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APPLICATIONS OF FUBINI TYPE THEOREM TO THE TENSOR PRODUCTS OF C*-ALGEBRAS

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Let A and B be C^*-algebras and $A\underset{\alpha}{\otimes}B$ their C^*-tensor product with α-norm ([16]). We consider the following linear mapping from $A\underset{\alpha}{\otimes}B$ to A and B defined by

$$R_\varphi\left(\sum_{i=1}^{n} a_i\otimes b_i\right)=\sum_{i=1}^{n}<a_i,\varphi>b_i \quad \left(\text{resp. } L_\psi\left(\sum_{i=1}^{n} a_i\otimes b_i\right)=\sum_{i=1}^{n}<b_i,\psi>a_i\right)$$

for a bounded functional φ of A (resp. ψ of B). This mapping satisfies the following relation

$$<x,\varphi\otimes\psi> = <L_\psi(x),\varphi> = <R_\varphi(x),\psi>$$

for every $x\in A\underset{\alpha}{\otimes}B$.

Now the above relation may be considered, in some sense, as the non-commutative version of Fubini theorem in iterated integrals and it is the purpose of our present discussions to clarify the utility of this result in the tensor products of C^*-algebras settling all type problems of product algebras (Theorem 2) by using this mapping, and deriving various structure theorems for them, some of which are regarded as the extension of several results in [7], [19].

The above mapping is also useful to more general situations (cf. [10], [15]), since it can be defined in any tensor product of Banach algebras whenever the defining cross-norm is not less than Schatten's λ-norm ([13]).

Through the discussions $S(A)$ means the set of all states of a C^*-algebra A and $P(A)$ means the set of all pure states of A. The value of a linear functional φ on x is always denoted as $<x,\varphi>$. Let $A\odot B$ be algebraic tensor product of A and B, then the norm α is given by

$$\left\|\sum_{i=1}^{n} a_i\otimes b_i\right\| = \sup \frac{<\left(\sum_{j=1}^{m} x_j\otimes y_j\right)^*\left(\sum_{i=1}^{n} a_i\otimes b_i\right)^*\left(\sum_{i=1}^{n} a_i\otimes b_i\right)\left(\sum_{j=1}^{m} x_j\otimes y_j\right), \varphi\otimes\psi>^{1/2}}{<\left(\sum_{j=1}^{m} x_j\otimes y_j\right)^*\left(\sum_{j=1}^{m} x_j\otimes y_j\right), \varphi\otimes\psi>^{1/2}}$$

where φ and ψ run over the set of all states of A and B and $\sum_{j=1}^{m} x_j\otimes y_j$ runs over $A\odot B$. This norm is also defined as the C^*-enveloping norm of the product representations $\pi_1\otimes\pi_2$ of $A\odot B$ where π_1 and π_2 run over irreducible representations of A and B, respectively.

1. We state the above cited non-commutative version of Fubini theorem in the following manner.

THEOREM 1. *Let A and B be C^*-algebras, then for any bounded linear functional φ of A (resp. ψ of B) we can define a linear continuous mapping R_φ (resp. L_ψ) from $A\underset{\alpha}{\otimes}B$ onto B (resp. A) such as*

$$R_\varphi\left(\sum_{i=1}^{n} a_i\otimes b_i\right) = \sum_{i=1}^{n} < a_i, \varphi > b_i \quad \left(resp.\ L_\psi\left(\sum_{i=1}^{n} a_i\otimes b_i\right) = \sum_{i=1}^{n} < b_i, \psi > a_i\right),$$

satisfying the relation

$$< x, \varphi\otimes\psi > = < R_\varphi(x), \psi > = < L_\psi(x), \varphi >$$

for every x in $A\underset{\alpha}{\otimes}B$. Moreover the family $\mathfrak{F}_R = \{R_\varphi \mid \varphi\in S(A)\}$ is the total family of positive linear mappings, that is, for any non-zero element x in $A\underset{\alpha}{\otimes}B$ (not necessarily positive) there exists a state φ with $R_\varphi(x)\neq 0$. The same holds for the family $\mathfrak{F}_L = \{L_\psi \mid \psi\in S(B)\}$ and also holds for the families $\{R_\psi \mid \varphi\in P(A)\}$ and $\{L_\psi \mid \psi\in P(B)\}$.

