REMARKS ON 4-DIMENSIONAL DIFFERENTIABLE MANIFOLDS

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Let X_4 be 4-dimensional differentiable manifold and let $B(X_4, Y, G)$ be an arbitrary tensor bundle over X_4, where Y is a linear space of dimension 4^{p+q} with coordinates $(y^{i_1 \cdots i_p}_{j_1 \cdots j_q})$[1]. It is well known ([1]) that the structural group G of $B(X_4, Y, G)$ is reducible to the orthogonal group $O(4)$. And if X_4 is orientable, then it is easily seen that G is reducible to $SO(4)$ or one of its subgroups. If especially Y is a 4^2-dimensional linear space with coordinates (y^i_j), then the matrix representation of $SO(4)$ or its subgroup operates on Y as matrix transformations.

The purpose of this note is first to show the existence of two intrinsic (1-1)-type tensor bundles over X_4, which are subbundles of $B(X_4, Y, G)$ and to show the existence or non existence of cross sections of the two intrinsic subbundles wholly depends on the group G (§2). These are owing to the speciality of $SO(4)$.

Secondly, we classify X_4 following the structural group G and study further on each classes case by case (§3 $\sim$ §7).

1. Preliminary. The local subgroups of $SO(4)$ are treated by Ôtsuki [2] in the standpoint of holonomy groups of 4-dimensional Riemannian manifolds. And the classification of structural equations of all connected subgroups of $SO(4)$ is done by Ishihara [3] making use of the structural equation of $SO(4)$ indicated by Chern [4]. We will consider it in another point of view and will do the classification of the connected subgroups of $SO(4)$ in a different way.

As is known, $SO(4)$ is locally represented as $SO(4) = SU(2) \otimes SU(2)$. $SU(2)$ leaves invariant an anti-involution of the second kind and $SU(2) \otimes SU(2)$ leaves invariant that of the first kind which is the Kronecker product of the anti-involutions left invariant by the two $SU(2)$ (Cartan [5] ; Berger [6]). $SO(4)$ is the real representation of the group $SU(2) \otimes SU(2)$ restricted on the double element (real dimension 4) of the anti-involutions (see Appendix 1°). Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be the complexifications of the Lie algebras of the first and the second $SU(2)$. $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are of complex dimension 3. Then $\mathfrak{S} = \mathfrak{S}_1 + \mathfrak{S}_2$ (direct sum)

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1) Throughout this paper, the indices i_1, j_1, $i, j, a, b, \cdots$ run from 1 to 4, unless otherwise stated. This tensor is of type $(p-q)$.

is the complexification of the Lie algebra of $SO(4)$. Let $\pi_1 : \mathfrak{S} \to \mathfrak{S}_1$ and $\pi_2 : \mathfrak{S} \to \mathfrak{S}_2$ be the natural projections, so that $\pi_1(\mathfrak{S}) = \mathfrak{S}_1, \pi_2(\mathfrak{S}) = \mathfrak{S}_2$.

First, we consider a connected subgroup G of $SO(4)$ irreducible in real number field. If G is reducible in complex number field, we get $G = U(2)$ or $SU(2)$ (real rep.) and any other cases can not occur. For, if G is a proper subgroup of $SU(2)$, its dimension is $\leqq 2$ and hence G is integrable. In this case G leaves invariant a real direction or real 2-dimensional plane[2], but this is impossible. Consider the case G is still irreducible in complex number field (absolutely irreducible). Let $\mathfrak{g}$ be the Lie algebra of G and we denote the complexification of $\mathfrak{g}$ by $\mathfrak{g}^*$. As is well known (Cartan [7]), $\mathfrak{g}^*$ is semi-simple or semi-simple mod $\mathfrak{t}^1$, where $\mathfrak{t}^1$ is the Lie algebra of the complex homothetic group (complex dimension 1). We consider the case $\mathfrak{g}^*$ is semi-simple. Then, $\pi_1(\mathfrak{g}^*) \subseteqq \mathfrak{S}_1$ and the kernel $\mathfrak{g}_1 = \pi_1^{-1}(0)$ $(\subseteqq \mathfrak{S}_2)$ is an ideal in $\mathfrak{g}^*$. If the dimension of this kernel is equal to 1 or 2, then it is integrable. Since we now consider the case where $\mathfrak{g}^*$ is semi-simple, we must have $\mathfrak{g}_1 = 0$ or $\mathfrak{S}_2$ (in the case where dim $\mathfrak{g}_1 = 3$, we have $\mathfrak{g}_1 = \mathfrak{S}_2$). It is analoguous for the kernel $\mathfrak{g}_2 = \pi_2^{-1}(0) : \mathfrak{g}_2 = 0$ or $\mathfrak{S}_1$. If $\mathfrak{g}_1 = \mathfrak{S}_2$ and $\mathfrak{g}_2 = \mathfrak{S}_1$, we get $\mathfrak{g}^* = \mathfrak{S}_1 + \mathfrak{S}_2$, hence $G = SO(4)$. If $\mathfrak{g}_1 = 0$, $\mathfrak{g}_2 = \mathfrak{S}_1$ (resp. $\mathfrak{g}_1 = \mathfrak{S}_2$, $\mathfrak{g}_2 = 0$), we get $\mathfrak{g}^* = \mathfrak{S}_1$ (resp. $\mathfrak{g}^* = \mathfrak{S}_2$), hence $G = SU(2)$ (real rep.), which is the case where G is reducible in complex number field. Consider the case $\mathfrak{g}_1 = \mathfrak{g}_2 = 0$. If dim $\mathfrak{g}^* < 3$, then $\mathfrak{g}^*$ is integrable, which is impossible. If dim $\mathfrak{g}^* = 3$, we can verify that G leaves invariant a real direction (see Appendix 2°), whose case is omitted in the present consideration. If $\mathfrak{g}^*$ is not semi-simple, $\mathfrak{g}^*$ contains the Lie algebra $\mathfrak{t}^1$. In this case, it is possible only one case: $G = SU(2) \otimes T^1 = U(2)$ (real rep.), where T^1 is the one dimensional torus group. But, this is the case where G is reducible in complex number field, which is already considered.

