ON HARMONIC TENSORS IN COMPACT SASAKIAN SPACES

SHUN-ICHI TACHIBANA

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A Riemannian space with a contact form \( \eta = \eta_\lambda dx^\lambda \) is called Sasakian if the contact structure satisfies certain conditions (Cf. § 3). In § 4 and § 5 of this paper we shall discuss harmonic vectors in compact Sasakian spaces and obtain some analogous results to compact Kählerian spaces (Cf. Theorem 4.2, 4.3, 5.1). In § 7 we shall prove that any harmonic \( p \)-form in \( n \) dimensional compact Sasakian spaces is orthogonal to \( \eta^\lambda = g^{\lambda\mu} \eta_{\mu} \) if \( p < (1/2)(n+1) \). In § 8, we shall introduce an operator \( \Phi \) and prove that \( \Phi u \) is harmonic for a harmonic \( p \)-form \( u \) if \( p < (1/2)(n+1) \). Preliminary facts and lemmas are given in the other sections.

1. Preliminaries.\(^1\) Consider an \( n \) dimensional Riemannian space \( M \) whose positive definite metric tensor is given by \( g_{\mu\nu} \). We denote by \( R_{\lambda\mu\nu}^\rho \) the Riemannian curvature tensor

\[
R_{\lambda\mu\nu}^\rho = \partial_\lambda \left\{ \frac{\rho}{\mu\nu} \right\} - \partial_\mu \left\{ \frac{\rho}{\lambda\nu} \right\} + \left\{ \frac{\rho}{\lambda\mu} \right\} \left( \frac{\lambda}{\mu\nu} \right) - \left\{ \frac{\rho}{\mu\nu} \right\} \left( \frac{\alpha}{\lambda\nu} \right), \quad \partial_\lambda = \partial/\partial x^\lambda,
\]

and by \( R_{\mu\nu} = R_{\lambda\mu}^\lambda \) the Ricci tensor, where \( \left\{ \frac{\lambda}{\mu\nu} \right\} \) are the Christoffel’s symbols. The operator of covariant derivation with respect to \( \left\{ \frac{\lambda}{\mu\nu} \right\} \) is denoted by \( \nabla_\lambda \).

It is easy to have

\[
R_{\lambda\mu\nu}^\rho u^{\lambda\mu\nu} = 0
\]

for any skew-symmetric tensor \( u^{\lambda\mu\nu} \), by virtue of the first Bianchi’s identity.

If a vector field \( \eta^\lambda \) satisfies

\[
\theta(\eta) g_{\lambda\mu} \equiv \nabla_\lambda \eta_\mu + \nabla_\mu \eta_\lambda = 0, \quad (\eta_\lambda \equiv g_{\lambda\mu} \eta^\mu),
\]

then it is called a Killing vector, where \( \theta(\eta) \) is the operator of Lie derivation with respect to \( \eta^\lambda \). It is well known that the equations

\(^1\) As to the notations we follow Yano, K., [5].
\[ \nabla^a \nabla_a \eta_\lambda + R^a_{\lambda a} = 0, \quad \nabla^3 \eta_\lambda = 0 \]

are valid for any Killing vector \( \eta_\lambda \).

We shall recall various operators on differential forms. A skew-symmetric tensor \( u_{\lambda_1 \ldots \lambda_p} \) may be regarded as the coefficients of a differential \( p \)-form

\[ u = \frac{1}{p!} u_{\lambda_1 \ldots \lambda_p} dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_p}. \]

We represent this fact by

\[ u : u_{\lambda_1 \ldots \lambda_p}. \]

If \( p = 0 \), then a \( p \)-form \( u \) is nothing but a scalar function which we shall usually denote by \( f \).

The exterior differential \( du \) and the codifferential \( \delta u \) of \( u \) are given by

\[
\begin{align*}
\text{du :} & \quad \left\{ \begin{array}{ll}
\nabla^a u_{\lambda_1 \ldots \lambda_p} - \sum_{i=1}^{p} \nabla^a \eta_{\lambda_1 \ldots \lambda_{i-1} \lambda_{i+1} \ldots \lambda_p}, & p \geq 1, \\
\nabla^a f, & p = 0,
\end{array} \right. \\
\delta u : & \quad \left\{ \begin{array}{ll}
\nabla^a u_{\lambda_1 \ldots \lambda_p}, & p \geq 1, \\
0, & p = 0.
\end{array} \right.
\end{align*}
\]

The Laplacian operator \( \Delta \) is given by \( \Delta = d \delta + \delta d \). For a \( p \)-form \( u \) we have explicitly

\[
(\Delta u)_{\lambda_1 \ldots \lambda_p} = \nabla^a \nabla_a u_{\lambda_1 \ldots \lambda_p} - \sum_{i=1}^{p} R^a_{\lambda_1 a} u_{\lambda_{i+1} \ldots \lambda_p} \]

\[
- \sum_{j<i} R_{\lambda_j \lambda_i} u_{\lambda_1 \ldots \lambda_j \ldots \lambda_p}, \quad p \geq 2,
\]

where the subscripts \( \sigma \) appears at the \( i \)-th position and \( \rho \) at the \( j \)-th position, and

\[
(\Delta u)_\lambda = \nabla^a \nabla_a u_\lambda - R^a_\lambda u_a, \quad p = 1,
\]

\[
\Delta f = \nabla^a \nabla_a f, \quad p = 0.
\]