PROOF. We proceed along with R_φ. Since α-norm is not less than Schatten's λ-norm,

$$\left\|R_\varphi\left(\sum_{i=1}^{n} a_i\otimes b_i\right)\right\| = \sup_{\|\psi\|\leqq 1}\left\|<\sum_{i=1}^{n} < a_i, \varphi > b_i, \psi>\right\| = \sup_{\|\psi\|\leqq 1}\left\|\sum_{i=1}^{n} <a_i, \varphi> < b_i, \psi>\right\|$$

$$= \sup_{\|\psi\|\leqq 1}\left|<\sum_{i=1}^{n} a_i\otimes b_i, \varphi\otimes\psi>\right| \leqq \sup_{\|\psi\|\leqq 1}\left\|\sum_{i=1}^{n} a_i\otimes b_i\right\| \|\varphi\|\|\psi\| = \|\varphi\|\left\|\sum_{i=1}^{n} a_i\otimes b_i\right\|.$$

so that R_φ can be extended to the mapping from $A\underset{\alpha}{\otimes}B$ onto B ($\|R_\varphi\|\leqq\|\varphi\|$).

The required relation is derived from the identity;

$$< \sum_{i=1}^{n} a_i \otimes b_i, \varphi \otimes \psi > = \sum_{i=1}^{n} < a_i, \varphi > < b_i, \psi > = < \sum_{i=1}^{n} < a_i, \varphi > b_i, \psi >$$

$$= < R_\varphi \left(\sum_{i=1}^{n} a_i \otimes b_i \right), \psi > = < \sum_{i=1}^{n} < b_i, \psi > a_i, \varphi > = < L_\psi \left(\sum_{i=1}^{n} a_i \otimes b_i \right), \varphi > .$$

The last half of the theorem is easily seen once we notice that the family of all product functional $\varphi \otimes \psi$ is total on $A \underset{\alpha}{\otimes} B$ and a product functional is a linear combination of product states. The case of pure states is also easily manageable.

We shall show two fundamental properties of these mapping at first.

LEMMA 1. *Suppose I is a closed ideal in $A \underset{\alpha}{\otimes} B$, then $\overline{R_\varphi(I)}$, the closure of $R_\varphi(I)$, (resp. $\overline{L_\psi(I)}$) is a closed ideal in B (resp. in A).*

PROOF. Let R_φ be a fixed map and take an element x in I and an element b in B. For an arbitrary positive number ε, choose an element $\sum_{i=1}^{n} x_i \otimes y_i$ in $A \odot B$ so that $\left\| x - \sum_{i=1}^{n} x_i \otimes y_i \right\| < \frac{\varepsilon}{\|b\| \|\varphi\|}$. By using approximate units in A we can find element e such that

$$\left\| \sum_{i=1}^{n} x_i \otimes b y_i - \sum_{i=1}^{n} e x_i \otimes b y_i \right\| < \frac{\varepsilon}{\|\varphi\|}$$

Since,

$$b R_\varphi \left(\sum_{i=1}^{n} x_i \otimes y_i \right) = \sum_{i=1}^{n} < x_i, \varphi > b y_i = R_\varphi \left(\sum_{i=1}^{n} x_i \otimes b y_i \right),$$

it can be shown that

$$\left\| b R_\varphi(x) - R_\varphi(e \otimes bx) \right\| \leqq \| b R_\varphi(x) - b R_\varphi \left(\sum_{i=1}^{n} x_i \otimes y_i \right) \| + \| R_\varphi \left(\sum_{i=}^{n} x_i \otimes b y_i \right)$$

$$- R_\varphi \left(\sum_{i=1}^{n} e x_i \otimes b y_i \right) \| + \| R_\varphi \left(\sum_{i=1}^{n} e x_i \otimes b y_i \right) - R_\varphi(e \otimes bx) \| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,$$

and $R(e \otimes bx) \in R_\varphi(I)$. Hence $b R_\varphi(x) \in \overline{R_\varphi(I)}$ and similarly $R_\varphi(x) b \in \overline{R_\varphi(I)}$. The same argument goes through L_ψ.

A composition series $\{I_\lambda\}_{0\leqq\lambda\leqq\lambda_0}$ of a C^*-algebra A means a well ordered ascending series of closed ideals I_λ beginning with 0 and ending with A, and for any limit ordinal λ, I_λ is the closure of the union of the preceding I's.

LEMMA 2. *Let* $\{I_\lambda\}_{0\leqq\lambda\leqq\lambda_0}$ *be a composition series of a closed ideal* I *in* $A\underset{\alpha}{\otimes}B$. *Then, collecting each coinciding part of* $\{\overline{R_\varphi(I_\lambda)}\}_{0\leqq\lambda\leqq\lambda_0}$ (*resp.* $\{\overline{L_\psi(I_\lambda)}\}_{0\leqq\lambda\leqq\lambda_0}$) *we get a composition series in* $\overline{R_\varphi(I)}$(*resp. in* $\overline{L_\psi(I)}$).