Summing up, if a connected subgroup of $SO(4)$ is irreducible in real number field, then G is one of the followings:

$$SO(4),\ U(2),\ SU(2).$$

If G is reducible in real number field, then either it leaves invariant mutually orthogonal 1- and 3-dimensional planes, or two 2-dimensional planes.

Hence we get the following lemma.

LEMMA 1.1. *We can sum up all connected Lie subgroups of $SO(4)$ as follows*:
(I) (*irreducible in real number field*); $SO(4)$, $U(2)$, $SU(2)$;

2) When G leaves invariant a complex direction $\mathbf{z}$, then G also leaves invariant the conjugate direction $\bar{\mathbf{z}}$. Hence the 2-dimensional real plane spanned by $\mathbf{z}$ and $\bar{\mathbf{z}}$ is left invariant by G.

(II) (*reducible in real number field*): $1 \times SO(3)$, $SO(2) \times SO(2)$, $1 \times SO(2)$, $SO(2) \underline{\times} SO(2)$, $SO(2) \underline{\underline{\times}} SO(2)$, 1.

The notations are as follows. The Lie algebras of $SO(2) \times SO(2)$, $SO(2) \underline{\times} SO(2)$, $SO(2) \underline{\underline{\times}} SO(2)$ are given by matrices of the form :

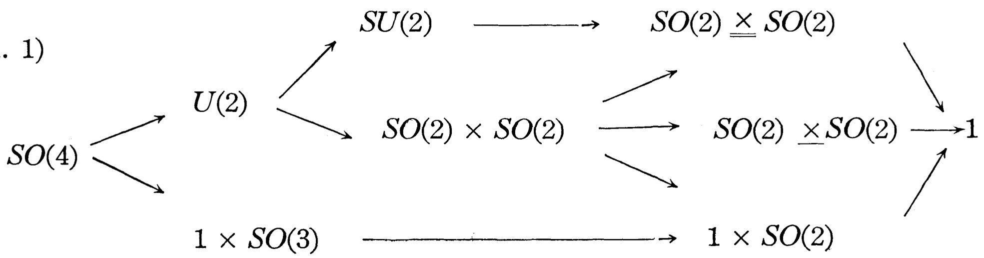
$$SO(2)\times SO(2): \quad \left(\begin{array}{cc|cc} 0 & \lambda & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -\lambda & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & \mu \\ & & -\mu & 0 \end{array}\right) \quad (\lambda,\mu: \text{independent}),$$

$$SO(2)\underline{\times} SO(2): \quad \left(\begin{array}{cc|cc} 0 & \lambda & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -\lambda & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & k\lambda \\ & & -k\lambda & 0 \end{array}\right) \quad (k: \text{const.} \neq 0, \pm 1),$$

$$SO(2)\underline{\underline{\times}} SO(2): \quad \left(\begin{array}{cc|cc} 0 & \lambda & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -\lambda & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & \lambda \\ & & -\lambda & 0 \end{array}\right).$$

If $k = -1$ in the case of $SO(2) \underline{\times} SO(2)$, we consider the frame with opposite orientation, then we get the case of $SO(2) \underline{\underline{\times}} SO(2)$. $SO(2) \times SO(2)$, $SO(2) \underline{\times} SO(2)$, $1 \times SO(2)$ are subgroups of $U(2)$, but not of $SU(2)$. $SO(2) \underline{\underline{\times}} SO(2)$ is a subgroup of $SU(2)$ (see Appendix 3°). The relations among them are summed up in the following table.

(1. 1)



Now, we get the following lemma.

LEMMA 1.2. *Let X_4 be an orientable 4-dimensional differentiable manifold and denote an arbitrary tensor bundle over X_4 by $B(X_4, Y, G)$, where Y is a linear space of dimension 4^{p+q} with coordinates $(y^{i_1\cdots i_p}_{j_1\cdots j_q})$. Then the group G is reducible to one of the groups indicated in Lemma* 1.1.

2. Two intrinsic (1-1)-tensor bundles associated X_4. First, let I_1, J_1, K_1 and I_2, J_2, K_2 be the matrices such that

$$(2.1)\qquad I_1 = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right), \quad J_1 = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right),$$

$$K_1 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right);$$

$$(2.2)\qquad I_2 = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right), \quad J_2 = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right),$$

$$K_2 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right).$$

We remark that if we put

$$\lambda = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}, \tag{2. 3}$$

then we have

$$I_2 = \lambda I_1 \lambda^{-1}, \; J_2 = \lambda J_1 \lambda^{-1}, \; K_2 = \lambda K_1 \lambda^{-1}. \tag{2. 4}$$

These I_1, J_1, K_1 and I_2, J_2, K_2 satisfy the quaternic relations:

$$\begin{cases} I_1^2 = J_1^2 = K_1^2 = -1; \; I_1 J_1 = -J_1 I_1 = K_1, \; J_1 K_1 = -K_1 J_1 = I_1, \\ \qquad K_1 I_1 = -I_1 K_1 = J_1; \\ I_2^2 = J_2^2 = K_2^2 = -1; \; I_2 J_2 = -J_2 I_2 = K_2, \; J_2 K_2 = -K_2 J_2 = I_2, \\ \qquad K_2 I_2 = -I_2 K_2 = J_2. \end{cases} \tag{2. 5}$$

And we also remark that each I_1, J_1, K_1 is commutative with each I_2, J_2, K_2.

Now, any transformation of $SO(4)$ decomposes into

$$(2.\ 6)_1 \begin{cases} x' = a_0x - a_1y - a_2u - a_3v \\ y' = a_1x + a_0y - a_3u + a_2v \\ u' = a_2x + a_3y + a_0u - a_1v \\ v' = a_3x - a_2y + a_1u + a_0v \\ \qquad (a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1) \end{cases}, \quad (2.\ 6)_2 \begin{cases} x' = b_0x - b_1y - b_2u - b_3v \\ y' = b_1x + b_0y + b_3u - b_2v \\ u' = b_2x - b_3y + b_0u + b_1v, \\ v' = b_3x + b_2y - b_1u + b_0v \\ \qquad (b_0^2 + b_1^2 + b_2^2 + b_3^2 = 1) \end{cases}$$

where (x, y, u, v) is a real vector in the 4-dimensional Euclidean space E^4 with respect to orthogonal bases. These equations are indicated in Chern [4] (see Appendix 1°).