A \( p \)-form \( u \) is called to be harmonic if \( du = 0 \) and \( \delta u = 0 \) are satisfied. Thus if \( u \) is harmonic, then we have \( \Delta u = 0 \).
Let \( \eta = \eta_\lambda \, dx^\lambda \) be a 1-form, then we shall naturally identify \( \eta \) with the contravariant vector field \( \eta^\lambda = \eta_\mu g^{\lambda \mu} \). Hence, for instance, if \( \eta \) is closed or \( \eta^\lambda \) is a Killing vector, then we shall say that \( \eta^\lambda \) is closed or the 1-form \( \eta = \eta_\lambda \, dx^\lambda \) is a Killing form, respectively.

For a Killing form \( \eta \) we have

\[
(\Delta \eta)_\lambda = -2R^i \xi^\eta, \quad \delta \eta = 0.
\]

For a 1-form \( \eta \) the operator \( i(\eta) \) is defined by

\[
(i(\eta) u)_{\lambda_1 \ldots \lambda_p} = \eta^a u_{a \lambda_1 \ldots \lambda_p}, \quad i(\eta) f = 0
\]
for any \( p \)-form \( u \) and any scalar function \( f \).

The Lie derivation with respect to \( \eta \) satisfies

\[
\theta(\eta) u = (di(\eta) + i(\eta)d) u
\]
for any \( p \)-form \( u \), \( (p \geq 0) \). For a \( p \)-form \( u \), \( \theta(\eta) u \) is given explicitly by

\[
(\theta(\eta) u)_{\lambda_1 \ldots \lambda_p} = \eta^a \nabla_a u_{\lambda_1 \ldots \lambda_p} + \sum_{i=1}^p u_{\lambda_i \ldots \alpha \ldots \lambda_p} \nabla_{\lambda_i} \eta^\alpha,
\]

\[
\theta(\eta) f = \eta^a \nabla_a f, \quad p = 0.
\]

The exterior product of a 1-form \( \eta \) or a 2-form \( \varphi \) with a \( p \)-form \( u \), \( (p \geq 2) \), is given respectively by

\[
\eta \wedge u : \eta_a u_{\lambda_1 \ldots \lambda_p} = \sum_{j=1}^p \eta_{\lambda_j} u_{\lambda_1 \ldots \alpha \ldots \lambda_p},
\]

\[
\varphi \wedge u : \varphi_{a \beta} u_{\lambda_1 \ldots \lambda_p} = \sum_{i} \varphi_{a \lambda_i \ldots \beta \ldots \lambda_p} - \sum_{j} \varphi_{\lambda_j \alpha} u_{\lambda_1 \ldots \alpha \ldots \lambda_p} + \sum_{j<i} \varphi_{\lambda_j \lambda_i} u_{\lambda_1 \ldots \alpha \ldots \beta \ldots \lambda_p},
\]

where the subscripts \( \alpha \) appears at the \( j \)-th position and \( \beta \) at the \( i \)-th position.

2. Harmonic tensors in a compact orientable Riemannian space. In this section we shall always consider a compact orientable Riemannian space \( M \). For any \( p \)-forms \( u \) and \( v \) the global inner product \( (u, v) \) is defined by
(u, v) = \frac{1}{p!} \int_M u_{\lambda_1 \ldots \lambda_p} v^{\lambda_1 \ldots \lambda_p} \, d\sigma,

where \( d\sigma \) means the volume element of \( M \).

For any \( p \)-form \( u \) and \( (p+1) \)-form \( v \) the following integral formulae are well known

(2.1) \[ (du, v) + (u, \delta v) = 0, \]

(2.2) \[ (\Delta u, u) + (du, du) + (\delta u, \delta u) = 0. \]

If a \( p \)-form \( u \) satisfies \((\Delta u, u) \geq 0\), then \( u \) is harmonic, by virtue of (2.2).

The following lemma is also well known.

**Lemma 2.1.** In a compact orientable Riemannian space

\[ \theta(\eta) u = 0 \]

is valid for any Killing vector \( \eta^\lambda \) and any harmonic \( p \)-form \( u \).

From this lemma and (1.3) we have easily the following

**Lemma 2.2.** In a compact orientable Riemannian space,

\[ i(\eta) u \]

is closed

for any Killing vector \( \eta^\lambda \) and any harmonic \( p \)-form \( u \).

