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A CONFORMAL TRANSFORMATION OF CERTAIN CONTACT RIEMANNIAN MANIFOLDS

SHÛKICHI TANNO

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1. For any contact manifold M with a contact form η, we can find an associated Riemannian metric g, a (1.1)-tensor ϕ and a unit vector field ξ such that ϕ, ξ, η and g are the tensors of a contact metric structure. They satisfy the following relations:

$$\phi\xi=0, \quad \eta(\xi)=1, \quad \phi^2X=-X+\eta(X)\cdot\xi, \tag{1. 1}$$

$$\eta(X)=g(\xi, X), \quad g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{1. 2}$$

$$d\eta(X, Y)=2g(X, \phi Y)=2w(X, Y) \tag{1. 3}$$

for any vector fields X and Y on M. A contact structure is said to be regular if the distribution defined by ξ is regular. A contact metric structure is a K-contact metric structure if ξ is a Killing vector field, and furthermore it is a normal contact metric one if the following relation is satisfied

$$(\nabla_Z w)(X, Y) = \eta(X)g(Z, Y) - \eta(Y)g(Z, X)$$

for any vector fields X, Y and Z on M, where ∇ denotes the Riemannian connection by g. For the details see [4], [6] and [7].

In this note we prove the following

THEOREM. *In a compact, connected, regular and normal contact Riemannian $m(>3)$-dimensional manifold M, if M admits a non-isometric conformal transformation, then M is isometric with a unit sphere.*

In this direction, M. Okumura [5] proved the following

(A) Let M be a complete, normal contact Riemannian $m(>3)$-dimensional connected manifold. If it admits a non-isometric infinitesimal conformal tansformation, then M is isometric with a unit sphere.

Denote by $C(M)$ or $I(M)$ the groups of conformal transformations or isometries of M, and by $C_0(M)$ or $I_0(M)$ their identity components. To prove our Theorem, it is enough to verify the following

PROPOSITION. *In a compact, connected, regular K-contact Riemannian manifold M, suppose that* $C_0(M)=I_0(M)$. *Then we have* $C(M)=I(M)$.

In fact, assume that M is not isometric with a unit sphere, then by (A) M does not admit any non-isometric infinitesimal conformal transformation, i.e. $C_0(M)=I_0(M)$. By this proposition we have $C(M)=I(M)$, this means that M does not admit any non-isometric conformal transformation.

2. Proof of the Proposition. In a K-contact Riemannian manifold, the Riemannian curvature tensor R satisfies the identity (see [2]):

$$g(R(X,\xi)Y,\xi)=g(X,Y)-\eta(X)\cdot\eta(Y) \tag{2. 1}$$

for any vector fields X and Y on M, where

$$-R(X,\xi)Y=\nabla_X\nabla_\xi Y-\nabla_\xi\nabla_X Y-\nabla_{[X,\xi]}Y.$$

Let φ be a conformal transformation, then we have $\varphi^*g=\sigma g$ for some scalar function σ. As ξ is a Killing vector field, it generates a 1-parameter group of isometries ϕ_t of M. Then, denoting by φ also the differential of φ, $\varphi\xi$ and $\varphi^{-1}\xi$ generate $\varphi\cdot\phi_t\cdot\varphi^{-1}$ and $\varphi^{-1}\cdot\phi_t\cdot\varphi$ respectively (see p.7, [3]). By the fact that $\varphi\cdot\phi_t\cdot\varphi^{-1}$ and $\varphi^{-1}\cdot\phi_t\cdot\varphi$ are conformal transformations and by the assumption that $C_0(M)=I_0(M)$, $\varphi\xi$ and $\varphi^{-1}\xi$ are Killing vector fields. If one operates the Lie derivation $L(\xi)$ to $\sigma g=\varphi^*g$, one gets

$$\begin{aligned}(L(\xi)\sigma)g&=L(\xi)(\varphi^*g)\\&=\lim_{t\to0}\left(\frac{1}{t}\right)(\varphi^*\varphi^{-1*}\phi_t{}^*\varphi^*g-\varphi^*g)\\&=\varphi^*(L(\varphi\xi)g)=0,\end{aligned}$$

since $(\varphi\cdot\phi_t\cdot\varphi^{-1})^*=\varphi^{-1*}\cdot\phi_t{}^*\cdot\varphi^*$. This shows that $L(\xi)\sigma=0$. As for the Lie derivation $L(\varphi^{-1}\xi)$, we have $L(\varphi^{-1}\xi)\sigma=0$.