We can see that under the transformation $(2.6)_1$, I_1, J_1, K_1 are left invariant and under the transformation $(2.6)_2$, each of them is transformed into a linear combination of I_1, J_1, K_1. Similarly, I_2, J_2, K_2 are left invariant by $(2.6)_2$ and each of them is transformed into a linear combination of I_2, J_2, K_2 by $(2.6)_1$. That is, by $SO(4)$, the matrices I_1, J_1, K_1 (resp. I_2, J_2, K_2) are transformed into the matrices I_1', J_1', K_1' (resp. I_2', J_2', K_2'), such that

$$(2.7)_1 \quad \begin{cases} I_1' = \alpha_1 I_1 + \beta_1 J_1 + \gamma_1 K_1 \\ J_1' = \alpha_1' I_1 + \beta_1' J_1 + \gamma_1' K_1 \\ K_1' = \alpha_1'' I_1 + \beta_1'' J_1 + \gamma_1'' K_1 \end{cases}, \qquad (2.7)_2 \quad \begin{cases} I_2' = \alpha_2 I_2 + \beta_2 J_2 + \gamma_2 K_2 \\ J_2' = \alpha_2' I_2 + \beta_2' J_2 + \gamma_2' K_2 \\ K_2' = \alpha_2'' I_2 + \beta_2'' I_2 + \gamma_2'' K_2 \end{cases}$$

The matrices $\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_1' & \beta_1' & \gamma_1' \\ \alpha_1'' & \beta_1'' & \gamma_1'' \end{pmatrix}$ and $\begin{pmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_2' & \beta_2' & \gamma_2' \\ \alpha_2'' & \beta_2'' & \gamma_2'' \end{pmatrix}$ are *orthogonal matrices*, which are easily verified from (2.5) and from the same relations among I_1', J_1', K_1' (resp. I_2', J_2', K_2').

A transformation of $SO(4)$ in a neighborhood of the identity is given by $\exp\alpha$, where

$$\alpha = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & f & 0 \end{pmatrix}$$ [3]

is a matrix in a neighborhood of the 0-matrix. For this α, we can verify that

$$(2.8) \quad \begin{cases} \alpha I_1 - I_1\alpha = (c-d)J_1 - (b+e)K_1 \\ \alpha J_1 - J_1\alpha = (a-f)K_1 - (c-d)I_1 \\ \alpha K_1 - K_1\alpha = (b+e)I_1 - (a-f)J_1, \end{cases}$$

$$(2.9) \quad \begin{cases} \alpha I_2 - I_2\alpha = -(c+d)J_2 - (b-e)K_2 \\ \alpha J_2 - J_2\alpha = (a+f)K_2 + (c+d)I_2 \\ \alpha K_2 - K_2\alpha = (b-e)I_2 - (a+f)J_2. \end{cases}$$

LEMMA 2.1. *The necessary and sufficient condition that a* (4×4)*-matrix* A *satisfy* $A^2 = -1$ *is that* $A = \alpha I_1 + \beta J_1 + \gamma K_1$ $(\alpha^2 + \beta^2 + \gamma^2 = 1)$ *or* $A = \alpha' I_2 + \beta' J_2 + \gamma' K_2$ $(\alpha'^2 + \beta'^2 + \gamma'^2 = 1)$, *where* I_1, J_1, K_1 *or* I_2, J_2, K_2 *are given by* (2.1), (2.2).

3) This matrix decomposes into the form (6) in the Appendix 1°.

PROOF. The sufficiency easily follows from (2. 5). Conversely, suppose that A satisfy $A^2 = -1$. By an orthogonal transformation M, we can transform A into $A' = MAM^{-1}$ which is just the same as I_1 in (2.1). First, suppose that $\det|M| = 1$. Under the present transformation by M, I_1, J_1, K_1 are transformed into I_1', J_1', K_1' such that (see $(2.7)_1$)

$$\begin{cases} A'(=I_1) = \alpha_1 I_1' + \alpha_1' J_1' + \alpha_1'' K_1' \\ J_1 \qquad\quad = \beta_1 I_1' + \beta_1' J_1' + \beta_1'' K_1' \\ K_1 \qquad\quad = \gamma_1 I_1' + \gamma_1' J_1' + \gamma_1'' K_1' \quad . \end{cases}$$

If we consider the first equation with respect to the original coordinate system, we see that $A = \alpha_1 I_1 + \alpha_1' J_1 + \alpha_1'' K_1$ and $\alpha_1^2 + \alpha_1'^2 + \alpha_1''^2 = 1$.

If $\det|M| = -1$, we can put $M = \lambda\ M_0$, where λ is given by (2. 3) and $\det|M_0| = 1$. From $MAM^{-1} = I_1$, we have $M_0AM_0^{-1} = \lambda^{-1}I_1\lambda = I_2$. In this case, we get $A = \alpha_2 I_2 + \alpha_2' J_2 + \alpha_2'' K_2$ $(\alpha_2^2 + \alpha_2'^2 + \alpha_2''^2 = 1)$. Q.E.D.

Now, let Y be a linear space of dimension 4^2 with coordinates (y_j^i) $(i, j = 1, 2, 3, 4)$. We denote the subspace of Y which is the set of all matrices $\alpha I_1 + \beta J_1 + \gamma K_1$ $(\alpha^2 + \beta^2 + \gamma^2 = 1)$ by Y_1. Similarly, we denote the subspace of Y which is the set of all matrices $\alpha' I_2 + \beta' J_2 + \gamma' K_2$ $(\alpha'^2 + \beta'^2 + \gamma'^2 = 1)$ by Y_2. Any matrix A of Y_1 or Y_2 satisfies $A^2 = -1$ by Lemma 2. 1 and we can write symbolically $\lambda Y_1 \lambda^{-1} = Y_2$, taking account of (2. 4).

