all points $p \in M$ such that $\lambda_X(p) < \infty$. Then M^0 is open in M.*

PROOF. Let $p_0 \in M^0$. Then there exists a system of coordinates $\{x_1, x_2, \cdots, x_n\}$ on an open neighbourhood U of p_0 such that (i) (ii) of Def. 2 are satisfied. Put $\lambda_X(p_0) = \lambda_0$. We can assume that $|x_1| < \lambda_0$ on U and that $X = \dfrac{\partial}{\partial x_1}$ on U. By the continuity of $\exp \lambda_0 X$ we can find an open neighborhood V of p_0 contained in U such that $\lambda_0 \cdot V \subset U$. We shall show that $V \subset M^0$. Take a point $p \in V$. Then $\lambda_0 \cdot p \in U$. Now by the property (ii) there exists a real number t such that $\lambda_0 \cdot p = t \cdot p$, $|t| < \lambda_0$. Hence $(\lambda_0 - t) \cdot p = p$ holds. Since $\lambda_0 - t > 0$, it follows $p \in M^0$, which proves $V \subset M^0$, q.e.d.

LEMMA 4. *Let X be a regular, proper vector field on M. Let $p_0 \in M$ such that $\lambda_X(p_0) = \infty$. Then for each positive number K there exists an open neighborhood U of p_0 such that $\lambda(p) \geqq K$ for any $p \in U$.*

PROOF. Let $\{x_1, x_2, \cdots, x_n\}$ be a system of coordinates on an open neighbourhood U_0 of p_0 satisfying (i) (ii) of Def.2. We can assume that there exists a positive number $K_1 < K$ such that $|x_1| < K_1$ on U_0 and that $X = \dfrac{\partial}{\partial x_1}$ on U. Put $A = \{t \cdot p_0 | \dfrac{K_1}{2} \leqq t \leqq K\}$. Since $p_0 \notin A$, there exist open sets W and U_1 such that $A \subset W$, $p_0 \in U_1 \subset U_0$ and $W \cap U_1 = \emptyset$. Now it is easy to find an open neighbourhood U of p_0 contained in U_1 such that $t \cdot p \in W$ for $\dfrac{K_1}{2} \leqq t \leqq K$ and $p \in U$. We shall show that $\lambda(p) > K$ for $p \in U$. Take a point $p \in U$ and t such that $0 < t \leqq K$. If $t \leqq \dfrac{K_1}{2}$, then $t \cdot p \neq p$ holds by (ii). If $t \geqq \dfrac{K_1}{2}$, then $t \cdot p \in W$, so $t \cdot p \notin U_1$. Hence $t \cdot p \neq p$ for $0 < t \leqq K$, which shows $\lambda(p) \geqq K$, q. e. d.

THEOREM 6. *Let (ϕ, ξ, η) be a normal almost contact structure on a connected manifold M, such that ξ is a regular, proper vector field on M. Suppose that $\lambda_\xi(p_0) < \infty$ for some $p_0 \in M$, then $\lambda_\xi(p) < \infty$ for any $p \in M$,* i. e. *ξ is a closed vector field on M.*

PROOF. Let M^0 be the set of points p for which $\lambda_\xi(p) < \infty$, then M^0 is open in M by Lemma 3. Put $\lambda_0 = \lambda_\xi(p_0)$, and $M_1 = \{p \in M | \lambda_\xi(p) = \lambda_0\}$. Let (ϕ^0, ξ^0, η^0) be the restriction of (ϕ, ξ, η) to the open submanifold M^0. Since (ϕ^0, ξ^0, η^0) is normal, we can apply Theorem 5. Hence $\lambda_{\xi^0} = \lambda_\xi | M^0$ is constant on each connected component of M^0. From this it follows that M_1 is open in M. Next we shall prove that M_1 is closed in $.M$. Take a sequence of points $p_\nu \in M_1$ $(\nu = 1, 2, \cdots)$ such that $p_\nu \to q \in M (\nu \to \infty)$. If $\lambda_\xi(q) = \infty$, then by Lemma 4 for $K = 2\lambda_0$, we can find an open neighbourhood U of q such that $\lambda_\xi(p) \geqq 2\lambda_0$ for $p \in U$. Since $p_{\nu_0} \in U$ for a sufficiently large ν_0, $\lambda_0 = \lambda_\xi(p_{\nu_0}) \geqq 2\lambda_0$ implies a contradiction. Hence $\lambda_\xi(q) < \infty$, so $q \in M^0$. Now, since $\lambda_{\xi^0}(p)$ is a continuous function on M^0 by Lemma 1, $\lambda_\xi(q) = \lim \lambda_\xi(p_\nu) = \lambda_0$. Hence $q \in M_1$, which proves that M_1 is closed in M. Since M is connected, the non empty open, closed set M_1 coincides with M, which proves Theorem 6.

