A VERSION OF THE CENTRAL LIMIT THEOREM FOR TRIGONOMETRIC SERIES

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1. The central limit problem in the theory of probability is to determine the conditions under which the distribution functions of sums of random variables should converge [2]. In the case of independent variables the problem was solved completely. The studies on this problem for dependent variables are separated into two ways. One is to specify the dependency of variables by their conditional probabilities and one of the best known result is due to S. Bernstein [1]. The other is to specify the functional form of the variables and in this direction R. Salem and A. Zygmund proved the central limit theorem for lacunary trigonometric series [3].

THEOREM OF SALEM AND ZYGMUND. *Let*

$$L_N(t) = \sum_{k=1}^{N} c_k \cos 2\pi m_k(t+\phi_k), \quad m_{k+1}/m_k \geqq q > 1,$$

where $\{c_k\}$ *and* $\{\phi_k\}$ *be arbitrary sequences of real numbers for which*

$$\| L_N \| = \left(\frac{1}{2} \sum_{k=1}^{N} c_k^2 \right)^{1/2} \to +\infty \quad \text{and} \quad c_N = o(\| L_N \|), \quad \text{as} \quad N \to +\infty.$$

Then we have, for any set $E \subset [0,1]$ *of positive measure and any real number* ω,

$$\lim_{N\to\infty} \frac{1}{|E|} \left| \{t;\ t \in E,\ L_N(t)/\| L_N \| \leqq \omega\} \right| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega} \exp\left(-\frac{u^2}{2}\right) du.^{*)}$$

This theorem shows that the asymptotic behavior of a lacunary trigonometric series resembles that of independent random variables.

The purpose of the present note is to prove a version of the central limit theorem for trigonometric series not necessarily lacunary. Throughout this

*) For a measurable set E, $|E|$ denotes its Lebesgue measure.

note we set

$$(1.1)\qquad S_N(t)=\sum_{k=1}^{N} a_k\cos 2\pi k(t+\alpha_k)\quad\text{and}\quad A_N=\left(\frac{1}{2}\sum_{k=1}^{N}a_k^2\right)^{1/2},$$

and assume that

$$(1.2)\qquad A_N\to+\infty,\quad \text{as}\quad N\to+\infty.$$

In §§2-6 we prove the following

THEOREM 1. *Let $\{n_k\}$ be a lacunary sequence of positive integers, that is,*

$$(1.3)\qquad n_{k+1}/n_k\geqq q>1,$$

and let us put

$$(1.4)\qquad \Delta_1(t)=S_{n_1}(t)\quad\textit{and}\quad \Delta_k(t)=S_{n_k}(t)-S_{n_{k-1}}(t),\quad \textit{for}\quad k>1.$$

If

$$(1.5)\qquad \sup_t|\Delta_k(t)|=o(A_{n_k}),\quad \textit{as}\quad k\to+\infty,$$

and, for some function $g(t)$

$$(1.6)\qquad \lim_{k\to\infty}\int_0^1\left|\frac{1}{A_{n_k}^2}\sum_{m=1}^{k}\{\Delta_m^2(t)+2\Delta_m(t)\Delta_{m+1}(t)\}-g(t)\right|dt=0,$$

then the function $g(t)$ is non-negative and we have, for any set $E\subset[0,1]$ of positive measure and any real number $\omega\neq 0$,

$$(1.7)\qquad \lim_{N\to\infty}\frac{1}{|E|}\,|\{t\,;\,t\in E,S_N(t)/A_N\leqq\omega\}|=\frac{1}{\sqrt{2\pi}|E|}\int_E dt\int_{-\infty}^{\omega/\sqrt{g(t)}}\exp\left(-\frac{u^2}{2}\right)du,$$

where $\omega/0$ denotes $+\infty$ or $-\infty$ according as $\omega>0$ or $\omega<0$.

If $S_N(t)$ is lacunary, then our theorem is that of Salem and Zygmund. In [4] we proved this theorem under somewhat restricted conditions.

For any given lacunary sequence $\{n_k\}$, we can construct a trigonometric series $S_N(t)$ such that $\lim_{k\to\infty}A_{n_k}^{-2}\sum_{m=1}^{k}\{\Delta_{m+1}(t)\Delta_m(t)\}$ exists for all t and does not vanish identically. Therefore, the above theorem shows that the sequence of functions $\{\Delta_k(t)\}$ is asymptotically one-dependent, that is, $n-k>1$ implies the asymptotic independence of two sets of functions

$$\{\Delta_1(t), \Delta_2(t), \cdots, \Delta_k(t)\} \quad \text{and} \quad \{\Delta_n(t), \Delta_{n+1}(t), \cdots, \Delta_m(t)\}.$$

Let $F_E(\omega)$ be the distribution function on the right hand side of (1.7), then we have, for any real number λ,

$$\int_{-\infty}^{\infty} e^{i\lambda\omega}\, dF_E(\omega) = \frac{1}{|E|}\int_E \exp\left\{-\frac{\lambda^2}{2}\, g(t)\right\} dt.$$

Therefore, for the proof of our theorem, it is sufficient to show that for any fixed real number λ

$$\lim_{N\to\infty}\int_E \exp\left\{\frac{i\lambda}{A_N}\, S_N(t)\right\} dt = \int_E \exp\left\{\frac{-\lambda^2}{2}\, g(t)\right\} dt. \tag{1.8}$$

In the same way we can prove a corresponding limit theorem for the remainder terms of the Fourier series of a squarely integrable function.