By virtue of (2. 7), these subspaces Y_1 and Y_2 are invariant under $SO(4)$.

DEFINITION. Let Y, Y_1, Y_2 be as in the above and let $B(X_4, Y, G)$ be the (1-1)-type tensor bundle over X_4, where G is $SO(4)$ or one of its connected subgroups which are indicated in §1. As is well known, with the same base space X_4 and group G, there exist two subbundles of $B(X_4, Y, G)$ with fibre Y_1 and Y_2. We denote these subbundles by $B_1(X_4, Y_1, G)$ and $B_2(X_4, Y_2, G)$ respectively.

THEOREM 2. 1. *Let X_4 be an orientable 4-dimensional differentiable manifold. Then we can associate to X_4 intrinsically two (1-1)-type tensor bundles $B_1(X_4, Y_1, G)$ and $B_2(X_4, Y_2, G)$, where G is $SO(4)$ or one of its connected subgroups.*

And with respect to the cross sections we can state as follows.[4]

1° *Any of the two bundles does not admit cross sections if and only if $G = SO(4)$, $1 \times SO(3)$.*

2° *One of the two bundles and only one admits at least a cross section*

4) Hereafter, if we denote $G = U(2)$ for instance, then we mean that the G of X_4 is reducible to $U(2)$, but not to any connected proper subgroup of $U(2)$.

if and only if $G = U(2)$, $SU(2)$.

3° *Both of them admit cross sections if and only if* $G = SO(2)\times SO(2)$, $SO(2)\underline{\times} SO(2)$, $SO(2)\underline{\underline{\times}} SO(2)$, $1 \times SO(2)$, 1.

In the cases 2° *and* 3°, X_4 *admits at least an almost complex structure.*

PROOF. a) In order that the bundle $B_1(X_4, Y_1, G)$ or $B_2(X_4, Y_2, G)$ admits a cross section, it is necessary and sufficient that $G \subseteqq U(2)$ (i.e. X_4 admits an almost complex structure), which follows at once from Lemma 2.1. This proves 1° and a part of 2°.

b) It is remained for us only to prove that if the bundles B_1 and B_2 admit cross sections simultaneously, then $G \neq U(2)$, $SU(2)$. If B_1 and B_2 together admit cross sections, then X_4 admits two almost complex structures $a(x)$ and $b(x)$ ($x \in X_4$), where $a^2 = b^2 = -1$. And we see that $a \neq \pm b$ by virtue of (2.1) and (2.2). Hence the tensor field $c(x) = a(x)\cdot b(x)$ over X_4 gives a non-trivial almost product structure: $c^2 = 1$. This means that G can be reducible to a group reducible in real number field, so that G is one of the groups indicated in 3°. This proves 2° and 3°. Q. E. D.

In the general tensor bundle $B(X_4, Y, G)$, G is one of the subgroups indicated in (1.1). In the following, we will consider such X_4's, the coordinate neighborhood being given by (x^i) ($i = 1, 2, 3, 4$).

3. X_4 **with** $G = 1 \times SO(3)$. If $G = 1 \times SO(3)$, G leaves invariant a matrix of the form

$$Y^* = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix},$$

with respect to a suitable orthogonal coordinate system. And $B(X_4, Y^*, G)$ is a subbundle of $B(X_4, Y, G)$. This subbundle admits a cross section, which is an *almost product structure*: $a(x) = (a_j^i(x))$ over X_4 so that $a^2 = 1$. If we put $p = \frac{1}{2}(1 - a)$, $q = \frac{1}{2}(1 + a)$, that is,

$$p_j^i = \frac{1}{2}(\delta_j^i - a_j^i), \quad q_j^i = \frac{1}{2}(\delta_j^i + a_j^i),$$

then $p = (p_j^i)$, $q = (q_j^i)$ are projection tensors so that $p^2 = p$, $q^2 = q$, $p + q = 1$

(Walker, [9], [10]). They define two complementary distributions D, D' over X_4 respectively. The rank of (p_j^i) and hence the dimension of D is 1. D is always integrable. The rank of (q_j^i) and hence the dimension of D' is 3. On the other hand in order that the distribution D' defined by $p_j^i dx^j = 0$ be completely integrable, it is necessary and sufficient that $\partial_{[a} p_{b]}{}^i q_j{}^a q_k{}^b = 0$ or

$$N_{jk}{}^i + N_{ja}{}^i a_k{}^a = 0, \tag{3. 1}$$

where $N_{jk}{}^i$ is the Nijenhuis tensor of a_j^i:

$$N_{jk}{}^i = \frac{1}{2} [a_{[j}{}^a \partial_{|a|} a_{k]}{}^i - a_{[j}{}^a \partial_{k]} a_a{}^i].$$

The condition (3. 1) is equivalent to $N_{jk}{}^i = 0$, since the relation $N_{jk}{}^i - N_{ja}{}^i a_k{}^a = 0$ corresponding to the integrability condition for $q_j^i dx^j = 0$ is always satisfied.

Summing up, the X_4 under consideration is as follows:

(i) *There exists an almost product structure.*

(ii) *There exist two complementary distributions D, D' of dimension* 1 *and* 3 *respectively. The distribution D is always integrable.*

(iii)

$N_{jk}{}^i \equiv 0$; *the distribution D' is also integrable.*	$N_{jk}{}^i \not\equiv 0$; *the distribution D' is not integrable.*

Furthermore, in this manifold, there exist a non singular symmetric tensor field a_{ij} with signature $(+++-)$ and two symmetric tensor fields of rank 1 and 3.

An example is $R^1 \times S^3$. In this case, $N_{jk}{}^i \equiv 0$.

