SUMMABILITY OF DOUBLE FOURIER SERIES

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(Received November 8, 1964 and in revised form, February 24, 1965)

1. Let \( f(x,y) \) be integrable in the square \(|x| \leq \pi, |y| \leq \pi\), let its Fourier series be

\[
\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{m,n} e^{i(mx + ny)},
\]

where

\[
c_{m,n} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) e^{-i(mx + ny)} \, dx \, dy,
\]

and let its \((m,n)\)-th partial sums be

\[
s_{m,n}(x,y) = \sum_{k=-m}^{m} \sum_{l=-n}^{n} c_{k,l} e^{i(kx + ly)}.
\]


\[
[s_{n,n}(x,y)]
\]

and obtained the following Theorems A and B.

**THEOREM A.** If \( f(x,y) \in L^p \) \((p > 1)\), then

\[
\lim_{n \to \infty} H_n(x,y) = 0 \quad \text{almost everywhere},
\]

and

\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \{H_n(x,y)\}^p \, dx \, dy \leq A_p \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x,y)|^p \, dx \, dy,
\]

where

\[
H_n(x,y) = \left( \frac{1}{n+1} \sum_{k=0}^{n} |s_{k,n}(x,y) - f(x,y)|^2 \right)^{1/2}.
\]
and

\[ H^a(x, y) = \sup_n H_n(x, y). \]

**Theorem B.** If \( f(x, y) \in L \), then

\[ \lim_{n \to \infty} \sigma_n(x, y) = f(x, y) \text{ almost everywhere}, \]

and for any \( t > 0 \)

\[ \left| \{ \sigma^a(x, y) > t \} \right| \leq \frac{A}{t} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x, y)| \, dx \, dy, \]

where

\[ \sigma_n(x, y) = \frac{1}{n+1} \sum_{k=0}^{n} s_k(x, y), \]

and

\[ \sigma^a(x, y) = \sup_n |\sigma_n(x, y)|. \]

When \( \varphi(x, y) \in L \) has a Fourier series of power series type, i.e.

\[ \varphi(x, y) \sim \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_{k, l} e^{i(kx + ly)}, \]

we denote its partial sums by

\[ t_{m, n}(x, y) = \sum_{k=0}^{m} \sum_{l=0}^{n} c_{k, l} e^{i(kx + ly)}, \]

its \((C, 1, 1)\) means by

\[ \tau_{m, n}(x, y) = \frac{1}{(m+1)(n+1)} \sum_{k=0}^{m} \sum_{l=0}^{n} t_{k, l}(x, y), \]

and we put

\[ T^a(x, y) = \left( \sum_{n=0}^{\infty} \frac{\left| t_{n, n}(x, y) - \tau_{n, n}(x, y) \right|^r}{n+1} \right)^{1/s}. \]
Then we have the following theorems.

**THEOREM 1.** If \( \varphi(x,y) \log^+ |\varphi(x,y)| \in L \) and its Fourier series is of power series type, then we have for \( r \geq 2 \)

\[
(i) \quad \left( \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \{T^*(x,y)\}^s \, dx \, dy \right)^{1/\mu} \leq A_{\mu,r} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\varphi(x,y)| \log^+ |\varphi(x,y)| \, dx \, dy + A_{r,r}, \quad 0 < \mu < 1.
\]

\[
(ii) \quad \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \{T^*(x,y)\} \, dx \, dy \leq B_r \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\varphi(x,y)| (\log^+ |\varphi(x,y)|)^s \, dx \, dy + B_r.
\]

**THEOREM 2.** If \( f(x,y) \in L \), then we have

\[
\lim_{n \to \infty} \gamma_n(x,y) = f(x,y) \text{ almost everywhere},
\]

and

\[
|\{\gamma^*(x,y) > t\}| \leq \frac{A}{t} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x,y)| \, dx \, dy,
\]

where

\[
p_n = [n^s], \quad 0 < k \leq 1,
\]

\[
\gamma_n(x,y) = \frac{1}{n+1} \sum_{m=0}^{n} s_{p_n,m}(x,y),
\]

and

\[
\gamma^*(x,y) = \sup_n |\gamma_n(x,y)|.
\]

This theorem is due to G. Maruyama [4] in one variable case.