2. From now on we assume that the conditions of Theorem 1 are satisfied. Let us put

$$U_l(t) = \sum_{k=(l-1)r+1}^{lr} \Delta_k(t) = S_{n_{lr}}(t) - S_{n_{(l-1)r}}(t), \tag{2.1}$$

$$B_l^2 = \int_0^1 U_l^2(t)\, dt \quad \text{and} \quad C_N^2 = \sum_{l=1}^{N} B_l^2 = A_{n_{Nr}}^2, \tag{2.2}$$

where r is a fixed positive integer satisfying

$$q^r(1-q^{-1}) > 6. \tag{2.3}$$

Then we have, by the conditions of the theorem,

$$C_N \to +\infty, \quad B_N = o(C_N), \tag{2.4}$$

$$\sup_t |U_N(t)| \leqq \sup_t \sum_{k=(N-1)r+1}^{Nr} |\Delta_k(t)| = o(C_N), \quad \text{as} \quad N\to+\infty, \tag{2.5}$$

and, for any m satisfying $n_{(N-1)r} < m \leqq n_{Nr}$,

$$(2.\ 6)\qquad \int_0^1 \left| S_m(t) - \sum_{l=1}^{N} U_l(t) \right|^2 dt \leqq B_N^2 = o(C_N^2), \qquad \text{as} \quad N \to +\infty ,$$

LEMMA 1. *We have*

$$\lim_{N\to\infty} \int_0^1 \left| \frac{1}{C_N^2} \sum_{l=1}^{N} \{U_l^2(t) + 2U_l(t)U_{l+1}(t)\} - g(t) \right| dt = 0 .$$

PROOF. We have

$$(2.\ 7)\qquad U_l^2(t) + 2U_l(t)U_{l+1}(t) = \sum_{k=(l-1)r+1}^{lr} \{\Delta_k^2(t) + 2\Delta_k(t)\Delta_{k+1}(t)\} + 2W_l(t) ,$$

where $W_l(t) = X_l(t) + Y_l(t) + Z_l(t)$ and

$$(2.\ 8)\qquad \begin{cases} X_l(t) = \displaystyle\sum_{k=(l-1)r+3}^{lr} \Delta_k(t) \sum_{j=(l-1)r+1}^{k-2} \Delta_j(t) , \\ Y_l(t) = \displaystyle\sum_{k=lr+2}^{(l+1)r} \Delta_k(t) \sum_{j=(l-1)r+1}^{lr} \Delta_j(t) , \\ Z_l(t) = \Delta_{lr+1}(t) \displaystyle\sum_{j=(l-1)r+1}^{lr-1} \Delta_j(t) . \end{cases}$$

Let w_l(or w_l') denotes the maximum (or minimum) frequency of terms of a trigonometric polynomial $X_l(t)$, then we have, by (1. 3) and (2. 1),

$$2n_{lr} > w_l \geqq w_l' \geqq \operatorname{Min}\{n_k - n_{k-1} ;\ (l-1)r + 2 \leqq k \leqq lr\} > n_{(l-1)r+2}(1-q^{-1}) .$$

From (1. 3) and (2. 3), it is easily seen that

$$w_{l+2}'/w_l > q^{r+2}(1-q^{-1})2^{-1} > 1 .$$

This implies that if $|k-l| \geqq 2$, then $X_l(t)$ and $X_k(t)$ are orthogonal. Hence, by the Schwarz inequality, we have

$$(2.\ 9)\qquad \int_0^1 \left\{ \sum_{l=1}^{N} X_l(t) \right\}^2 dt \leqq 2 \sum_{l=1}^{N} \int_0^1 X_l^2(t)dt = 2 \sum_{l=1}^{N} \int_0^1 \left\{ \sum_{k=(l-1)r+3}^{lr} \Delta_k(t) \sum_{j=(l-1)r+1}^{k-2} \Delta_j(t) \right\}^2 dt .$$

In the same way we can see that $\Delta_k(t) \sum_{j=(l-1)r+1}^{k-2} \Delta_j(t)$ and $\Delta_m(t) \sum_{j=(l-1)r+1}^{m-2} \Delta_j(t)$ are orthogonal if $|m-k|>c(q)$, where $c(q)$ is a constant depending only on q. Hence we have, for some constant K depending only on q,

$$\int_0^1 X_l^2(t)dt \leqq K \sum_{k=(l-1)r+3}^{lr} \int_0^1 \Delta_k^2(t) \left\{\sum_{j=(l-1)r+1}^{k-2} \Delta_j(t)\right\}^2 dt$$

$$\leqq K\left(\sup_t \sum_{j=(l-1)r+1}^{lr} |\Delta_j(t)|\right)^2 \sum_{k=(l-1)r+3}^{lr} \int_0^1 \Delta_k^2(t)\,dt = o(C_l^2)\,B_l^2\,, \quad \text{as} \quad l \to +\infty\,.$$