4. X_4 with $G = U(2)$ or $SU(2)$. A transformation T of $U(2)$ decomposes into (4) and (7) in the Appendix 1°. In this case, we can easily verify that

$$\begin{cases} TI_1T^{-1} = I_1 \\ TJ_1T^{-1} = \quad lJ_1 + mK_1 \qquad (l^2 + m^2 = 1). \\ TK_1T^{-1} = -mJ_1 + lK_1 \end{cases}$$

Hence I_1 is invariant by $U(2)$ and this gives rise a cross section in $B(X_4, Y_1, G)$ which is an almost complex structure $\phi(x) = (\phi_j{}^i(x))$ in X_4. On the other hand, if we put

$$A = \alpha J_1 + \beta K_1 \qquad (\alpha^2 + \beta^2 = 1),$$

then $A^2 = -1$, and we denote the set of all such A's by Y'. There exists a subbundle $B_1'(X_4, Y_1', G)$ of $B_1(X_4, Y_1, G)$. If this subbundle admits a cross section, then it gives rise another almost complex structure $\psi(x) = (\psi_j{}^i(x))$ in X_4 and we can easily see that $\tau(x) = \phi(x)\cdot\psi(x) = -\psi(x)\cdot\phi(x)$ gives the third almost complex structure. In this case, G is reducible to $SU(2)$.

Consequently, *if the structural group G is $U(2)$ or one of its subgroups, then we can associate a* (1-1)*-type tensor bundle $B_1'(X_4, Y_1', G)$, which is a subbundle of $B_1(X_4, Y_1, G)$. If this subbundle admits a cross section, then G is reducible to $SU(2)$ or one of its subgroups and vice versa.*

1) $G = U(2)$. According to the vanishing or non vanishing of the Nijenhuis tensor $N_{jk}{}^i$ of $\phi_j{}^i$ we can classify X_4 into two classes, which is well known.

Furthermore, since there exist Riemannian metrics such that $g_{ab}\phi_i{}^a\phi_j{}^b = g_{ij}$, we put $\phi_{ij} = g_{ja}\phi_i{}^a$ and $\phi_{ijk} = \partial_{[k}\phi_{ij]}$. *With respect to such a metric g_{ij}, X_4 is classified according to the vanishing or non vanishing of ϕ_{ijk}.*

An example is the two dimensional complex projective space (in its real representation). In this case, $N_{jk}{}^i = \phi_{ijk} = 0$, the Riemannian metric g_{ij} being kählerian to the complex structure $\phi_j{}^i$.

2) $G = SU(2)$. In this case, as has been shown, there are three almost complex structures $\phi = (\phi_j{}^i)$, $\psi = (\psi_j{}^i)$, $\tau = (\tau_j{}^i)$ such that $\phi\psi = -\psi\phi = \tau$, $\psi\tau = -\tau\psi = \phi$, $\tau\phi = -\phi\tau = \psi$. The set of ϕ,ψ,τ is the so-called almost quaternion structure. Let $N_{jk}{}^i(\phi)$, $N_{jk}{}^i(\psi)$, $N_{jk}{}^i(\tau)$ be the Nijenhuis tensor of ϕ,ψ,τ respectively, then the following theorem is known ([11], Cor. 2 to Thm. 10. 4):

THEOREM. $N_{jk}{}^i(\phi)$, $N_{jk}{}^i(\psi)$, $N_{jk}{}^i(\tau)$ vanish identically if any two of them vanish identically.

Hence, X_4 is classified into one of the followings:

1) *Any one of $N_{jk}{}^i(\phi)$, $N_{jk}{}^i(\psi)$, $N_{jk}{}^i(\tau)$ does not vanish.*
2) *One and only one of the above three Nijenhuis tensors vanish.*
3) *All of them vanish.*

Now, since it is known that there exist Riemannian metrics hermitian with respect to all ϕ,ψ,τ ([11]), we put $\psi_{ij} = g_{ja}\psi_i{}^a$, $\tau_{ij} = g_{ja}\tau_i{}^a$, and $\psi_{ijk} = \partial_{[k}\psi_{ij]}$, $\tau_{ijk} = \partial_{[k}\tau_{ij]}$. The following theorem is known.

THEOREM. $N_{jk}{}^i(\phi)$, ψ_{ijk}, τ_{ijk} vanish identically if any two of them vanish identically ([12], Thm. 5. 3).

Hence, with respect to such a Riemannian metric g_{ij}, X_4 is classified into one of the following types.

$$\begin{cases} \text{(i) } \textit{Any one of } N_{jk}^{\ i}(\phi),\ \psi_{ijk},\ \tau_{ijk} \textit{ does not vanish.} \\ \text{(ii) } \textit{One and only one of the above three tensors vanish.} \\ \text{(iii) } \textit{All of them vanish.} \end{cases}$$

An example is the manifold of the tangent bundle of a 2-dimensional differentiable manifold (cf. the last part of §8).

5. X_4 **with** $G = SO(2)\times SO(2)$. As mentioned in §1, the Lie algebra of G is given by the matrices of the form

$$\left(\begin{array}{cc|cc} 0 & \lambda & & \\ -\lambda & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & \mu \\ \multicolumn{2}{c|}{0} & -\mu & 0 \end{array}\right) \qquad (\lambda,\ \mu \text{ independent})$$

with respect to a suitable orthogonal coordinate system. And G leaves invariant the matrices I_1 and I_2 in (2.1) and (2.2). I_1 and I_2 are commutative: $I_1I_2=I_2I_1$, and these I_1,I_2 give rise cross sections in $B_1(X_4, Y_1, G)$, $B_2(X_4, Y_2, G)$, which are almost complex structures $\phi=(\phi_j^{\ i})$, $\phi'=(\phi_j'^{\ i})$ in X_4. And we see that $\phi\phi'=\phi'\phi$. If we put $\pi=-\phi\phi'$, that is, $\pi_j^{\ i}=-\phi_j^{\ a}\phi_a'^{\ i}$, then we see that π is an almost product structure in X_4. The normal form of π is such that

$$\pi=\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}.$$

There are relations as follows:

$$(5.1)\qquad \phi^2=\phi'^2=-1,\ \pi^2=1;\ \phi\phi'=\phi'\phi=-\pi,\ \phi'\pi=\phi,\ \pi\phi=\phi'.$$

This system $(\phi,\ \phi',\ \pi)$ is the so-called *almost complex product structure* (of the second kind) ([13], p. 394).

We can sum up the general properties of X_4 as in the followings, where b), c) are easily verified as in the case $G = 1\times SO(3)$.