3. Family of contact bundles over a complex manifold. We shall define the product of two contact bundles (M, Σ) and $(\bar{M}, \bar{\Sigma})$ over a complex manifold M_0, (cf. Def. 1). First we recall the definition of the product (or called sometimes sum) of the principal circle bundles M and $\bar{M}$ (cf. e.g. [3]). Let $M = M(M_0, S^1, \pi)$ and $\bar{M} = \bar{M}(M_0, S^1, \bar{\pi})$. $\Delta(M \times \bar{M})$ denotes the set of all elements $(p, \bar{p}) \in M \times \bar{M}$ such that $\pi(p) = \bar{\pi}(\bar{p})$. We say that two elements $(p, \bar{p})$ and $(q, \bar{q})$ of $\Delta(M \times \bar{M})$ are equivalent if there exists an element $a \in S^1$ such that

$$p \cdot a = q, \ \bar{p} \cdot a^{-1} = \bar{q}.$$

We denote by $M \cdot \bar{M}$ the quotient space of $\Delta(M \times \bar{M})$ by this equivalence relation. The projection from $\Delta(M \times \bar{M})$ onto M_0 induces a projection from

$M\cdot\bar{M}$ onto M_0, which we shall denote by $\tilde{\pi}=\pi\cdot\bar{\pi}$. The action of S^1 on $\Delta(M\times\bar{M})$ defined by $(p,\bar{p})\cdot a=(p\cdot a,\bar{p})$, $(p,\bar{p})\in\Delta(M\times\bar{M})$, $a\in S^1$ preserves the equivalence relation, hence it defines the action of S^1 on $M\cdot\bar{M}$. The bundle $M\cdot\bar{M}(M_0,S^1,\tilde{\pi})$ is, by definition, the product of M and $\bar{M}$. It is known that the family of circle bundles over M_0 form a multiplicative abelian group by this multiplication, the unit element being the trivial circle bundle over M_0.

Let now $\Sigma=(\phi,\xi,\eta)$ and $\bar{\Sigma}=(\bar{\phi},\bar{\xi},\bar{\eta})$. We define a linear differential form $\eta\times\bar{\eta}$ on $M\times\bar{M}$ as follows:

$$\eta\times\bar{\eta}=\rho^*(\eta)+\bar{\rho}^*(\bar{\eta}),$$

where ρ and $\bar{\rho}$ are the natural projection from $M\times\bar{M}$ onto M and $\bar{M}$ respectively. We denote also by $\eta\times\bar{\eta}$ the restriction of $\eta\times\bar{\eta}$ to $\Delta(M\times\bar{M})$. Then there exists an unique differential form $\tilde{\eta}$ on $M\cdot\bar{M}$ such that

$$\mu^*(\tilde{\eta})=\eta\times\bar{\eta}$$

where μ is the natural projection of $\Delta(M\times\bar{M})$ onto $M\cdot\bar{M}$. We can see that $\tilde{\eta}$ defines a connection on $M\cdot\bar{M}$ and the 2-forms Ω, $\bar{\Omega}$ and $\tilde{\Omega}$ associated to the connections η, $\bar{\eta}$ and $\tilde{\eta}$ respectively satisfy

$$\tilde{\Omega}=\Omega+\bar{\Omega}. \tag{3. 1}$$

(For the proof, cf. [3] p. 32). We denote $\tilde{\eta}=\eta\cdot\bar{\eta}$.