Now let \( \eta \) be a Killing form and \( u \) be a harmonic \( p \)-form, then we have

\[ (\delta(\eta \wedge u))_{\lambda_1 \ldots \lambda_p} = \nabla^\alpha(\eta \wedge u)_{\alpha \lambda_1 \ldots \lambda_p} \]
\[ = \nabla^\alpha(\eta_\alpha u_{\lambda_1 \ldots \lambda_p} - \sum \eta_\alpha u_{\lambda_1 \ldots \alpha \ldots \lambda_p}) \]
\[ = \eta^\alpha \nabla_\alpha u_{\lambda_1 \ldots \lambda_p} - \sum u_{\lambda_1 \ldots \alpha \ldots \lambda_p} \nabla^\alpha \eta_\alpha \]
\[ = (\theta(\eta) u)_{\lambda_1 \ldots \lambda_p}. \]

Thus taking account of Lemma 2.1 we get

**Lemma 2.3.** In a compact orientable Riemannian space,
for any Killing form \( \eta \) and any harmonic \( p \)-form \( u \).

By Lemma 2.2 and Lemma 2.3 we can obtain

**Lemma 2.4.** In a compact orientable Riemannian space, if

\[
i(\eta) u \quad \text{is coclosed}
\]

for a Killing form \( \eta \) and a harmonic \( p \)-form \( u \), then the \( p \)-form

\[
\eta \wedge i(\eta) u \quad \text{is coclosed .}
\]

3. **Identities in a Sasakian space.**

An \( n \) dimensional Sasakian space (or normal contact metric space) is a Riemannian space which admits a unit Killing vector field \( \eta^\lambda \) satisfying

\[
(3.1) \quad \nabla_\lambda \nabla_\mu \eta_\nu = \eta_\nu g_{\lambda \mu} - \eta_\mu g_{\lambda \nu}.
\]

It is well known that a Sasakian space is orientable and odd dimensional.

In this section we prepare identities in an \( n \) dimensional Sasakian space.

Now if we define \( \varphi^\mu_\nu \) by

\[
(3.1) \quad \varphi^\mu_\nu = \nabla_\mu \eta^\nu,
\]

then we have

\[
(3.2) \quad \varphi^\lambda_\mu \varphi^\mu_\nu = -\delta^\lambda_\nu + \eta^\nu \eta^\lambda, \quad \varphi^\lambda_\nu \eta^\mu = 0,
\]

\[
(3.3) \quad \varphi_{\mu \nu} = -\varphi_{\nu \mu}, \quad (\varphi_{\mu \nu} \equiv \varphi^\sigma_\mu g_{\sigma \nu}).
\]

(3.1) is then written as

\[
(3.2) \quad \nabla_\lambda \varphi_{\mu \nu} = \eta_\mu g_{\lambda \nu} - \eta_\nu g_{\lambda \mu}
\]

and we have easily

\[
(3.3) \quad \nabla^\lambda \varphi_{\lambda \mu} = -(n-1)\eta_\mu,
\]

2) Okumura, M., \[1\], Sasaki, S and Y. Hatakeyama, \[2\]. Examples of Sasakian space have been given in \[2\] and \[4\].
Applying the Ricci’s identity to $\eta$, we have

$$\nabla_\nu \nabla_\mu \eta_\lambda - \nabla_\mu \nabla_\nu \eta_\lambda = - R_{\nu\lambda\mu} \eta_\nu,$$

from which we can get

$$R_{\nu\lambda\mu} \eta_\nu = \eta_\nu g_{\lambda\mu} - \eta_\mu g_{\nu\lambda}, \quad (3.5)$$

$$R_{\nu} \eta_\nu = (n-1) \eta_\nu \cdot \quad (3.6)$$

Next, applying the Ricci’s identity to $\phi^z$ we have

$$\nabla_\rho \nabla_\sigma \phi^z - \nabla_\sigma \nabla_\rho \phi^z = R_{\rho\sigma\xi} \phi^z - R_{\rho\xi} \phi^z - \phi^z g_{\rho\sigma}, \quad (3.7)$$

Substituting (3.2) into the left hand member of the last equation we get

$$R_{\rho\sigma\xi} \phi^z - R_{\rho\xi} \phi^z = \phi^z \delta_\sigma - \phi^z g_{\rho\sigma} - \phi^z g_{\rho\xi} + \phi^z g_{\rho\sigma} g_{\rho\xi}, \quad (3.8)$$

from which it follows that

$$\phi^z R_{\rho\sigma\pi} = - R_{\rho\sigma\xi} \phi^z + \phi^z g_{\rho\sigma} g_{\rho\xi} - \phi^z g_{\rho\sigma} g_{\rho\xi} + \phi^z g_{\rho\sigma} g_{\rho\xi}. \quad (3.9)$$

Contracting (3.7) with respect to $\rho$ and $\alpha$ we can get

$$\frac{1}{2} \phi_{\rho\beta} R_{\alpha\rho\beta} = R_{\rho\alpha} \phi^z + (n-2) \phi^z \rho, \quad (3.10)$$

For $\phi^z$ we have

$$R_{\rho\sigma} \phi^z = - R_{\rho\sigma} \phi^z, \quad R_{\rho} \phi^z = R_{\alpha} \phi^z. \quad (3.11)$$

for any vector $u_\mu$.

4. Harmonic vectors in a compact Sasakian space.\(^3\)) Let $u$ be a harmonic 1-form in a compact Sasakian space, then we have $df = 0$ for the scalar $f$ defined by $f = i(\eta) u$, by virtue of Lemma 2.2.