2. To prove Theorem 1, we need several lemmas.

**LEMMA 1.** Let \( f_n(x) \in L \), and let its conjugate function be \( \tilde{f}_n(x) \). Then we have for \( r > 1 \)
LEMMA 2. Let $f_n(x) \in L$, and let the $k_n$-th partial sum of its Fourier series be $s_{n,k_n}(x)$. Then we have for $r > 1$,

(i) \[ \int_{-\pi}^{\pi} \left( \sum_n |s_{n,k_n}(x)|^r \right)^{\mu/r} \, dx \]
\[ \leq A_{r,\mu} \left( \int_{-\pi}^{\pi} \left( \sum_n |f_n(x)|^r \right)^{1/r} \, dx \right)^{\mu}, \quad 0 < \mu < 1, \]

(ii) \[ \int_{-\pi}^{\pi} \left( \sum_n |s_{n,k_n}(x)|^r \right)^{1/r} \, dx \]
\[ \leq A_r \int_{-\pi}^{\pi} \left( \sum_n |f_n(x)|^r \right)^{1/r} \log^r \left( \sum_n |f_n(x)|^r \right)^{1/r} \, dx + A_r. \]

This is derived from Lemma 1 in the usual way.

LEMMA 3. Let $\phi(z) = \sum_{n=0}^{\infty} c_n z^n$, $z = \rho e^{ix}$, be regular in the interior of a unit circle. If we put

\[ g_r(x) = \left( \int_0^1 (1-\rho)^{r-1} |\phi(z)|^r \, d\rho \right)^{1/r}, \]

then we have for $r \geq 2$

\[ \int_{-\pi}^{\pi} [g_r(x)]^p \, dx \leq A_p \int_{-\pi}^{\pi} |\phi(e^{ix})|^p \, dx, \quad p > 1. \]
This is due to G. Sunouchi [6].

LEMMA 4. Let \( \varphi(x) \in L \) have a Fourier series of power series type, i.e.

\[
\varphi(x) \sim \sum_{n=0}^{\infty} c_n e^{inx}.
\]

We denote the partial sums and \((C, 1)\) means of \((3)\) by \(t_n(x)\) and \(\tau_n(x)\), respectively. If we put

\[
T^k(x) = \left( \sum_{n=0}^{\infty} \frac{|t_n(x) - \tau_n(x)|^r}{n+1} \right)^{1/r},
\]

then for \(r \geq 2\), we have

\[
\text{(i) } \left\{ \int_{-\pi}^{\pi} (T^k(x))^\mu \, dx \right\}^{1/\mu} \leq A_r \int_{-\pi}^{\pi} |\varphi(x)| \, dx
\]

\[
\text{(ii) } \int_{-\pi}^{\pi} T^k(x) \, dx \leq B_r \int_{-\pi}^{\pi} |\varphi(x)| \log^+ |\varphi(x)| \, dx + B_r.
\]

This lemma is proved by the same way as in A. Zygmund [7; p. 237-239] with exponent \(r \geq 2\), using Lemmas 2 and 3.

Now after J. Marcinkiewicz [1], we can prove Theorem 1. We have

\[
t_{n,n}(x,y) - \tau_{n,n}(x,y) = t^{(1)}_{n,n}(\varphi) - \tau^{(1)}_{n,n}(\varphi)
\]

\[
+ \tau^{(2)}_{n}(\varphi) - \tau^{(2)}_{n}(\varphi)
\]

\[
= I_n(x,y) + J_n(x,y),
\]

where, in general, \(t^{(1)}(g)\) or \(\tau^{(1)}(g)\) means the partial sums or \((C, 1)\) means of Fourier series of \(g(x,y)\) considered as a function of \(x\) only for any fixed \(y\), respectively. \(t^{(2)}(g)\) or \(\tau^{(2)}(g)\) means those of \(y\) only for any fixed \(x\), respectively. Then we have by Lemma 2(i) and Lemma 4(ii)

\[
\left( \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left\{ \sum_{k=0}^{\infty} \frac{|I_n|^r}{n+1} \right\} \, dx \, dy \right)^{1/\mu}
\]

\[
\leq (2\pi)^{1-\mu} \int_{-\pi}^{\pi} \left\{ A_r^{1/r} \int_{-\pi}^{\pi} \left( \sum_{k=0}^{\infty} |t^{(2)}(\varphi) - \tau^{(2)}(\varphi)|^r / (k+1) \right)^{1/r} \, dx \right\} \, dy
\]
Concerning $J_n$, we have a similar calculation.

The proof of Theorem 1, (ii) is similar to that of above and we may omit it.