We want now to define the product $\tilde{\phi}=\phi\cdot\bar{\phi}$ of ϕ and $\bar{\phi}$ as follows. As usual we denote by $T_p(M)$ the tangent space of M at p. Then we see that the tangent space $T_{(p,\bar{p})}(\Delta(M\times\bar{M}))$ can be identified with the subspace $T^0_{(p,\bar{p})}(M\times\bar{M})$ of $T_{(p,\bar{p})}(M\times\bar{M})$ defined by

$$T^0_{(p,\bar{p})}(M\times\bar{M})=\{(X,\bar{X})\in T_{(p,\bar{p})}(M\times\bar{M})\,|\,\pi X=\bar{\pi}\bar{X}\}.$$

LEMMA 5. *Let* $(X_p,\bar{X}_{\bar{p}})\in T_{(p,\bar{p})}(\Delta(M\times\bar{M}))$ *and* $(X'_q,\bar{X}'_{\bar{q}})\in T_{(q,\bar{q})}(\Delta(M\times\bar{M}))$. *Then,* $\mu(X_p,\bar{X}_{\bar{p}})=\mu(X'_q,\bar{X}'_{\bar{q}})$ *implies* $\mu((J\pi X)^*_p,(J\pi X)^*_{\bar{p}})=\mu((J\pi X'_q)^*_q,(J\pi X'_q)^*_{\bar{q}})$, *where* Y^*_p $(Y^*_{\bar{p}}$ *resp.*) *denotes the lift of* Y $(\bar{Y}$ *resp.*) *at* p (*at* $\bar{p}$ *resp.*) *with respect to the connection* η ($\bar{\eta}$ *resp.*).

PROOF. Let θ be the projection of $\Delta(M\times\bar{M})$ onto M_0. For an element $a\in S^1$ we define the map Q_a of $\Delta(M\times\bar{M})$ onto itself by

$$Q_a(p,\bar{p})=(pa,\ \bar{p}a^{-1})$$

for $(p,\bar{p})\in\Delta(M\times\bar{M})$. Then clearly $\mu\circ Q_a=\mu$, and $\theta=\tilde{\pi}\circ\mu$ hold. Now by the assumption

$$\tilde{\pi}\mu(X_p,\ \bar{X}_{\bar{p}})=\tilde{\pi}\mu(X'_q,\ \bar{X}'_{\bar{q}}).$$

Hence

$$\pi X_p=\theta(X_p,\bar{X}_{\bar{p}})=\theta(X'_q,\ \bar{X}'_{\bar{q}})=\pi X'_q.$$

Since $\mu(p, \bar{p}) = \mu(q, \bar{q})$, there exists an element $a \in S^1$ such that $q = p\cdot a$, $\bar{q} = \bar{p}\cdot a^{-1}$. Therefore we have

$$(J\pi X'_q)^*_q = (J\pi X'_q)^*_{p\cdot a} = R_a(J\pi X_p)^*_p$$

and

$$(J\pi X'_q)^{\bar{*}}_{\bar{q}} = R_{a^{-1}}(J\pi X_p)^{\bar{*}}_{\bar{p}}.$$

Hence

$$\begin{aligned}\mu((J\pi X'_q)^*_q,\ (J\pi X'_q)^{\bar{*}}_{\bar{q}}) &= \mu(R_a(J\pi X_p)^*_p,\ R_{a^{-1}}(J\pi X_p)^{\bar{*}}_{\bar{p}})\\ &= \mu\circ Q_a((J\pi X_p)^*_p,\ (J\pi X_p)^{\bar{*}}_{\bar{p}})\\ &= \mu((J\pi X_p)^*_p,\ (J\pi X_p)^{\bar{*}}_{\bar{p}}), \qquad \text{q. e. d.}\end{aligned}$$

By virtue of Lemma 5 we can define a tensor field $\tilde{\Phi}$ of type (1. 1) on $M\cdot\bar{M}$ as follows:

$$\tilde{\Phi}(\mu(X_p, \bar{X}_{\bar{p}})) = \mu((J\pi X_p)^*_p,\ (J\pi X)^{\bar{*}}_{\bar{p}}) \tag{3. 2}$$

for $(X_p, \bar{X}_{\bar{p}}) \in T_{(p,\bar{p})}(\Delta(M \times \bar{M}))$.