\(^3\) Throughout the paper we assume that $n = \dim M > 1$. 
Hence $f$ is constant. If we define $\beta$ by

\[(4.1)\quad u = f \eta + \beta,\]

then $\beta$ is a 1-form orthogonal to $\eta$, i.e., $i(\eta)\beta = 0$. Operating $\Delta$ to (4.1) we get $\Delta\beta = -f\Delta\eta$ and as $\eta$ is Killing we have

\[(\Delta\beta)_\lambda = -f(\Delta\eta)_\lambda = 2f R^i_\lambda \eta_i = 2(n-1)f \eta_i\]

by virtue of (1.2) and (3.6). Hence $\beta$ is harmonic, because we have $(\Delta\beta, \beta) = 0$. Thus we have $f=0$ and obtain the following

**Theorem 4.1.** Any harmonic 1-form $u$ in a compact Sasakian space is orthogonal to $\eta$, i.e., $i(\eta)u = 0$.

Next we introduce an operator $J : p$-form $u \rightarrow$ tensor of type $(0, p)$, $\tilde{u} = J u$ by

\[
\tilde{u}_{\lambda_1 \ldots \lambda_p} = \varphi_{\lambda_1}^a u_{a \lambda_2 \ldots \lambda_p}
\]

and we shall compute $\Delta \tilde{u}$ for a harmonic 1-form $u$ in a compact Sasakian space.

Taking account of $i(\eta)u = 0$, $\Delta u = 0$, (3.4) and (3.10) we can get

\[
\nabla^a \nabla_a \tilde{u}_\lambda = -2 \nabla_\lambda (\eta^a u_a) + \varphi_\lambda^a \nabla^a \nabla_a u_a = \varphi_\lambda^a R^a_s u_s = R^a_s \tilde{u}_a.
\]

Thus we have $\Delta \tilde{u} = 0$ and hence we get

**Theorem 4.2.** In a compact Sasakian space, $\tilde{u} = J u$ is a harmonic 1-form for any harmonic 1-form $u$.

As a corollary to this theorem we have

**Theorem 4.3.** The first Betti number of a compact Sasakian space is zero or even.

5. **$C$-analytic 1-form.** It is known that in a compact Kählerian space a harmonic 1-form is holomorphic (i.e., covariant analytic) and vice versa. In this section we shall consider an analogous fact.
Now, in a Sasakian space, let \( u \) be a 1-form satisfying

\[(5.1)\quad Jdu - dJu = 0,\]

which is written explicitly as follows

\[(5.2)\quad \phi^\alpha_{\lambda}(\nabla_\alpha u_\mu - \nabla_\mu u_\alpha) - [\nabla_\lambda(\phi^\alpha_{\mu} u_\alpha) - \nabla_\mu(\phi^\alpha_{\lambda} u_\alpha)] = 0.\]

It is easy to see that (5.2) is equivalent to the following equation

\[(5.3)\quad \phi^\alpha_{\lambda} \nabla_\alpha u_\mu - \phi^\alpha_{\mu} \nabla_\lambda u_\alpha + \eta_\lambda u_\mu - \eta_\mu u_\lambda = 0.\]

If we transvect \( \eta^1 \) to (5.2), we have

\[(5.4)\quad u_\lambda = (u_\alpha \eta^\alpha)\eta_\lambda + \phi^\alpha_{\lambda} \eta^\alpha \nabla_\alpha u_\alpha\]

and transvecting (5.2) with \( g^{\lambda\mu} \) we obtain

\[(5.5)\quad \phi^\alpha_{\beta} \nabla_\alpha u_\beta = 0.\]

Now we define a \textit{C-analytic} 1-form\(^4\) as a 1-form \( u \) satisfying (5.1) and \( i(\eta)u = 0 \). Then we have

\textbf{THEOREM 5.1.} \textit{A necessary and sufficient condition for a 1-form in a compact Sasakian space to be harmonic is that it is C-analytic.}

\textbf{PROOF.} For a harmonic 1-form \( u \), we have \( du = 0, dJu = 0 \) and \( i(\eta)u = 0 \) by virtue of Theorem 4.1 and Theorem 4.2. Hence it is C-analytic.

Conversely let \( u \) be C-analytic, then we have \( i(\eta)u = \eta^\alpha u_\alpha = 0 \). Taking account of (3.3) and (5.5) we can get

\[(5.6)\quad \eta^\alpha \nabla^\lambda \nabla_\lambda u_\alpha = 0.\]

Next, transvecting (5.3) with \( \phi^\alpha_{\mu} \) we have

\[\nabla_\lambda u_\alpha + \phi^\alpha_{\beta} \nabla_\beta u_\alpha - \eta_\mu \eta^\alpha \nabla_\lambda u_\alpha + \eta_\lambda \phi^\alpha_{\mu} u_\alpha = 0.\]

Operating \( \nabla^\lambda = g^{\lambda\mu} \nabla_\mu \) to the last equation we get \( \Delta u = 0 \), by virtue of (5.6) and (3.11). Thus \( u \) is harmonic.