By (2. 9) and the above relation, we have

$$\int_0^1 \left\{\sum_{l=1}^{N} X_l(t)\right\}^2 dt = o(C_N^4)\,, \quad \text{as} \quad N \to +\infty.$$

In the same way, we can obtain

$$\int_0^1 \left\{\sum_{l=1}^{N} Y_l(t)\right\}^2 dt = o(C_N^4) \quad \text{and} \quad \int_0^1 \left\{\sum_{l=1}^{N} Z_l(t)\right\}^2 dt = o(C_N^4), \quad \text{as} \quad N \to +\infty\,.$$

By the above relations, (2. 7) and (1. 6), we can prove the lemma.

3. From (2.5) it is seen that there exists a non-decreasing sequence of positive integers $\{\phi(k)\}$ such that

(3. 1) $\phi(N) \to +\infty$ and $\phi(N) \operatorname*{Max}_{l\leqq N} \sup_t |U_l(t)| = o(C_N)$, as $N \to +\infty$.

Putting $\psi(k) = \sum_{m=1}^{k} \phi(m)$, we can take a sequence of integers $\{p(k)\}$ satisfying the following conditions;

(3 .2)
$$\begin{cases} p(0) = 1, \quad \text{and} \quad \psi(2k-1) < p(k) \leqq \psi(2k) \\ \text{and} \quad B^2_{p(k)-1} \leqq \{\phi(k)\}^{-1} \sum_{l=\psi(2k-1)}^{\psi(2k)-1} B_l^2\,, \quad \text{for} \quad k \geqq 1\,. \end{cases}$$

Since $\psi(2k-1) < p(k)$ and $\phi(k) \to +\infty$, as $k \to +\infty$, we have

(3. 3) $$\sum_{k=1}^{m} B^2_{p(k)-1} \leqq \sum_{k=1}^{m-1} \{\phi(k)\}^{-1} \sum_{l=\psi(2k-1)}^{\psi(2k)-1} B_l^2 + B^2_{p(m)-1} = o(C^2_{p(m)})\,, \quad \text{as} \quad m \to +\infty\,.$$

If we put

$$D_m = C_{p(m)} \quad \text{and} \quad T_k(t) = \sum_{l=p(m-1)}^{p(m)-2} U_l(t)\,,$$

then we have, by (3. 3),

$$\int_0^1 \left| \sum_{k=1}^{m} T_k(t) - \sum_{l=1}^{p(m)} U_l(t) \right|^2 dt = o(D_m^2), \quad \text{as} \quad m \to +\infty, \tag{3. 4}$$

and, by (3. 1) and (3. 2),

$$|T_k(t)| \leqq \sum_{l=p(k-1)}^{p(k)} |U_l(t)| \leqq 3\phi(2k) \operatorname*{Max}_{l \leqq p(k)} \sup_t |U_l(t)| \tag{3. 5}$$

$$\leqq 3\phi(p(k)) \operatorname*{Max}_{l \leqq p(k)} \sup_t |U_l(t)| = o(D_k)^{*)}, \quad \text{as} \quad k \to +\infty.$$

By (2. 6), (3. 4) and (3 .5) we have, for any k satisfying $n_{rp(m-1)} < k \leqq n_{rp(m)}$,

$$\int_0^1 \left| S_k(t) - \sum_{k=1}^{m} T_k(t) \right|^2 dt = o(D_m^2), \quad \text{as} \quad m \to +\infty.$$

Hence by (1. 8), to prove our theorem it is sufficient to show that for any fixed real number λ,

$$\lim_{m\to\infty} \int_E \exp\left\{ \frac{i\lambda}{D_m} \sum_{k=1}^{m} T_k(t) \right\} dt = \int_E \exp\left\{ -\frac{\lambda^2}{2} g(t) \right\} dt. \tag{3. 6}$$

LEMMA 2. *We have*

$$\lim_{N\to\infty} \int_0^1 \left| \frac{1}{D_N^2} \sum_{k=1}^{N} T_k^2(t) - g(t) \right| dt = 0.$$

REMAEK. By this lemma we can see that $g(t)$ is non-negative.