Next we define a vector field $\tilde{\xi}$ on $M\cdot\bar{M}$ as follows:

$$\tilde{\xi}_{\mu(p,\bar{p})} = \mu(\xi_p, 0_{\bar{p}}), \tag{3. 3}$$

where $0_{\bar{p}}$ denotes the zero tangent vector of $\bar{M}$ at $\bar{p}$. We see easily that $\tilde{\xi}$ is well defined by (3. 3) and $\tilde{\xi}$ is a vertical vector field of $M\cdot\bar{M}$ such that

$$\tilde{\eta}(\tilde{\xi}) = 1.$$

Now we have the following proposition

PROPOSITION 1. *$\tilde{\Sigma} = (\tilde{\Phi}, \tilde{\xi}, \tilde{\eta})$ is a normal almost contact structure on $M\cdot\bar{M}$.*

PROOF. For $X_{p_0} \in T_{p_0}(M_0)$ and $\pi p = p_0$ we denote by $\tilde{X^*}_{\mu(p,\bar{p})}$ the lift of X_{p_0} at $\mu(p, \bar{p})$ with respect to the connection $\tilde{\eta}$. Then it is easily seen that $\tilde{X^*}_{\mu(p,\bar{p})} = \mu(X^*_p, X^{\bar{*}}_{\bar{p}})$. Hence by the definition (3. 2) of $\tilde{\Phi}$ we have

$$\tilde{\Phi}(\tilde{X^*}_{\mu(p,\bar{p})}) = \tilde{\Phi}(\mu(X^*_p,\ X^{\bar{*}}_{\bar{p}})) = \mu((JX_{p_0})^*_p,\ (JX_{p_0})^{\bar{*}}_{\bar{p}}) = (JX_{p_0})^{\tilde{*}}_{\mu(p,\bar{p})},$$

which shows that

$$\tilde{\Phi}(X^{\tilde{*}}) = (JX)^{\tilde{*}}$$

for $X \in \mathfrak{X}(M_0)$. By virtue of (3. 1) we see that the associated 2-form $\tilde{\Omega}$ to $\tilde{\eta}$ is of type (1. 1) with respect to J. Now in the same way as the proof of Theorem 6[4] we can prove that $\tilde{\Sigma}$ is a normal almost contact structure of $M\cdot\bar{M}$. We shall not repeat the proof in detail.

DEFINITION 4. $\tilde{\Sigma}$ in Proposition 1 will be called *the product of* Σ *and* $\bar{\Sigma}$ and denoted by $\tilde{\Sigma} = \Sigma\cdot\bar{\Sigma}$.

DEFINITION 5. Let $L_0(M_0, S^1, \pi_0)$ be the trivial circle bundle over M_0, i. e. $L_0 = M_0 \times S^1$ and π_0 is the usual projection from L_0 onto M_0. We define the normal almost contact structure $\Sigma_0 = (\phi_0, \xi_0, \eta_0)$ on L_0 as follows:

$$\phi_0(X, Y) = (JX, 0) \qquad X \in T_{p_0}(M_0),\ Y \in T_a(S^1),$$

$$\xi_0 = \left(0, \frac{d}{dt}\right)$$

$$\eta_0 = (0\,,\ dt)$$

where t denotes the coordinates of S^1.

LEMMA 6. *Let (M, Σ), $(\overline{M}, \overline{\Sigma})$ be contact bundles over the same complex manifold M_0. Let $\Sigma = (\phi, \xi, \eta)$ and $\overline{\Sigma} = (\overline{\phi}, \overline{\xi}, \overline{\eta})$. Suppose that there is a bundle isomorphism f of M onto $\overline{M}$ such that $f^*\overline{\eta} = \eta$. Then $f^{-1}\overline{\phi}f = \phi$ holds, i. e. Σ and $\overline{\Sigma}$ are isomorphic by f.*

PROOF. Put $\phi' = f^{-1}\overline{\phi}f$. First we note that $f(X^*) = X^{\overline{*}}$ for $X \in \mathfrak{B}(M_0)$, since f is a bundle isomorphism such that $f^*\overline{\eta} = \eta$. Then for $X \in \mathfrak{B}(M_0)$ we have

$$\phi'(X^*) = f^{-1}\overline{\phi}fX^* = f^{-1}\overline{\phi}X^{\overline{*}} = f^{-1}(JX)^{\overline{*}} = (JX)^* = \phi(X^*).$$

On the other hand we have $\phi'(\xi) = \phi(\xi) = 0$. Hence $\phi'(X) = \phi(X)$ for any $X \in \mathfrak{B}(M)$, which proves the lemma.