Q.E.D.

\(^4\) For an almost complex space, see Tachibana, S., [3].
6. A lemma. Consider a harmonic \( p \)-form \( u \), \((p \geq 2)\), in an \( n \) dimensional Sasakian space and define a \((p-2)\)-form \( v \) by

\[
v : \quad v_{\lambda_1, \ldots, \lambda_p} = \varphi^{\lambda_1 \lambda_2} u_{\lambda_1, \ldots, \lambda_p}.
\]

In the following we shall compute \( \Delta v \) and obtain a lemma.

First as we have

\[
\nabla^\varphi \nabla^\varphi v_{\lambda_1, \ldots, \lambda_p} = \nabla^\varphi \nabla^\varphi \varphi^{\lambda_1 \lambda_2} u_{\lambda_1, \ldots, \lambda_p} + 2 \nabla^\varphi \varphi^{\lambda_1 \lambda_2} \nabla^\varphi u_{\lambda_1, \ldots, \lambda_p} + \varphi^{\lambda_1 \lambda_2} \nabla^\varphi \nabla^\varphi u_{\lambda_1, \ldots, \lambda_p},
\]

if we take account of (3.2) and (3.4) we can get

\[
(6.1) \quad \nabla^\varphi \nabla^\varphi v_{\lambda_1, \ldots, \lambda_p} = -2v_{\lambda_1, \ldots, \lambda_p} + \varphi^{\lambda_1 \lambda_2} \nabla^\varphi \nabla^\varphi u_{\lambda_1, \ldots, \lambda_p}.
\]

As \( u \) satisfies \( \Delta u = 0 \), the last term becomes as follows:

\[
(6.2) \quad \varphi^{\lambda_1 \lambda_2} \nabla^\varphi \nabla^\varphi u_{\lambda_1, \ldots, \lambda_p} = \varphi^{\lambda_1 \lambda_2} \left( \sum_{i=1}^{p} R_{\lambda_1} \omega_{\lambda_1, \ldots, \lambda_p} + \sum_{j<\ell} R_{\lambda_1 \lambda_j} \omega_{\lambda_1, \ldots, \lambda_p} \right)
\]

\[
= \sum_{i=1}^{p} R_{\lambda_1} \omega_{\lambda_1, \ldots, \lambda_p} + \sum_{2<j<\ell} R_{\lambda_1 \lambda_j} \omega_{\lambda_1, \ldots, \lambda_p}
\]

\[
+ 2\varphi^{\lambda_1 \lambda_2} R_{\lambda_1} \omega_{\lambda_1, \ldots, \lambda_p} + \varphi^{\lambda_1 \lambda_2} \omega_{\lambda_1, \ldots, \lambda_p} + 2\varphi^{\lambda_1 \lambda_2} \sum_{2<j<\ell} R_{\lambda_1 \lambda_j} \omega_{\lambda_1, \ldots, \lambda_p}.
\]

On the other hand we have

\[
(6.3) \quad \varphi^{\lambda_1 \lambda_2} R_{\lambda_1 \lambda_j} \omega_{\lambda_1, \ldots, \lambda_p} = \varphi^{\lambda_1 \lambda_2} R_{\lambda_1 \lambda_j} \omega_{\lambda_1, \ldots, \lambda_p}
\]

\[
= 2[R_{\lambda_1} \varphi^{\lambda_1} + (n-2)\varphi_{\lambda_1}] \omega_{\lambda_1, \ldots, \lambda_p}
\]

\[
= -2\varphi^{\lambda_1 \lambda_2} R_{\lambda_1} \omega_{\lambda_1, \ldots, \lambda_p} + 2(n-2)v_{\lambda_1, \ldots, \lambda_p}
\]

by virtue of (3.9) and (3.10) and taking account of (3.8) and (1.1) we have

\[
(6.4) \quad \varphi^{\lambda_1 \lambda_2} R_{\lambda_1 \lambda_j} \omega_{\lambda_1, \ldots, \lambda_p} = \varphi^{\lambda_1 \lambda_2} R_{\lambda_1 \lambda_j} \omega_{\lambda_1, \ldots, \lambda_p}
\]

\[
= -R_{\lambda_1 \lambda_j} \varphi_{\lambda_1} \omega_{\lambda_1, \ldots, \lambda_p}
\]

\[
+ (\varphi_{\lambda_1} g_{\lambda_1} - \varphi_{\lambda_1} g_{\lambda_1} - \varphi_{\lambda_1} g_{\lambda_1} + \varphi_{\lambda_1} g_{\lambda_1}) \omega_{\lambda_1, \ldots, \lambda_p}
\]