3. We now prove Theorem 2. We may suppose that $0 < k < 1$, because that the case $k = 1$ of Theorem 2 was proved by G. Grünwald [2]. We have

\( \gamma_n(x, y) = \frac{1}{n+1} \sum_{k=0}^{n} s_{n,v}(x, y) \)

where

\[ D_n(u, v) = \sum_{k=0}^{n} \sin \left( p_k + \frac{1}{2} \right) u \sin \left( p_k + \frac{1}{2} \right) v. \]

We consider two functions of a variable $s$ such that

\( g_i(s) = p_i + \frac{1}{2} \) for \( i \leq s < i + 1, \)

\( g_i(s) = q_i + 1 + (s-i-1)(q_{i+1} - q_i) \) for \( i \leq s < i + 1, \)

where

\( q_i = i^k, \quad 0 < k < 1. \)

Then we have

\[ D_n(u, v) = \int_{0}^{q_{n+1}} \sin(g_1(s)u) \sin(g_1(s)v) \, ds. \]

If we put

\[ D_n(u, v) = \int_{0}^{q_{n+1}} \sin(g_2(s)u) \sin(g_2(s)v) \, ds, \]
\begin{align*}
\gamma_n(x, y) &= \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + u, y + v) \left[ \frac{\tilde{D}_n(u, v)}{4(n+1) \sin \frac{u}{2} \sin \frac{v}{2}} \right] du \, dv \\
&\quad + \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + u, y + v) \left[ \frac{D_n(u, v) - \tilde{D}_n(u, v)}{4(n+1) \sin \frac{u}{2} \sin \frac{v}{2}} \right] du \, dv \\
&= \gamma_n^{(a)}(x, y) + \gamma_n^{(b)}(x, y).
\end{align*}

To estimate the last expressions we need the following lemma and some calculations.

**Lemma 5.** If \( f(x, y) \in L \), then we have the inequalities

\[ \sigma(x, y) \leq A(f^*(x, y) + f^{**}(x, y)), \]

and

\[ |\{f^*(x, y) > t\}| \leq \frac{A}{t} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x, y)| \, dx \, dy, \]

\[ |\{f^{**}(x, y) > t\}| \leq \frac{A}{t} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x, y)| \, dx \, dy, \]

where

\[ f^*(x, y) = \sup_t \{f^*_t(x, y) \, 2^{-\lfloor t \rfloor}\}, \quad s = 0, \pm 1, \pm 2, \cdots \]

\[ f^{**}(x, y) = \sup_t \{f^{**}_t(x, y) \, 2^{-\lfloor t \rfloor}\}, \quad s = 0, \pm 1, \pm 2, \cdots \]

and

\[ f^*_*(x, y) = \sup_{h > 0} \frac{1}{4\pi h^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x + u, y + v)| \, du \, dv, \]

\[ f^{**}_*(x, y) = \sup_{h > 0} \frac{1}{4\pi h^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x + u, y + v)| \, du \, dv. \]

The proof of this lemma is found in G. Grünwald [2].
Now we estimate $\gamma_n^{(3)}(x, y)$ in (7) first. We have

$$D_n(u, v) - \widehat{D}_n(u, v) = \int_0^{n+1} \sin (g_i(s) u) \sin (g_i(s) v) - \sin (g_i(s) u) \sin (g_i(s) v) \, ds$$

$$= - \int_0^{n+1} \{g_i(s) - g_i(s)\} \{u \cos (\xi(s) u) \sin (\xi(s) v) + v \sin (\xi(s) u) \cos (\xi(s) v)\} \, ds,$$

where

$$g_i(s) < \xi(s) < g_i(s),$$

and from (7) we have

$$|\gamma_n^{(3)}(x, y)| \leq \frac{A}{n+1} \int_0^{n+1} |g_i(s) - g_i(s)| \, ds \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x+u, y+v)| \times|u \cos (\xi(s) u) \sin (\xi(s) v) + v \sin (\xi(s) u) \cos (\xi(s) v)| \, du \, dv$$

$$\leq \frac{A'}{n+1} \int_0^{n+1} f_2^*(x, y) \, ds$$

(8)

$$\leq A'' f_2^*(x, y),$$

since $g_i(s) - g_i(s)$ is bounded from the definition (6).

Finally, to estimate $\gamma_n^{(3)}(x, y)$, we proceed as follows. From (6) and (7) we have

$$\widehat{D}_n(u, v) = \frac{1}{2} \int_0^{n+1} \cos \{g_i(s)(u-v)\} - \cos \{g_i(s)(u+v)\} \, ds$$

$$= \frac{1}{2} \sum_{i=0}^{n} \int_{-\pi}^{\pi} \{\cos (q_{i+1}+1+(s-i-1)(q_{i+1}-q_i))(u-v)$$

$$- \cos (q_{i+1}+1+(s-i-1)(q_{i+1}-q_i))(u+v)\} \, ds$$

$$= \frac{1}{2} \sum_{i=0}^{n} \left\{ \frac{\sin (q_{i+1}+1)(u-v) - \sin (q_i+1)(u-v)}{(q_{i+1}-q_i)(u-v)}$$

$$- \frac{\sin (q_{i+1}+1)(u+v) - \sin (q_i+1)(u+v)}{(q_{i+1}-q_i)(u+v)} \right\}$$
by partial summation