PROOF. We have

$$T_k^2(t) = \sum_{l=p(k-1)}^{p(k)-2} \{U_l^2(t) + 2U_l(t)U_{l+1}(t)\} - 2U_{p(k)-2}(t)U_{p(k)-1}(t) + 2V_k(t), \tag{3. 7}$$

where

$$V_k(t) = \sum_{l=p(k-1)+2}^{p(k)-2} U_l(t) \sum_{j=p(k-1)}^{l-2} U_j(t). \tag{3. 8}$$

Estimating the maximum and minimum frequencies of terms of a trigonometric polynomial $V_l(t)$ in the same way as those of $X_l(t)$ in Lemma 1, we can see

*) From (3. 2), it is seen that $p(k) > \psi(2k-1) \geqq (2k-1)\phi(1) \geqq (2k-1)$.

that $\{V_k(t)\}$ is orthogonal. Hence we have, by (3.8),

$$\int_0^1 \left\{\sum_{k=1}^m V_k(t)\right\}^2 dt = \sum_{k=1}^m \int_0^1 \left\{\sum_{l=p(k-1)+2}^{p(k)-2} U_l(t) \sum_{j=p(k-1)}^{l-2} U_j(t)\right\}^2 dt\,,$$

and in the same way we can see that $U_l(t)\sum_{j=p(k-1)}^{l-2} U_j(t)$ and $U_s(t)\sum_{j=p(k-1)}^{s-2} U_j(t)$ are orthogonal if $|s-l|\geqq 2$. Therefore we obtain, by the Schwarz inequality and (3.5),

$$\int_0^1 \left\{\sum_{k=1}^m V_k(t)\right\}^2 dt \leqq 2\sum_{k=1}^m \sum_{l=p(k-1)+2}^{p(k)-2} \int_0^1 U_l^2(t)\left\{\sum_{j=p(k-1)}^{l-2} U_j(t)\right\}^2 dt$$

$$\leqq 2\sum_{k=1}^m \left\{\sup_t \sum_{j=p(k-1)}^{p(k)} |U_j(t)|\right\} \sum_{l=p(k-1)+2}^{p(k)-2} \int_0^1 U_l^2(t)dt$$

$$= \sum_{k=1}^m o(D_k^2) \sum_{l=p(k-1)}^{p(k)-1} B_l^2 = o(D_m^4)\,, \quad \text{as} \quad m\to +\infty\,.$$

On the other hand we have, by (3.7),

$$(3.\ 9) \quad \sum_{k=1}^m T_k^2(t) = \sum_{l=1}^{p(m)} \{U_l^2(t)+2U_l(t)U_{l+1}(t)\} - U_{p(m)}^2(t) - 2U_{p(m)}(t)U_{p(m)+1}(t)$$

$$+ \sum_{k=1}^m V_k(t) - \sum_{k=1}^m [U_{p(k)-1}^2(t)+2U_{p(k)-1}(t)\{U_{p(k)-2}(t)+U_{p(k)}(t)\}]\,.$$

By (3.3) and (3.1), it is easily seen that

$$\int_0^1 \sum_{k=1}^m U_{p(k)-1}^2(t)\, dt = o(D_m^2)\,, \quad |U_{p(m)}^2(t)+2U_{p(m)}(t)U_{p(m)+1}(t)| = o(D_m^2),$$

and

$$\int_0^1 \left|\sum_{k=1}^m U_{p(k)-1}(t)\{U_{p(k)-2}(t) + U_{p(k)}(t)\}\right| dt$$

$$\leqq \left(\sum_{k=1}^m B_{p(k)-1}^2\right)^{1/2} \left\{\sum_{k=1}^m (B_{p(k)-2}^2 + B_{p(k)}^2)\right\}^{1/2} = o(D_m^2)\,, \quad \text{as} \quad m\to +\infty.$$

Thus by Lemma 1 and (3.9), we can prove this lemma.

4. Since $D_N \uparrow +\infty$, (3.5) implies that $\operatorname{Max}_{k\leqq N} \sup_t | T_k(t)/D_N | = o(1)$, as $N \to +\infty$. Hence without loss of generality we may assume that $\operatorname{Max}_{k\leqq N} \sup_t | T_k(t)/D_N| < 1/2$ for all N, that is,

(4. 1) $$\lim_{N\to\infty} \epsilon_N = 0 \quad \text{and} \quad 0 < \epsilon_N < 1/2 \quad \text{for all} \quad N,$$

where

(4. 1′) $$\epsilon_N = \operatorname{Max}_{k\leqq N} \sup_t | T_k(t)/D_N| .$$

If we put, for $M \geqq 1$

(4. 2) $$P_{0,N}(t, M) = 1 \text{ and } P_{k,N}(t, M) = \prod_{m=1}^{k} \left\{1 - \frac{T_m^2(t)}{MD_N^2}\right\}, \quad \text{for} \quad k \geqq 1,$$

then we have, by (4.1),

(4. 2′) $$0 \leqq 1 - P_{N,N}(t, M) = \frac{1}{MD_N^2} \sum_{k=1}^{N} T_k^2(t) P_{k-1,N}(t, M) \leqq 1 ,$$

and

(4. 2″) $$1 - P_{N,N}^2(t, M) = \sum_{k=1}^{N} \left\{ \frac{2T_k^2(t)}{MD_N^2} - \frac{T_k^4(t)}{M^2 D_N^4} \right\} P_{k-1,N}^2(t, M) .$$