\[
= -2v_{\lambda_1, \ldots, \lambda_p}.
\]
Substituting (6.3) and (6.4) into (6.2) we have
\[ \varphi^{i_{1} \cdots i_{s}} \nabla_{i_{1}} \cdots \nabla_{i_{s}} u_{\lambda_{1} \cdots \lambda_{s}} = \sum_{i=3}^{p} R_{\lambda_{1} \cdots \lambda_{s}}^{i_{1} \cdots i_{s}} v_{\lambda_{1} \cdots \lambda_{s}} + \sum_{i<j<l} R_{\lambda_{1} \lambda_{2} \lambda_{3}}^{i_{1} i_{2} i_{3}} v_{\lambda_{1} \lambda_{2} \lambda_{3}} + 2(n-2p+2)v_{\lambda_{1} \cdots \lambda_{s}} \]
and from (6.1) we get
\[ \Delta v = 2(n+1-2p)v. \]
Hence if our space is compact, we have an integral formula
\[ 2(n+1-2p)(v, v) + (dv, dv) + (\delta v, \delta v) = 0. \]
Thus we have \( v=0 \) if \( 2p<n+1 \) and \( v \) is harmonic if \( 2p=n+1 \). Now if we define \( w \) by
\[ w = i(\eta)u : \eta^{a}u_{a_{\lambda_{1} \cdots \lambda_{s}}}, \]
then we have \( \delta w = \delta i(\eta)u = -v \). Hence when \( 2p=n+1 \), \( v \) being harmonic, we have \( v=0 \), too. Consequently we obtain

**Lemma 6.1.** Let \( u \) be a harmonic \( p \)-form, \( p \geq 2 \), in an \( n \)-dimensional compact Sasakian space. If \( p \leq (1/2)(n+1) \), then
\[ i(\eta)u \quad \text{is coclosed.} \]

7. **Harmonic tensors in a compact Sasakian space.** In this section we shall prove the following

**Theorem 7.1.** In an \( n \)-dimensional compact Sasakian space, any harmonic \( p \)-form \( u \) is orthogonal to \( \eta \), i.e., \( i(\eta)u = 0 \), provided that \( p < (1/2)(n+1) \).

If \( p = 1 \), this is nothing but Theorem 4.1, so we shall assume \( p \geq 2 \). To prove this theorem we introduce \( w, \alpha \) and \( \beta \) by
\[ w = i(\eta)u : \quad w_{\lambda_{1} \cdots \lambda_{s}} = \eta^{a}u_{a_{\lambda_{1} \cdots \lambda_{s}}}, \]
\[ \alpha = \eta \wedge w : \quad \alpha_{\lambda_{1} \cdots \lambda_{s}} = \sum_{i=1}^{p} (-1)^{i-1} \eta_{i} \omega_{\lambda_{1} \cdots \lambda_{s}}^{i}, \]
\[ \beta = u - \alpha : \quad u_{\lambda_{1} \cdots \lambda_{s}} = \alpha_{\lambda_{1} \cdots \lambda_{s}} + \beta_{\lambda_{1} \cdots \lambda_{s}}. \]
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where \( \hat{\lambda}_i \) means that \( \lambda_i \) is omitted.

We can easily see that \( \beta \) is orthogonal to \( \eta \), \( i(\eta)\beta = 0 \), and \( \alpha \) being coclosed by Lemma 2.4 and Lemma 6.1 \( \beta \) is coclosed, too.

We shall need the following

**Lemma 7.2.** In an \( n \) dimensional compact Sasakian space, \( p \)-form \( \alpha = \eta \wedge i(\eta)u \) and \( \beta = u - \alpha \) are harmonic for any harmonic \( p \)-form \( u \), provided that \( p \leq (1/2)(n+1) \).

**Proof.** As we have \( \Delta \beta = -\Delta \alpha \), it holds that

\[
\beta^{i_1 \cdots i_p}(\Delta \beta)_{\lambda_1 \cdots \lambda_p} = -\beta^{i_1 \cdots i_p}(\Delta \alpha)_{\lambda_1 \cdots \lambda_p},
\]

On the other hand we have, taking account of (3.5), (3.6) and \( i(\eta)\beta = 0 \),

\[
\beta^{i_1 \cdots i_p} R_{i_1} \alpha_{i_1 \cdots i_p} = \beta^{i_1 \cdots i_p} R_{i_1} \left[ \sum_{k \neq i} (-1)^{i-1} \eta_{i_1} w_{i_1 \cdots i_p} + (-1)^{i-1} \eta_\epsilon w_{i_1 \cdots i_p} \right]
\]

\[
= (-1)^{i-1} \beta^{i_1 \cdots i_p} R_{i_1} \eta_i w_{i_1 \cdots i_p} = 0,
\]

\[
\beta^{i_1 \cdots i_p} R_{i_1} \alpha_{i_1 \cdots i_p} = \beta^{i_1 \cdots i_p} R_{i_1} \left[ \sum_{k \neq i} (-1)^{i-1} \eta_{i_1} \alpha_{i_1 \cdots i_p} + (-1)^{i-1} \eta_\epsilon \alpha_{i_1 \cdots i_p} \right]
\]

\[
+ (-1)^{i-1} \eta_\epsilon \alpha_{i_1 \cdots i_p} = 0.
\]