\[
\begin{align*}
&= \frac{1}{2} \sum_{i=0}^{n-1} \sin (q_{i+1} + 1)(u-v) - \sin (q_{i} + 1)(u-v) \cdot \frac{1}{u-v} \\
&\quad - \frac{1}{2} \sum_{i=0}^{n-1} \sin (q_{i+1} + 1)(u+v) - \sin (q_{i} + 1)(u+v) \cdot \frac{1}{u+v} \\
&\quad + \frac{1}{2} \frac{\sin (q_{n+1} + 1)(u-v)}{u-v} \frac{1}{q_{n+1} - q_{n}} - \frac{1}{2} \frac{\sin (q_{n+1} + 1)(u+v)}{u+v} \frac{1}{q_{n+1} - q_{n}},
\end{align*}
\]

and we have

\[
\begin{align*}
\widetilde{D}_{n}(u, v) &= \frac{1}{2} \sum_{i=0}^{n-1} \left\{ \frac{\sin (q_{i+1} + 1)(u-v)}{u-v} - \frac{\sin (q_{i+1} + 1)(u+v)}{u+v} \right\} \Delta \left( \frac{1}{q_{i+1} - q_{i}} \right) \\
(9) &\quad - \frac{1}{2} \left\{ \frac{\sin (q_{0} + 1)(u-v)}{u-v} - \frac{\sin (q_{0} + 1)(u+v)}{u+v} \right\} \sum_{i=0}^{n-1} \Delta \left( \frac{1}{q_{i+1} - q_{i}} \right) \\
&\quad + \frac{1}{2} \left\{ \frac{\sin (q_{n+1} + 1)(u-v)}{u-v} - \frac{\sin (q_{n+1} + 1)(u+v)}{u+v} \right\} \frac{1}{q_{n+1} - q_{n}}.
\end{align*}
\]

If we put

\[
\widetilde{\sigma}_{n}(x, y) = \frac{1}{\pi^{2}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + u, y + v) \frac{1}{8(n+1) \sin \frac{u}{2} \sin \frac{v}{2}} \times \left\{ \frac{\sin (n+1)(u-v)}{u-v} - \frac{\sin (n+1)(u+v)}{u+v} \right\} du dv
\]

and

\[
\widetilde{\sigma}^{*}(x, y) = \sup_{n} \left| \widetilde{\sigma}_{n}(x, y) \right|
\]

then we have from (9)

\[
\begin{align*}
\gamma_{n}^{(1)}(x, y) &= \frac{1}{n+1} \left[ \sum_{i=0}^{n-1} \widetilde{\sigma}_{q_{i+1}}(x, y)(q_{i+1} + 1) \Delta \left( \frac{1}{q_{i+1} - q_{i}} \right) \\
&\quad - \widetilde{\sigma}_{q_{n+1}}(x, y)(q_{n+1} + 1) \sum_{i=0}^{n-1} \Delta \left( \frac{1}{q_{i+1} - q_{i}} \right) + \widetilde{\sigma}_{q_{n+1}}(x, y)(q_{n+1} + 1) \frac{1}{q_{n+1} - q_{n}} \right].
\end{align*}
\]
and
\[ |\gamma_n^{(1)}(x, y)| \leq \widetilde{\sigma}(x, y) \sum_{i=0}^{n-1} (q_{i+1} + 1) \Delta \left( \frac{1}{q_{i+1} - q_i} \right) + (q_n + 1) \sum_{i=0}^{n-1} \Delta \left( \frac{1}{q_{i+1} - q_i} \right) + (q_{n+1} + 1) \frac{1}{q_{n+1} - q_n}. \]

It is easily seen that the expression in the bracket is finite since \(q_i = i^k, 0 < k < 1\), so we have

(10) \[ |\gamma_n^{(1)}(x, y)| \leq A \tilde{\sigma}(x, y) \leq A \sigma(x, y), \]

since \(\tilde{\sigma}(x, y)\) is essentially equal to

\[ \sigma_n(x, y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + u, y + v) \frac{1}{8(n+1)} \sin \frac{u}{2} \sin \frac{v}{2} \]
\[ \times \left\{ \frac{\sin(n+1)(u-v)}{2 \sin \frac{u-v}{2}} - \frac{\sin(n+1)(u+v)}{2 \sin \frac{u+v}{2}} \right\} \\ \]
\[ du \, dv. \]

Combining (7), (8) and (10) with Lemma 5 we get the inequality

\[ \gamma_n(x, y) \leq A \{ f^{**}(x, y) + f^{**}(x, y) \}. \]

And finally we have, again by Lemma 5,

\[ |[\gamma_n(x, y) > t]| \leq A \frac{1}{t} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x, y)| \, dx \, dy. \]

The convergence almost everywhere of \(\gamma_n(x, y)\) to \(f(x, y)\) is derived from the above inequality by the usual way. Thus we complete the proof.

REFERENCES


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