(5.2) $\left\{\begin{array}{l}\text{a) } \textit{There exists a so-called almost complex product structure of the 2nd kind.} \\ \text{b) } \textit{There exist two complementary distributions } D, D' \textit{ of dimension } 2.\end{array}\right.$

In this manifold there exist a non singular symmetric tensor field with signature $(++--)$ and two symmetric tensor fields of rank 2.

Next, we will classify the X_4. Let $N_{jk}{}^i(\phi)$, $N_{jk}{}^i(\phi')$, $N_{jk}{}^i(\pi)$ be the Nijenhuis tensor of ϕ, ϕ', π respectively. Then we know that the vanishing of any two of $N_{jk}{}^i(\phi)$, $N_{jk}{}^i(\phi')$, $N_{jk}{}^i(\pi)$ implies the vanishing of the remaining one ([14]).

The integrability conditions of the distributions D and D' are given by the followings respectively:

$$n_{jk}{}^i(D) \equiv N_{jk}{}^i(\pi) - N_{ja}{}^i(\pi)\ \pi_k{}^a = 0, \quad n_{jk}{}^i(D') \equiv N_{jk}{}^i(\pi) + N_{ja}{}^i(\pi)\pi_k{}^a = 0.$$

The X_4 is one of the following types.

(5. 3)

- (i) *Any of the tensors* $N_{jk}{}^i(\phi)$, $N_{jk}{}^i(\phi')$, $N_{jk}{}^i(\pi)$, $n^i_{jk}(D)$, $n^i_{jk}(D')$ *does not vanish.*
- (ii) $n_{jk}{}^i(D) = 0$; *the others do not vanish.* In this case, the distribution D is integrable.
- (iii) $N_{jk}{}^i(\pi) = 0$, $n_{jk}{}^i(D) = 0$, $n_{jk}{}^i(D') = 0$; *the others do not vanish.* In this case, the distributions D and D' are both integrable.
- (iv) $N_{jk}{}^i(\phi) = 0$; *the others do not vanish.* The almost complex structure ϕ is integrable.
- (v) $N_{jk}{}^i(\phi) = 0$, $n_{jk}{}^i(D) = 0$; *the others do not vanish.*
- (vi) *All tensors in* (i) *vanish.*

An example is $S^2 \times S^2$. This is the case (vi).

6. X_4 with $G = 1 \times SO(2)$. The Lie algebra of G is given by the matrices of the form:

$$\left(\begin{array}{c|cc} 0 & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & \lambda \\ & -\lambda & 0 \end{array}\right).$$

Since this is a subgroup of $SO(2)\times SO(2)$, there exists in X_4 the almost complex product structure of the 2nd kind (5. 1). Although the properties (5. 2) for $G=SO(2)\times SO(2)$ hold good, we can futhermore decompose the almost product structure $\pi=(\pi_j^i)$ as follows.

Evidently, G leaves invariant the matrices

$$(6.\ 1)\qquad \begin{pmatrix} 1 & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix},\quad \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix},\quad \begin{pmatrix} 0 & & & 0 \\ & 0 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix},$$

and $B(X_4, Y, G)$ admits cross sections corresponding to (6.1), which are tensor fields $p=(p_j^i)$, $q=(q_j^i)$, $r=(r_j^i)$ over X_4. We see that

$$(6.\ 2)\qquad p^2=p,\ q^2=q,\ r^2=r;\ pq=qr=rp=0,\ p+q+r=1,$$

and furthermore $p+q-r=\pi$. The tensor fields p, q, r define complementary distributions D, D', D'' of dimension 1, 1, 2 respectively. These distributions are defined by $(q_j^i+r_j^i)dx^j=0$, $(p_j^i+r_j^i)\,dx^j=0$, $(p_j^i+q_j^i)\,dx^j=0$ respectively. The 1-dimensional distributions D, D' are always integrable. The integrability condition of the distribution D'' is $n_{jk}{}^i(D'')\equiv(\partial_{[a}p_{b]}{}^i+\partial_{[a}q_{b]}{}^i)\,r_j{}^a r_k{}^b=0$, which is equivalent to $N_{jk}{}^i(\pi)=0$.

The general properties of X_4 are summed up as follows. They are special cases of (5. 2).

a) *All properties of* (5. 2) *hold good.*
b) *Especially the last property* c) *of* (5. 2) *is stated more precisely as follows: There exist three complementary distributions D, D', D'' defined by projection tensors p, q, r in* (6. 2), *where $p+q-r=\pi$. The 1-dimensional distributions D and D' are always integrable.*

And X_4 is classified into one of the following types:

(i) *Any of $N_{jk}{}^i(\phi)$, $N_{jk}{}^i(\phi')$, $N_{jk}{}^i(\pi)$ do not vanish.*
(ii) $N_{jk}{}^i(\pi)=0$; *the others do not vanish. The distribution D'' is integrable.*
(iii) $N_{jk}{}^i(\phi)=0$; *the others do not vanish. The almost complex structure ϕ is integrable.*
(iv) $N_{jk}{}^i(\phi)=N_{jk}{}^i(\phi')=N_{jk}{}^i(\pi)=0$.

An example of such an X_4 is $R^2 \times S^2$. This is the case (iv).

REMARK. In the present X_4, if we put $\pi_1 = p - q + r$, $\pi_2 = p - q - r$, then we can easily see that

$$\pi^2 = \pi_1^2 = \pi_2^2 = 1,\ \pi\pi_1 = \pi_1\pi = \pi_2,\ \pi_1\pi_2 = \pi_2\pi_1 = \pi,\ \pi_2\pi = \pi\pi_2 = \pi_1.$$

7. X_4 **with** $G = SO(2) \underline{\times}\, SO(2)$. This is the case $\mu = k\lambda$ ($k \neq 0, \pm 1$) in §5. Hence, *for the* X_4 *the general properties and the classification in* §5 *are valid in the present case.*

The X_4 can not be a global product manifold $X_2 \times X_2'$, where X_2 and X_2' are 2-dimensional differentiable manifolds. For, if $X_4 = X_2 \times X_2'$ (in the global sense), then the minimal connected subgroup containing the structural group is $SO(2) \times SO(2)$, $1 \times SO(2)$ or 1. But these are impossible (cf. footnote 4)).