Thus we can get

\[
(7.1) \quad \beta^{i_1 \cdots i_p}(\Delta \beta)_{\lambda_1 \cdots \lambda_p} = -\beta^{i_1 \cdots i_p} \nabla^k \nabla_\epsilon \alpha_{\lambda_1 \cdots \lambda_p}
\]

\[
= 2 \beta^{i_1 \cdots i_p} \sum_{i=1}^p (-1)^{i+1} \varphi_{i}^\epsilon \nabla_\epsilon w_{\lambda_1 \cdots \lambda_p}.
\]

As \( w \) is closed by Lemma 2.2, we have for a fixed \( i \),

\[
\nabla_\epsilon w_{\lambda_1 \cdots \lambda_p} = \sum_{i=1}^p \nabla_{\lambda_i} w_{\lambda_1 \cdots \lambda_p},
\]

where the subscript \( \epsilon \) appears at the \( j \)-th position if \( j < i \) and at the \( (j-1) \)-th position if \( j > i \). Hence if, for fixed \( i \) and \( j(i \neq j) \), we define \( A_{i_1}^{i} \) by

\[
A_{i_1}^{i} = \beta^{i_1 \cdots i_p} \varphi_{i}^\epsilon w_{\lambda_1 \cdots \lambda_p},
\]

then it follows that
Substituting the last equation into (7.1) we get \((\Delta \beta, \beta) = 0\), from which it follows that \(\beta\) and hence \(\alpha\) are harmonic. Q.E.D.

**PROOF OF THEOREM 7.1.** As \(\alpha = \eta \wedge \omega\) is harmonic, we have \(d \eta \wedge \omega = 0\). Hence it follows that

\[\varphi_{\lambda_1 \lambda_2 \ldots \lambda_{p+n}} - \sum_{i=1}^{p+1} \varphi_{\lambda_1 \lambda_i \ldots \lambda_{p+n}} - \sum_{j=3}^{p+1} \varphi_{\lambda_1 \lambda_j \ldots \lambda_{p+n}} + \sum_{j<k} \varphi_{\lambda_1 \lambda_j \ldots \lambda_k \ldots \lambda_{p+n}} = 0.\]

Transvecting the last equation with \(\varphi^{\lambda_1}_{\lambda_2}\) we can get \((n + 1 - 2p) \omega = 0\), so if \(n + 1 > 2p\), we have \(\omega = 0\). Q.E.D.

In the case when \(2p = n + 1\), we have from Lemma 2.2, 6.1 and 7.2 the following

**THEOREM 7.3.** In an \(n\) dimensional compact Sasakian space, if \(u\) is a harmonic \((1/2)(n + 1)\)-form, then \(i(\eta)u\) and \(\eta \wedge i(\eta)u\) are harmonic.

8. **An operator** \(\Phi\). In an \(n\) dimensional compact Sasakian space we shall introduce an operator

\[\Phi : u \longrightarrow \Phi u = u\]

which is defined by

\[\Phi u : u_{\lambda_1 \ldots \lambda_p} = \sum_{i=1}^{p} \varphi_{\lambda_i} u_{\lambda_1 \ldots \lambda_{i-1} \lambda_{i+1} \ldots \lambda_p},\]

where \(u\) is a \(p\)-form and the subscript \(\alpha\) appears at the \(i\)-th position.

The purpose of this section is to prove the following

**THEOREM 8.1.** In an \(n\) dimensional compact Sasakian space, if \(u\) is a harmonic \(p\)-form and \(p < (1/2)(n + 1)\), then \(\Phi u\) is harmonic too.
PROOF. If \( p = 1 \), this is nothing but Theorem 4.2, so we shall assume \( p \geq 2 \). Let \( u \) be harmonic and assume that \( p < (1/2)(n+1) \), then we have \( \Delta u = 0 \) and \( i(\eta)u = 0 \).

We shall show in the following that \((\Delta u)^{\lambda_1, \ldots, \lambda_p} u^{\lambda_1, \ldots, \lambda_p} = 0\). At first on taking account of \( \varphi_\alpha = \nabla_\alpha \eta^*, (3.2), (3.4) \) and \( i(\eta)u = 0 \) we can get easily

\[
\nabla^\varepsilon \nabla^*_\varepsilon u_{\lambda_1, \ldots, \lambda_p} = \sum_{i=1}^{p} \varphi_{\lambda_i}^* \nabla^\varepsilon \nabla^*_\varepsilon u_{\lambda_1, \ldots, \lambda_p}
\]

and by virtue of \( \Delta u = 0 \) we have

\[
\nabla^\varepsilon \nabla^*_\varepsilon u_{\lambda_1, \ldots, \lambda_p} = \sum_{i=1}^{p} \varphi_{\lambda_i}^* \left[ R_{\alpha}^\sigma u_{\lambda_1, \ldots, \lambda_p} + \sum_{j=i}^{p} R_{\lambda_j}^\sigma u_{\lambda_1, \ldots, \lambda_p} \right]
\]

\[
+ \sum_{k<i} R_{\lambda_i \lambda_k}^{\rho \sigma} u_{\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_p} + \sum_{k>i} R_{\lambda_i \lambda_k}^{\rho \sigma} u_{\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_p} + \sum_{k<i, j>i} R_{\lambda_i \lambda_j}^{\rho \sigma} u_{\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_p}.
\]