LEMMA 7. *Let (M, Σ), (M_1, Σ_1), $(\overline{M}, \overline{\Sigma})$ and $(\overline{M}_1, \overline{\Sigma}_1)$ be contact bundles over the same complex manifold M_0. If $\Sigma \simeq \Sigma_1$ and $\overline{\Sigma} \cong \overline{\Sigma}_1$, then $\Sigma\cdot\overline{\Sigma} \simeq \Sigma_1\cdot\overline{\Sigma}_1$.*

PROOF. Let f(g resp.) be an isomorphism of Σ to Σ_1 (of $\overline{\Sigma}$ to $\overline{\Sigma}_1$ resp.). Then the bundle isomorphisms f and g induce an bundle isomorphism h of $M \cdot \overline{M}$ onto $M_1 \cdot \overline{M}_1$ such that $h^*(\eta_1 \cdot \overline{\eta}_1) = \eta \cdot \overline{\eta}$, where η, for example, is the contact form of Σ, i. e. $\Sigma = (\phi, \xi, \eta)$. Hence $\Sigma \cdot \overline{\Sigma} \simeq \Sigma_1 \cdot \overline{\Sigma}_1$ follows from Lemma 6.

LEMMA 8. *Let (M, Σ), (M_1, Σ_1) and (M_2, Σ_2) be contact bundles over the same complex manifold M_0. Then*

$$(\Sigma \cdot \Sigma_1) \cdot \Sigma_2 \cong \Sigma \cdot (\Sigma_1 \cdot \Sigma_2). \tag{3. 4}$$

In fact, there exists a bundle isomorphism of $(M \cdot M_1) \cdot M_2$ onto $M \cdot (M_1 \cdot M_2)$ preserving the contact forms, hence (3. 4) follows from Lemma 6.

In the same way as above two lemmas we can prove easily the following Lemma.

LEMMA 9. *Let (L_0, Σ_0) be the trivial contact bundles over M_0 and (M, Σ) be a contact bundle over M_0. Then we have*

$$\Sigma \cdot \Sigma_0 \cong \Sigma_0 \cdot \Sigma \cong \Sigma.$$

We now define *the inverse* Σ^{-1} *of* Σ for a contact bundle (M, Σ). For this purpose, we first recall the definition of $M^{-1}(M_0, S^1, \pi)$. The bundle space M^{-1} is the same as M. The action of an element a of S^1 on M is $p \to p \cdot a^{-1}$ for $p \in M$. i.e. M^{-1} is different from M only in the action of the structure group S^1. Now we define $\Sigma^{-1} = (\phi, -\xi, -\eta)$ for M^{-1} if $\Sigma = (\phi, \xi, \eta)$ for M. Then we see that (M^{-1}, Σ^{-1}) is a contact bundle over M_0. The following Lemma can be proved in the same way as the preceding lemmas.

LEMMA 10. *Notations being as in Lemma* 9, *we have*

$$\Sigma \cdot \Sigma^{-1} \cong \Sigma^{-1} \cdot \Sigma \cong \Sigma_0.$$

PROPOSITION 2. *Let* (M, Σ) *and* $(\bar{M}, \bar{\Sigma})$ *be contact bundles over* M_0. *Suppose that the associated 2-forms to* (M, Σ) *and* $(\bar{M}, \bar{\Sigma})$ *both vanishes identically, and suppose that* M_0 *is simply connected. Then* $\Sigma \cong \bar{\Sigma}$.