8. X_4 **with** $G = SO(2) \underline{\underline{\times}}\, SO(2)$. The Lie algebra of G is given by the matrices of the form

$$\left(\begin{array}{cc|cc} 0 & \lambda & & \\ -\lambda & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & \lambda \\ \multicolumn{2}{c|}{0} & -\lambda & 0 \end{array}\right).$$

This is a special case of $G = SU(2)$ and $G = SO(2) \times SO(2)$, hence we can find in X_4 an almost quaternion structure (ϕ, ψ, τ) (see §4) and an almost complex product structure (ϕ, ϕ', π) (see §5). Furthermore, since ϕ' is commutative with all ϕ, ψ, τ (see §5 and §2), we put

$$\psi\phi' = \phi'\psi = -\pi_1,\quad \tau\phi' = \phi'\tau = -\pi_2.$$

Then (ψ, ϕ', π_1), (τ, ϕ', π_2) are also almost complex product structures. The normal forms of π, π_1, π_2 are as follows:

$$\pi = \left(\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & \multicolumn{2}{c}{0} \\ \hline & & -1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & -1 \end{array}\right),\qquad \pi_1 = \left(\begin{array}{cc|cc} & & 0 & -1 \\ \multicolumn{2}{c|}{0} & 1 & 0 \\ \hline 0 & 1 & & \\ -1 & 0 & \multicolumn{2}{c}{0} \end{array}\right),$$

$$\pi_2 = \left(\begin{array}{cc|cc} & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \\ \hline 1 & 0 & \multicolumn{2}{c}{0} \\ 0 & 1 & & \end{array}\right)$$

An example of such an X_4 is the manifold of the tangent bundle of a 2-dimensional differentiable manifold (the details will be appear in another paper).

APPENDIX

1° (see §1). In $SO(4) = SU(2) \otimes SU(2)$, the transformations of the first and the second $SU(2)$ in a complex 2-dimensional linear space C^2 are given by

$$(1)_1 \quad \begin{cases} z_1' = az_1 + bz_2 \\ z_2' = -\bar{b}z_1 + \bar{a}z_2 \\ \qquad (a\bar{a} + b\bar{b} = 1), \end{cases} \qquad (1)_2 \quad \begin{cases} w_1' = \alpha w_1 + \beta w_2 \\ w_2' = -\bar{\beta}w_1 + \bar{\alpha}w_2 \\ \qquad (\alpha\bar{\alpha} + \beta\bar{\beta}) = 1) \end{cases},$$

where $(z_1, z_2) \in C^2$ and $(w_1, w_2) \in C^2$. $(1)_1$ and $(1)_2$ leave invariant anti-involutions of the second kind: $Z_1 = \bar{z}_2$, $Z_2 = -\bar{z}_1$ and $W_1 = \bar{w}_2$, $W_2 = -\bar{w}_1$ respectively. If we put

$$z_{ij} = z_i \otimes z_j \qquad (i, j = 1, 2),$$

then a transformation of $SU(2) \otimes SU(2)$ is given by

$$(2) \qquad \begin{cases} z_{11}' = a\alpha z_{11} + a\beta z_{12} + b\alpha z_{21} + b\beta z_{22} \\ z_{12}' = -a\bar{\beta}z_{11} + a\bar{\alpha}z_{12} - b\bar{\beta}z_{21} + b\bar{\alpha}z_{22} \\ z_{21}' = -\bar{b}\alpha z_{11} - \bar{b}\beta z_{12} + \bar{a}\alpha z_{21} + \bar{a}\beta z_{22} \\ z_{22}' = \bar{b}\bar{\beta}z_{11} - \bar{b}\bar{\alpha}z_{12} - \bar{a}\bar{\beta}z_{21} + \bar{a}\bar{\alpha}z_{22} \end{cases}.$$

This transformation leaves invariant an anti-involution of the first kind:

$$Z_{11} = \bar{z}_{22},\ Z_{12} = -\bar{z}_{21},\ Z_{21} = -\bar{z}_{12},\ Z_{22} = \bar{z}_{11}.$$

which is the Kronecker product of the preceding two anti-involutions. The double element of this anti-involution (real dimension 4) is defined by $z_1 = \bar{z}_{22}$, $z_{12} = -\bar{z}_{21}$. $SO(4)$ is the restriction of (2) on this double element. If $(1)_2$ is the identity, then (2) reduces to

$$(3)\qquad \begin{cases} z'_{11} = az_{11} + bz_{21} \\ z'_{12} = az_{12} + bz_{22} \\ z'_{21} = -\bar{b}z_{11} + \bar{a}z_{21} \\ z'_{22} = -\bar{b}z_{12} + \bar{a}z_{22}. \end{cases}$$

If we put

$z_{11} = \bar{z}_{22} = x + \sqrt{-1}\, y$, $z_{12} = -\bar{z}_{21} = u + \sqrt{-1}\, v$, $a = a_0 + \sqrt{-1}\, a_1$, $b = a_2 + \sqrt{-1}\, a_3$, then (3) becomes

$$(4)\qquad \begin{cases} x' = a_0x - a_1y - a_2u - a_3v \\ y' = a_1x + a_0y - a_3u + a_2v \\ u' = a_2x + a_3y + a_0u - a_1v \\ v' = a_3x - a_2y + a_1u + a_0v \end{cases} \qquad (a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1).$$

If $(1)_1$ is the identity and if we put $\alpha = b_0 + \sqrt{-1}\, b_1$, $\beta = -b_2 + \sqrt{-1}$ then we get similarly

$$(5)\qquad \begin{cases} x' = b_0x - b_1y - b_2u - b_3v \\ y' = b_1x + b_0y + b_3u - b_2v \\ u' = b_2x - b_3y + b_0u + b_1v \\ v' = b_3x + b_2y - b_1u + b_0v \end{cases} \qquad (b_0^2 + b_1^2 + b_2^2 + b_3^2 = 1).$$