If we take account of (3.10), then the last equation is written as the following form,

\[
(8.1) \quad \nabla^\varepsilon \nabla^*_\varepsilon u_{\lambda_1, \ldots, \lambda_p} - \sum_{i=1}^{p} R_{\lambda_i}^\sigma u_{\lambda_1, \ldots, \lambda_p} = \sum_{i=1}^{p} T_{\lambda_i}^{(i)} u_{\lambda_1, \ldots, \lambda_p},
\]

where we have put

\[
T_{\lambda_i}^{(i)} = - \sum_{k<i} \varphi_{\lambda_i}^* R_{\lambda_k \lambda_\rho} u_{\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_p} + \sum_{k=i}^{p} \varphi_{\lambda_i}^* R_{\lambda_k}^{\rho \sigma} u_{\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_p} + \sum_{k<i, j>i} \varphi_{\lambda_i}^* R_{\lambda_i \lambda_j}^{\rho \sigma} u_{\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_p}.
\]

where scripts \( \rho, \alpha \) and \( \sigma \) of \( u \) are at the \( k \)-th, \( i \)-th and \( j \)-th positions respectively.

Next we shall compute \( T_{\lambda_i}^{(i)} u_{\lambda_1, \ldots, \lambda_p} \). By virtue of the first Bianchi's identity, (3.8) and the skew-symmetric property of \( u \) and \( \nabla u \), we have

\[
\varphi_{\lambda_i}^* R_{\lambda_\rho \lambda_\sigma} u_{\lambda_1, \ldots, \lambda_{\rho-1}, \lambda_{\sigma+1}, \ldots, \lambda_{\lambda_1, \ldots, \lambda_p}}
\]

\[
= \varphi_{\lambda_i}^* (- R_{\lambda_\sigma \lambda_\rho} - R_{\lambda_\rho \lambda_\sigma}) u_{\lambda_1, \ldots, \lambda_{\rho-1}, \lambda_{\sigma+1}, \ldots, \lambda_{\lambda_1, \ldots, \lambda_p}}
\]

\[
= -2 \varphi_{\lambda_i}^* R_{\rho \sigma} u_{\lambda_1, \ldots, \lambda_{\rho-1}, \lambda_{\sigma+1}, \ldots, \lambda_{\lambda_1, \ldots, \lambda_p}}
\]

\[
= -2 [R_{\lambda_\rho \lambda_\sigma} \varphi_{\lambda_i}^* + \varphi_{\lambda_\rho} \varphi_{\lambda_\sigma} - \varphi_{\lambda_\rho} \varphi_{\lambda_\sigma} - \varphi_{\lambda_\rho} \varphi_{\lambda_\sigma} + \varphi_{\lambda_\rho} \varphi_{\lambda_\sigma}] u_{\lambda_1, \ldots, \lambda_{\rho-1}, \lambda_{\sigma+1}, \ldots, \lambda_{\lambda_1, \ldots, \lambda_p}}
\]

\[
= \varphi_{\lambda_i}^* \varphi_{\lambda_\rho} \varphi_{\lambda_\sigma} u_{\lambda_1, \ldots, \lambda_{\rho-1}, \lambda_{\sigma+1}, \ldots, \lambda_{\lambda_1, \ldots, \lambda_p}}
\]
Thus we can get

\[
\sum_{i=1}^{p} T^{\lambda_1\ldots\lambda_p}_{\lambda_1\ldots\lambda_p} = -\sum_{i} \sum_{k \neq i} R_{\lambda_i \lambda_i} \varphi_{\lambda_i} u_{\lambda_i} \ldots u_{\lambda_k} u^{\lambda_1\ldots\lambda_k} + \sum_{i} \sum_{k < j} R_{\lambda_i \lambda_j} \varphi_{\lambda_i} u_{\lambda_i} \ldots u_{\lambda_j} u^{\lambda_1\ldots\lambda_k} = 0 \]

Hence, from (8.1) and (8.2), we have \((\Delta u)^{\lambda_1\ldots\lambda_p}_{\lambda_1\ldots\lambda_p} = 0\) and this completes the proof. \(Q.E.D.\)

Let \(u\) be a harmonic \(p\)-form, \(p < (1/2)(n+1)\), in an \(n\) dimensional compact Sasakian space, then we know that \(\Phi u, \Phi^2 u, \ldots\) are harmonic \(p\)-forms and hence the following \(p\) \(p\)-forms

\[
\sum_{i=1}^{p} \varphi_{\lambda_i} u_{\lambda_i} \ldots u_{\lambda_k} + \sum_{j=1}^{p} \varphi_{\lambda_j} \varphi_{\lambda_j} u_{\lambda_j} \ldots u_{\lambda_k} + \ldots + \varphi_{\lambda_k} \varphi_{\lambda_k} \ldots u_{\lambda_k} = 0
\]

are harmonic.

**BIBLIOGRAPHY**


**OCHANOMIZU UNIVERSITY, TOKYO.**