PROOF. Let $\Sigma = (\phi, \xi, \eta)$, $\bar{\Sigma} = (\bar{\phi}, \bar{\xi}, \bar{\eta})$. By the assumptions, the holonomy groups of η and $\bar{\eta}$ are both reduced to the identity. Take and fix three points $p_1 \in M$, $\bar{p}_1 \in \bar{M}$ and $p_0 \in M_0$ such that $\pi(p_1) = p_0 = \bar{\pi}(\bar{p}_1)$, $\pi(\bar{\pi}$ resp.) being the projection of $M(\bar{M}$ resp.) onto M_0. Take a point $p \in M$. We want to correspond a point $\bar{p}$ in $\bar{M}$ to the point p. We choose a curve γ in M_0 joining $\pi(p)$ and p_0. Take the horizontal lift $\tilde{\gamma}$ of γ on M whose initial point is p. Let q denote the end point of $\tilde{\gamma}$. Then there exists an (unique) element $a \in S^1$ such that $q = p_1 \cdot a$. Now take the horizontal lift $\tilde{\gamma}'$ of γ^{-1} on $\bar{M}$ whose initial point is $\bar{p}_1 \cdot a$, where γ^{-1} denotes the inverse curve of γ. Then the end point $\bar{p}$ of $\tilde{\gamma}'$ is independent on the choice of the curve γ, since the holonomy groups with respect to η and $\bar{\eta}$ are both identity. We now denote $\bar{p} = f(p)$. Then we can verify that f gives rise to an isomorphism from Σ to $\bar{\Sigma}$. Since the proof is canonical, we shall omit the proof in detail.

COROLLARY. *Let* (M, Σ) *be a contact bundle over* M_0, *whose assciated 2-form vanishes. Suppose* M_0 *is simply connected. Then* $\Sigma \cong \Sigma_0$.

THEOREM 7. *Let* (M, Σ), $(\bar{M}, \bar{\Sigma})$ *be contact bundles over a simply connected complex manifold* M_0, *whose associated 2-forms are* Ω *and* $\bar{\Omega}$ *respectively. Suppose* $\Omega = \bar{\Omega}$, *then* $\Sigma \cong \bar{\Sigma}$.

PROOF. Consider the product $\Sigma \cdot \bar{\Sigma}^{-1}$ on $M \cdot \bar{M}^{-1}$. Then the associated 2-form of $\Sigma \cdot \bar{\Sigma}^{-1}$ vanishes identically by the formula (3.1). Hence by the corollary above, we have

$$\Sigma \cdot \bar{\Sigma}^{-1} \cong \Sigma_0.$$

Then by Lemma 8, 9 and 10 we have

$$\Sigma \cong \Sigma \cdot (\bar{\Sigma}^{-1} \cdot \bar{\Sigma}) \cong (\Sigma \cdot \bar{\Sigma}^{-1}) \cdot \bar{\Sigma} \cong \Sigma_0 \cdot \bar{\Sigma} \cong \bar{\Sigma},$$

which proves Theorem 7.

We shall now prove the following theorem[2] which is the converse to Theorem 2 when the base space is simply connected.

THEOREM 8. *Let* (M, Σ) $((\overline{M}, \overline{\Sigma})$ *resp.) be a contact bundle over a complex manifold* M_0 $(\overline{M}_0$ *resp.) whose associated 2-form is* Ω $(\overline{\Omega}$ *resp.). Suppose that there eixsts a diffeomorphism* f_0 *of* M_0 *onto* $\overline{M}_0$ *such that* $f_0^* \overline{\Omega} = \Omega$. *Suppose also that* M_0 *is simply connected. Then we can find an isomorphism* f *of* Σ *to* $\overline{\Sigma}$ *such that*

$$\overline{\pi} \circ f = f_0 \circ \pi, \tag{3. 5}$$

where π *and* $\overline{\pi}$ *denotes the projection of* M *and* $\overline{M}$ *onto* M_0 *and* $\overline{M}_0$ *respectively.*

PROOF. $M' = M(\overline{M}_0, S^1, f_0 \circ \pi)$ with Σ on M defines clearly a contact bundle over $\overline{M}_0$ whose associated 2-form is $f_0^{-1*}\Omega = \overline{\Omega}$. By virtue of Theorem 7 there exists a bundle isomorphism f of M' onto $\overline{M}$ such that f is an isomorphism of Σ to $\overline{\Sigma}$. (3. 5) is now clear by the definition of M' and Theorem 8 is proved.

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2) A similar theorem as Theorem 8 may be proved even when M_0 ($\overline{M}_0$ resp.) is an almost complex manifold.