Any transformation of $SO(4)$ decomposes into (4) and (5) with respect to a fixed oriented orthogonal frame. (cf. [4]). The Lie algebra of $SO(4)$ is given by the matrices of the form:

$$(6)\qquad \begin{pmatrix} 0 & -(\lambda_1 + \lambda'_1) & -(\mu_1 + \mu'_1) & -(\nu_1 + \nu'_1) \\ (\lambda_1 + \lambda'_1) & 0 & -(\nu_1 - \nu'_1) & (\mu_1 - \mu'_1) \\ (\mu_1 + \mu'_1) & (\nu_1 - \nu'_1) & 0 & -(\lambda_1 - \lambda'_2) \\ (\nu_1 + \nu'_1) & -(\mu_1 - \mu'_1) & (\lambda_1 - \lambda'_1) & 0 \end{pmatrix}.$$

If we put $\alpha\bar{\alpha}=1$, $\beta=0$ in $(1)_2$, then we obtain a transformation of $U(2)$. In this case, (5) turns into

$$(7)\qquad \begin{cases} x' = b_0x - b_1y \\ y' = b_1x + b_0y \\ u' = \qquad\qquad b_0u + b_1v \qquad (b_0^2+b_1^2=1). \\ v' = \qquad\qquad -b_1u + b_0v \end{cases}$$

That is, a transformation of $U(2)$ decomposes into (4) and (7) with respect to a fixed oriented orthogonal frame.

2° (see §1). If $\mathfrak{g}_1=\mathfrak{g}_2=0$ and $\dim \mathfrak{g}^*=3$, then $\pi_1(\mathfrak{g}^*)=\mathfrak{S}_1$ and $\pi_2(\mathfrak{g}^*)=\mathfrak{S}_2$. In this case, we can consider that the bases of the real Lie algebra $\mathfrak{g}$ are given by

$$X_1-(lY_1+mY_2+nY_3),\ X_2-(l'Y_1+m'Y_2+n'Y_3),$$
$$X_3-(l''Y_1+m''Y_2+n''Y_3),$$

where X_1, X_2, X_3 and Y_1, Y_2, Y_3 are bases of the Lie algebras of (4) and (5) respectively. Furthermore, we can consider that the X's and Y's are so chosen that

$$[X_1X_2]=X_3,\ [X_2X_3]=X_1,\ [X_3X_1]=X_2;\ [Y_1Y_2]=-Y_3,$$
$$[Y_2Y_3]=-Y_1,\ [Y_3Y_1]=-Y_2.$$

Hence we know that the matrix

$$(8)\qquad \begin{pmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{pmatrix}$$

is an orthogonal matrix and the determinant is equal to $+1$. In this case, among the constants of (6) there are relations such that

$$\begin{cases} -\lambda_1' = l\lambda_1 + l'\mu_1 + l''\nu_1 \\ -\mu_1' = m\lambda_1 + m'\mu_1 + m''\nu_1 \\ -\nu_1' = n\lambda_1 + n'\mu_1 + n''\nu_1 \ . \end{cases}$$

Since one of the characteristic roots of (8) is equal to + 1, there exists a real vector (x_0, y_0, z_0) such that

$$\begin{cases} (l-1)x_0 + my_0 + nz_0 = 0 \\ l'x_0 + (m'-1)y_0 + n'z_0 = 0 \\ l''x_0 + m''y_0 + (n''-1)z_0 = 0 . \end{cases}$$

Consequently, in the 4-dimensional Euclidean space E^4, the real vector $(0, x_0, y_0, z_0)$ is invariant under G, taking account of (6).

3°. a) $1 \times SO(3)$ *is not a subgroup of* $U(2)$. With respect to a suitable orthogonal coordinate system, a transformation of $G=1\times SO(3)$ in a neighborhood of the identity is given by exp σ, where σ is of the form

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & d & e \\ 0 & -d & 0 & f \\ 0 & -e & -f & 0 \end{pmatrix} .$$

If G is a subgroup of $U(2)$, then it leaves invariant a matrix A such that $A^2 = -1$. According to Lemma 2.1, we have $A = \alpha I_1 + \beta J_1 + \gamma K_1$ or $A = \alpha' I_2 + \beta' J_2 + \gamma' K_2$, for example, $A = \alpha I_1 + \beta J_1 + \gamma K_1$ $(\alpha^2 + \beta^2 + \gamma^2 = 1)$. From $\sigma A - A\sigma = 0$ and making use of (2.8), we see that G is of dimension 1 or 0, which is impossible.

b) $U(2) \supset SO(2)\times SO(2)$, *but* $SU(2) \not\supset SO(2)\times SO(2)$. We remark that if $G \subseteqq SU(2)$, then G leaves invariant all I_1, J_1, K_1 or all I_2, J_2, K_2. Then, with respect to a suitable orthogonal coordinate system, a transformation of $G = SO(2)\times SO(2)$ in a neighborhood of the identity is given by exp σ, where σ is given in § 5. We know that $\sigma I_1 - I_1\sigma = 0$ and $\sigma I_2 - I_2\sigma = 0$, hence $G \subset U(2)$. However since

$$\sigma J_1 - J_1\sigma = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & (\lambda-\mu) \\ & & -(\lambda-v) & 0 \\ \hline 0 & (\lambda-\mu) & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -(\lambda-\mu) & 0 & & \end{array}\right) ,$$

$$\sigma J_2 - J_2\sigma = \left(\begin{array}{cc|cc} & & 0 & -(\lambda+\mu) \\ \multicolumn{2}{c|}{0} & -(\lambda+\mu) & 0 \\ \hline 0 & (\lambda+\mu) & & \\ (\lambda+\mu) & 0 & \multicolumn{2}{c}{0} \end{array}\right)$$

we know that $G \not\subset SU(2)$. Moreover, if we consider the case $\mu = \lambda$, $\mu = k\lambda$ $(k \neq \pm 1)$, $\mu = 0$, respectively, then we see that $SU(2) \supset SO(2) \underline{\times} SO(2)$, $SU(2) \not\supset SO(2) \underline{\times} SO(2)$, and $SU(2) \not\supset 1 \times SO(2)$.

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