Since $0 \leqq P_{k,N}(t, M) \leqq 1$, the above relations imply that

(4. 3) $$\frac{1}{D_N^2} \sum_{k=1}^{N} T_k^2(t) P_{k-1,N}^2(t, M) \leqq M ,$$

(4. 3′) $$\frac{1}{D_N^3} \sum_{k=1}^{N} | T_k(t) P_{k-1,N}(t, M)|^3 \leqq \epsilon_N M ,$$

and

(4. 3″) $$\frac{1}{MD_N^4} \sum_{k=1}^{N} T_k^4(t) P_{k-1,N}^2(t, M) \leqq \epsilon_N^2 .$$

By (4.2″) and (4.3″) it is seen that

(4. 4) $$\left| \frac{2}{D_N^2} \sum_{k=1}^{N} T_k^2(t) P_{k-1,N}^2(t, M) - M\{1 - P_{N,N}^2(t, M)\} \right| < \epsilon_N^2 .$$

On the other hand estimating $\log P_{N,N}(t,M)$, we have, by (4.1),

$$\exp\left\{\frac{-(1+\epsilon_N^2)}{MD_N^2}\sum_{k=1}^{N}T_k^2(t)\right\}\leqq P_{N,N}(t,M)\leqq\exp\left\{\frac{-1}{MD_N^2}\sum_{k=1}^{N}T_k^2(t)\right\}.$$

Hence we have, for any $\alpha>0$,

$$(4.5)\qquad |P_{N,N}^{\alpha}(t,M)-e^{-\alpha g(t)/M}|$$
$$\leqq\left|\exp\left\{\frac{-\alpha}{MD_N^2}\sum_{k=1}^{N}T_k^2(t)\right\}-\exp\left\{\frac{-\alpha(1+\epsilon_N^2)}{MD_N^2}\sum_{k=1}^{N}T_k^2(t)\right\}\right|$$
$$+\left|\exp\left\{\frac{-\alpha}{MD_N^2}\sum_{k=1}^{N}T_k^2(t)\right\}-e^{-\alpha g(t)/M}\right|$$
$$\leqq\frac{\alpha\epsilon_N^2}{MD_N^2}\sum_{k=1}^{N}T_k^2(t)+\frac{\alpha}{M}\left|\frac{1}{D_N^2}\sum_{k=1}^{N}T_k^2(t)-g(t)\right|.^{*)}$$

LEMMA 3. *We have, for any $M\geqq 1$ and real number λ,*

$$\lim_{N\to\infty}\int_0^1\left|\exp\left\{\frac{-\lambda^2}{2D_N^2}\sum_{k=1}^{N}T_k^2(t)\,P_{k-1,N}^2(t,M)\right\}-\exp\left\{\frac{\lambda^2M}{4}(e^{-2g(t)/M}-1)\right\}\right|dt=0.$$

PROOF. The integrand of the above integral is not greater than

$$\psi_N(t,M)=\lambda^2\left|\frac{-1}{2D_N^2}\sum_{k=1}^{N}T_k^2(t)P_{k-1,N}^2(t,M)-\frac{M}{4}(e^{-2g(t)/M}-1)\right|.$$

By (4.4) and (4.5), we have

$$4\psi_N(t,M)\leqq\lambda^2\epsilon_N^2+\lambda^2M|P_{N,N}^2(t,M)-e^{-2g(t)/M}|$$
$$\leqq\lambda^2\epsilon_N^2+\frac{2\lambda^2\epsilon_N^2}{D_N^2}\sum_{k=1}^{N}T_k^2(t)+2\lambda^2\left|\frac{1}{D_N^2}\sum_{k=1}^{N}T_k^2(t)-g(t)\right|.$$

Therefore, (4.1) and Lemma 2 imply that $\int_0^1\psi_N(t,M)dt=o(1)$, as $N\to+\infty$.

LEMMA 4. *For any real number λ and $M\geqq 1$, we have*

$$\int_0^1\left|\exp\left\{\frac{i\lambda}{D_N}\sum_{k=1}^{N}T_k(t)\right\}-\exp\left\{\frac{i\lambda}{D_N}\sum_{k=1}^{N}T_k(t)\,P_{k-1,N}(t,M)\right\}\right|dt\leqq|\lambda|(\delta_N+\eta_M),$$

*) From (1.6) and Lemma 2 it is seen that $g(t)\in L(0,1)$ and $g(t)\geqq 0$.

where

$$\begin{cases} \delta_N = \left\{\epsilon_N^2 + 2\int_0^1 \left|\frac{1}{D_N^2}\sum_{k=1}^N T_k^2(t) - g(t)\right| dt\right\}^{1/2}, \\ \eta_M = \left\{\int_{\{g(t)>M\}} g(t)dt + \int_0^1 g(t)(1-e^{-g(t)/M})\,dt\right\}^{1/2}. \end{cases}$$

PROOF. Since $|e^{i\alpha}-e^{i\beta}| \leqq |\alpha-\beta|$, it is sufficient to show that

$$\int_0^1 \left|\frac{1}{D_N}\sum_{k=1}^N T_k(t)\{1-P_{k-1,N}(t,M)\}\right| dt \leqq (\delta_N + \eta_M).$$