On the other hand, as φ is a conformal transformation, the Riemannian curvature teneor ${}^{\varphi}R$ of φ^*g is given by the relation:

$$\begin{aligned}{}^{\varphi}R^i{}_{jkl}=R^i{}_{jkl}&+\delta^i_k(\nabla_j\alpha_l-\alpha_j\alpha_l)-\delta^i_l(\nabla_j\alpha_k-\alpha_j\alpha_k)\\&+(\nabla_k\alpha^i-\alpha_k\alpha^i)g_{jl}-(\nabla_l\alpha^i-\alpha_l\alpha^i)g_{jk}\\&+\alpha_r\alpha^r(\delta^i_kg_{jl}-\delta^i_lg_{jk})\end{aligned} \tag{2. 2}$$

in a local coordinate neighborhood, where $\alpha=(1/2)\log\sigma$ and $\alpha_k=\partial_k\alpha$. As M is compact, there exists a point x of M where σ takes the maximum. Then at x we have $d\alpha=0$ namely $\alpha_k=0$. Let y be the point φx, then by (2.1) we have

$$g_y(R(\varphi\xi, \xi)\varphi\xi, \xi) = g_y(\varphi\xi, \varphi\xi) - [\eta_y(\varphi\xi)]^2 \\ = \sigma_x - [\eta_y(\varphi\xi)]^2. \tag{2. 3}$$

Transvecting (2. 2) with $\xi^k(\varphi^{-1}\xi)^l\xi^j$, we have

$$g_x({}^{\varphi}R(\xi, \varphi^{-1}\xi)\xi, \varphi^{-1}\xi) = g_x(R(\xi, \varphi^{-1}\xi)\xi, \varphi^{-1}\xi),$$

where we have utilyzed $\alpha_k|_x=0$, $\xi^k\nabla_j\alpha_k|_x = -(\nabla_j\xi^k)\alpha_k|_x=0$ since $\xi^k\alpha_k=0$, and similar relation $(\varphi^{-1}\xi)^k\nabla_j\alpha_k|_x=0$. Thus we have

$$\begin{aligned} g_y(R(\varphi\xi, \xi)\varphi\xi, \xi) &= g_y(\varphi[\varphi^{-1}\cdot R(\varphi\xi, \xi)\varphi\xi], \varphi\varphi^{-1}\xi) \\ &= \sigma_x g_x({}^{\varphi}R(\xi, \varphi^{-1}\xi)\xi, \varphi^{-1}\xi) \\ &= \sigma_x g_x(R(\xi, \varphi^{-1}\xi)\xi, \varphi^{-1}\xi) \\ &= \sigma_x g_x(\varphi^{-1}\xi, \varphi^{-1}\xi) - \sigma_x[\eta_x(\varphi^{-1}\xi)]^2 \\ &= 1 - \sigma_k[\eta_x(\varphi^{-1}\xi)]^2. \end{aligned} \tag{2. 4}$$

However we have

$$\eta_y(\varphi\xi) = g_y(\xi, \varphi\xi) = (\varphi^* g)_x(\varphi^{-1}\xi, \xi) = \sigma_x\eta_x(\varphi^{-1}\xi).$$

Therefore by (2.3) and (2.4), we get

$$(\sigma_x - 1)(1 - \sigma_x[\eta_x(\varphi^{-1}\xi)]^2) = 0. \tag{2. 5}$$

Hence $\sigma_x=1$ or $1=\sigma_x[\eta_x(\varphi^{-1}\xi)]^2$ holds good. Suppose that $[\eta_x(\varphi^{-1}\xi)]^2=\sigma_x^{-1}$ holds, then as $g_x(\varphi^{-1}\xi, \varphi^{-1}\xi)=\sigma_x^{-1}$, by $(1. 2)_2$ we see that $\varphi_y^{-1}\xi_y$ is proportional to ξ_x. Let $l(x)$ be the leaf of ξ which passes through x, then $\varphi l(x)$ is the leaf $l(y)$ which passes through y. While each leaf of ξ is of the same length in a regular contact manifold ([1], [9]). But the relation $L(\xi)\sigma=0$ implies that σ is constant on $l(x)$, and hence $\sigma=1$ holds on $l(x)$. Thus (2.5) shows that $\sigma=1$ on $l(x)$, and as σ_x is the maximum, $\sigma=1$ must hold on M. This completes the proof.

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TÔHOKU UNIVERSITY.