From the definitions of $T_k(t)$, (2. 3) and (1. 3) it is seen that the maximum frequency z_k and minimum frequency z_k' of terms of T_k(t) satisfy

$$z_k \geqq z_k' \quad \text{and} \quad z_{k+1}'/z_k > 6. \tag{4. 6}$$

By (4. 2) and (4. 6) the frequency x_k of any term of $P_{k,M}(t,M)$ satisfies the conditions;

$$x_k \leqq 2\sum_{l=1}^k z_l < 2z_k \sum_{l=1}^k 6^{(l-k)} \leqq 12z_k/5 < 2z_{k+1}'/5 \leqq 2z_{k+1}/5.$$

Hence it is easily seen that

$$\text{(frequencies of terms of } T_k(t)P_{k-1,N}(t,M)) \in (3z_k'/5,\ 7z_k/5). \tag{4. 7}$$

Since (4. 6) implies that the intervals $(3z_k'/5,\ 7z_k/5)$, $k \geqq 1$, are disjoint, $[T_k(t)\{1-P_{k-1,N}(t)\}]$ is othogonal and we have

$$\left(\int_0^1 \left|\frac{1}{D_N}\sum_{k=1}^N T_k(t)\{1-P_{k-1,N}(t,M)\}\right| dt\right)^2 \leqq \int_0^1 \left|\frac{1}{D_N}\sum_{k=1}^N T_k(t)\{1-P_{k-1,N}(t,M)\}\right|^2 dt$$

$$= \int_0^1 \frac{1}{D_N^2}\sum_{k=1}^N T_k^2(t)\{1-P_{k-1,N}(t,M)\}^2 dt \leqq \int_0^1 \frac{1}{D_N^2}\sum_{k=1}^N T_k^2(t)\{1-P_{N,N}(t,M)\}\,dt$$

$$\leqq \int_0^1 g(t)\{1-P_{N,N}(t,M)\}\,dt + \int_0^1 \left|g(t) - \frac{1}{D_N^2}\sum_{k=1}^N T_k^2(t)\right| dt$$

$$\leqq \int_0^1 g(t)\{1-e^{-g(t)/M}\}dt + M\int_{\{g(t)\leqq M\}} |P_{N,N}(t,M) - e^{-g(t)/M}|\,dt$$

$$+ \int_{\{g(t)>M\}} g(t)\,dt + \int_0^1 \left|g(t) - \frac{1}{D_N^2}\sum_{k=1}^N T_k^2(t)\right| dt.$$

By (4. 5) we have

$$M\int_{\{g(t)\leqq M\}}|P_{N,N}(t,M)-e^{-g(t)/M}|\,dt$$
$$\leqq \epsilon_N^2\int_0^1\frac{1}{D_N^2}\sum_{k=1}^N T_k^2(t)\,dt+\int_0^1\left|\frac{1}{D_N^2}\sum_{k=1}^N T_k^2(t)-g(t)\right|dt$$
$$\leqq \epsilon_N^2+\int_0^1\left|\frac{1}{D_N^2}\sum_{k=1}^N T_k^2(t)-g(t)\right|dt\,.$$

From the above inequalities it is seen that

$$\left(\int_0^1\left|\frac{1}{D_N}\sum_{k=1}^N T_k(t)\{1-P_{k-1,N}(t,M)\}\right|dt\right)^2\leqq(\delta_N^2+\eta_M^2)\leqq(\delta_N+\eta_M)^2\,.$$

5. In this paragraph let λ and $M, M\geqq 1$, be any fixed real numbers. We put, for these numbers λ and M,

$$(5.\ 1)\quad Q_{0,N}(t)=1,\ \text{and}\ Q_{k,N}(t)=\prod_{m=1}^k\left\{1+\frac{i\lambda}{D_N}T_m(t)P_{m-1,N}(t,M)\right\},\ \text{for}\ k\geqq 1.$$

LEMMA 5. *For any $f(t)\in L(0,1)$, we have*

$$\lim_{N\to\infty}\int_0^1 f(t)\,Q_{N,N}(t)\,dt=\int_0^1 f(t)\,dt.$$

PROOF. Since (4. 3) and (5. 1) imply that $|Q_{N,N}(t)|^2\leqq e^{\lambda^2M}$, it is sufficient to show that for any measurable set $E\subset[0,1]$

$$\lim_{N\to\infty}\int_E\{Q_{N,N}(t)-1\}dt=0\,.$$

From (5. 1) it is easily seen that

$$Q_{N,N}(t)-1=\frac{i\lambda}{D_N}\sum_{k=1}^N T_k(t)P_{k-1,N}(t,M)\,Q_{k-1,N}(t)\,.$$

By (4. 7) and (5. 1) the frequency of any term of $Q_{k,N}(t)$ is less than $\sum_{m=1}^k 7z_m/5$. Thus by (4. 7) the frequencies of terms of $T_k(t)\,P_{k-1,N}(t,M)\,Q_{k-1,N}(t)$ belong to the interval $\left[3z_k'/5 - \sum_{m=1}^{k-1} 7z_m/5,\ 7z_k/5 + \sum_{m=1}^{k-1} 7z_m/5\right]$. Since (4.6) implies that $\sum_{m=1}^{k-1} 7z_m/5 < 7z_k'/25 \leqq 7z_k/25$, the frequencies of $T_k(t)P_{k-1,N}(t, M)Q_{k-1,N}(t)$ belong to the interval $[8z_k'/25,\ 42z_k/25]$ and these intervals are disjoint. Therefore $\sum_{m=1}^{k} T_m(t)P_{m-1,N}(t, M)\ Q_{m-1,N}(t)$ is a trigonometric sum whose order is not greater than $42z_k/25$ and not less than $8z_k'/25$, If we put, for the indicator function $\chi_E(t)$ of the set E

$$\chi_E(t) \sim |E| + \sum_{k=1}^{\infty} e_k \cos 2\pi k(t+\beta_k),$$

and

$$Q_{N,N}(t) - 1 \sim \sum_{k=1}^{\infty} (p_k^{(N)} + iq_k^{(N)}) \cos 2\pi kt, \text{*)}$$

then $q_k^{(N)}$ and $p_k^{(N)}$ are zero for sufficiently large k and

$$\frac{i\lambda}{D_N} \sum_{m=1}^{k} T_m(t)P_{m-1,N}(t, M)\, Q_{m-1,N}(t) = \sum_{m=1}^{42z_k/25} (p_m^{(N)} + iq_m^{(N)}) \cos 2\pi mt\,.$$

Hence we have, by the Parseval's relation,

$$\int_E \{Q_{N,N}(t) - 1\}\, dt$$

$$= \frac{i\lambda}{D_N} \int_E \sum_{k=1}^{N'} T_k(t)P_{k-1,N}(t, M)Q_{k-1,N}(t)\, dt + \frac{1}{2} \sum_{k>N''} e_k(p_k^{(N)} + iq_k^{(N)}) \cos 2\pi k\beta_k\,,$$

where

$$N' = [\epsilon_N^{-1/2}] \quad\text{and}\quad N'' = 42z_{N'}/25\,.$$

Since $|P_{k-1,N}(t, M)Q_{k-1,N}(t, M)| \leqq |Q_{N,N}(t)| < e^{\frac{\lambda^2 M}{2}}$ and $\lim_{N\to\infty} N'' = +\infty$, we have

$$\left|\frac{1}{D_N} \sum_{k=1}^{N'} T_k(t)P_{k-1,N}(t, M)Q_{k-1,N}(t)\right| \leqq \epsilon_N^{1/2}\, |Q_{N,N}(t)| = o(1), \quad\text{as}\quad N \to +\infty,$$

*) For simplicity of writing the formulas we assume that $S_N(t)$ contains cosine terms only. Hence $\{Q_{N,N}(t) - 1\}$ contains cosine terms only. The general case follows in the same way.

and

$$\sum_{k>N''} |e_k(p_k^{(N)} + iq_k^{(N)})| \leqq \left(\sum_{k>N''} e_k^2\right)^{1/2} \left(\sum_{k=1}^{\infty} |p_k^{(N)} + iq_k^{(N)}|^2\right)^{1/2}$$

$$\leqq \left(\sum_{k>N''} e_k^2\right)^{1/2} \left(2\int_0^1 |Q_{N,N}(t)-1|^2\, dt\right)^{1/2} = o(1), \quad \text{as} \quad N \to +\infty.$$

By the above relations we can prove the lemma.

LEMMA 6. *We have, for any measurable set* $E \subset [0,1]$,

$$\lim_{N\to\infty} \int_E \exp\left\{\frac{i\lambda}{D_N} \sum_{k=1}^{N} T_k(t) P_{k-1,N}(t, M)\right\} dt$$

$$= \int_E \exp\left\{\frac{\lambda^2 M}{4}(e^{-2g(t)/M} - 1)\right\} dt\,.$$

PROOF. If we put $e^{iz} = (1+iz)e^{-\frac{z^2}{2}+A(z)}$, for a real number z, then we have $|A(z)| < |z|^3$ if $|z| < 1/2$. Therefore for any sequence of real numbers z_k, $k=1,2,\cdots,N$, $|z_k| < 1/2$, we have

$$\left|e^{i\Sigma z_k} - \prod (1+iz_k)\, e^{-\frac{1}{2}\Sigma z_k^2}\right| \leqq e^{|\Sigma A(z_k)|} - 1 < e^{\Sigma |z_k|^3} - 1\,.$$

Thus we have, by (4. 3′),

(5. 2)

$$\left|\exp\left\{\frac{i\lambda}{D_N} \sum_{k=1}^{N} T_k(t) P_{k-1,N}(t, M)\right\} - Q_{N,N}(t) \exp\left\{\frac{-\lambda^2}{2D_N^2} \sum_{k=1}^{N} T_k^2(t) P_{k-1,N}^2(t, M)\right\}\right|$$

$$\leqq \exp\{|\lambda|^3 \epsilon_N M\} - 1 = o(1)\,, \quad \text{as} \quad N \to +\infty\,.$$

Since $|Q_{N,N}(t)| \leqq e^{\lambda^2 M/2}$, we have, by Lemma 3,

$$\left|\int_E Q_{N,N}(t)\left[\exp\left\{\frac{-\lambda^2}{2D_N^2}\sum_{k=1}^{N} T_k^2(t)P_{k-1,N}^2(t, M)\right\} - \exp\left\{\frac{\lambda^2 M}{4}(e^{-2g(t)/M} - 1)\right\}\right] dt\right|$$

$$\leqq e^{\lambda^2 M} \int_0^1 \left|\exp\left\{\frac{-\lambda^2}{2D_N^2}\sum_{k=1}^{N} T_k^2(t)P_{k-1,N}^2(t, M)\right\} - \exp\left\{\frac{\lambda^2 M}{4}(e^{-2g(t)/M} - 1)\right\}\right| dt$$

$$= o(1)\,, \quad \text{as} \quad N \to +\infty\,.$$

Further we have, by Lemma 5,

$$\lim_{N\to\infty}\int_E Q_{N,N}(t)\exp\left\{\frac{\lambda^2 M}{4}(e^{-2g(t)/M}-1)\right\}dt$$
$$=\int_E \exp\left\{\frac{\lambda^2 M}{4}(e^{-2g(t)/M}-1)\right\}dt\,.$$

By (5.2) and the above two relations we can prove the lemma.

6. In this paragraph let λ denote any fixed real number and $E\subset[0,1]$ any fixed set of positive measure. Further let ϵ be an arbitrary positive number. Since $g(t)\in L(0,1)$ and $g(t)\geqq 0$, we can take a positive integer M_0 such that

$$\left|\int_E \exp\left\{\frac{\lambda^2 M_0}{4}(e^{-2g(t)/M_0}-1)\right\}dt-\int_E\exp\left\{\frac{-\lambda^2}{2}g(t)\right\}dt\right|<\epsilon/4\,,$$

and

$$|\lambda|\left\{\int_{\{g(t)>M_0\}} g(t)+dt\ \int_0^1 g(t)\,(1-e^{-g(t)/M_0})\,dt\right\}^{1/2}<\epsilon/4\,.$$

By Lemma 6 an integer $N(M_0)$ exists such that $N>N(M_0)$ implies that

$$\left|\int_E \exp\left\{\frac{i\lambda}{D_N}\sum_{k=1}^{N}T_k(t)\,P_{k-1,N}(t,M_0)\right\}-\int_E\exp\left\{\frac{\lambda^2 M_0}{4}(e^{-2g(t)/M_0}-1)\right\}dt\right|<\epsilon/4\,,$$

and by (4.1) and Lemma 2 an integer N_0 exists such that $N>N_0$ implies that

$$|\lambda|\left\{\epsilon_N^2+2\int_0^1\left|\frac{1}{D_N^2}\sum_{k=1}^{N}T_k^2(t)-g(t)\right|dt\right\}^{1/2}<\epsilon/4\,.$$

Therefore, by Lemma 4, $N>\mathrm{Max}(N(M_0),\,N_0)$ implies that

$$\left|\int_E\exp\left\{\frac{i\lambda}{D_N}\sum_{k=1}^{N}T_k(t)\right\}dt-\int_E\exp\left\{-\frac{\lambda^2}{2}g(t)\right\}dt\right|<\epsilon\,.$$

This is (3.6) and our theorem is proved.

7. In this paragraph let $f(t)$ be a function of $L_2(0,1)$ and $R_N(t)$ the N-th remainder of the Fourier series of $f(t)$. Further let us put

$$\left\{\int_0^1 R_N^2(t)\,dt\right\}^{1/2} = E_N \quad \text{and} \quad \Delta_k(t) = R_{n_k}(t) - R_{n_{k+1}}(t)\,,$$

where $\{n_k\}$ is a lacunary sequence of positive integers. Then we can prove the following

THEOREM 2. *Suppose that*

$$\sup_t |\Delta_k(t)| = o(E_{n_k})\,, \quad \text{as} \quad k \to +\infty\,,$$

and, for some function $g(t)$

$$\lim_{k\to\infty}\int_0^1 \left|\frac{1}{E_{n_k}^2}\sum_{m=k}^{\infty}\{\Delta_m^2(t) + 2\Delta_m(t)\Delta_{m+1}(t)\} - g(t)\right| dt = 0\,,$$

then the function $g(t)$ *is non-negative and we have, for any set* $E \subset [0,1]$ *of positive measure and any real number* $\omega \neq 0$

$$\lim_{N\to\infty}\frac{1}{|E|}\,|\{t\,;\, t\in E,\ R_N(t)/E_N \leqq \omega\}| = \frac{1}{\sqrt{2\pi}\,|E|}\int_E dt\int_{-\infty}^{\omega/\sqrt{g(t)}} \exp\left(-\frac{u^2}{2}\right)du\,,$$

where $\omega/0$ *denotes* $+\infty$ *or* $-\infty$ *according as* $\omega>0$ *or* $\omega<0$.

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