The proof is almost a series of verbal checks, so is left to the reader.

Let A_1 and B_1 be C^*-algebras on a Hilbert space H_1 and A_2 and B_2 be those on H_2. Then both C^*-algebras $A_1\underset{\alpha}{\otimes}A_2$ and $B_1\underset{\alpha}{\otimes}B_2$ are considered to be operator algebras on a Hilbert space $H_1\otimes H_2$. In this case we have

LEMMA 3. *If* $B_1\underset{\alpha}{\otimes}B_2$ *is a subalgebra of* $A_1\underset{\alpha}{\otimes}A_2$, *both* B_1 *and* B_2 *are subalgebras of* A_1 *and* A_2 *respectively,*

PROOF. Suppose that B_1 is not contained in A_1 and let a be a non-zero element of B_1 not belonging to A_1. We can find a bounded linear functional φ on $B(H_1)$, the algebra of all bounded linear operators on H_1, such that

$$<a,\varphi>\neq 0 \text{ and } <x,\varphi>=0 \text{ for every } x\in A_1.$$

Take a bounded linear functional ψ on $B(H_2)$, the algebra of all bounded linear operators on H_2, whose restriction to B_2 does not reduce to zero functional.

The restriction of the product functional $\varphi\otimes\psi$ to $A_1\underset{\alpha}{\otimes}A_2$ is zero, whereas the restriction to $B_1\underset{\alpha}{\otimes}B_2$ does not reduce to zero, a contradiction. Hence $A_1\supset B_1$ and similarly $A_2\supset B_2$.

In the following, we denote by π_φ the canonical representation associated with a state φ of a C^*-algebra. H_π means the representation space of a representation π. We also denote by $C(H)$ the C^*-algebra of all compact operators on a Hilbert space H. An irreducible representation π of a C^*-algebra A is called normal if $\pi(A)$ contains a (non-zero) compact operator of H_π (hence necessarily $\pi(A)\supset C(H)$ by the result of [5]).

An immediate corollary of Lemma 3 is the following result concerning normal representations without the assumption of separability.

COROLLARY. *Let* π_1 *and* π_2 *be irreducible representations of* C^**-algebras* A *and* B. *The product representation* $\pi_1\otimes\pi_2$ *of* $A\underset{\alpha}{\otimes}B$ *is normal if and only if both* π_1 *and* π_2 *are normal.*

Now we shall decide all types of tensor products of C^*-algebras. Though proofs are separated we shall state all of them together for completeness in the next

THEOREM 2. *Let A and B be C^*-algebras. Then,*

(a) *$A \underset{\alpha}{\otimes} B$ is a C^*-algebras with continuous trace if and only if both A and B are C^*-algebras with continuous trace,*

(b) *$A \underset{\alpha}{\otimes} B$ is a generalized C^*-algebra with continuous trace (abbreviated by GTC-algebra) if and only if both A and B are GTC-algebras,*

(c) *$A \underset{\alpha}{\otimes} B$ is a CCR algebra if and only if both A and B are CCR algebras,*

(d) *$A \underset{\alpha}{\otimes} B$ is a GCR algebra if and only if both A and B are GCR algebras,*

(e) *$A \otimes B$ is an NGCR algebra if and only if either A or B is an NGCR algebra.*

A part of these results is found in Wulfsohn [18], [19] and Guichardet ([7] some of them with the assumption of separability). It is also to be noticed that using Sakai's recent result [12] we can prove the assertion (e) along with the same line as Guichardet [7] used the results of Glimm [6] in separable case, but our proof of (e) is more direct and uses only elementary properties avoiding to use the results of Glimm's difficult constructions.

Indeed the assertion (e) is an easy consequence of the following result which seems to have an interest of its own.

THEOREM 3. *Let I be a CCR ideal in $A \underset{\alpha}{\otimes} B$ and φ be a pure state of A (resp. B). Then $\overline{R_\varphi(I)}$ (resp. $\overline{L_\varphi(I)}$) is a CCR ideal in B (resp. in A). If I is a GCR ideal in $A \underset{\alpha}{\otimes} B$, $\overline{R_\varphi(I)}$(resp. $\overline{L_\psi(I)}$) is also GCR ideal in B (resp. in A).*

PROOF. We may assume that $R_\varphi(I) \neq 0$. Write $\varphi = {}^t\pi_\varphi(\omega_\xi)$ where ω_ξ is a vector state of $\pi_\varphi(A)$ by a vector $\xi(\|\xi\|=1)$ in H_{π_φ}. Let ψ be a pure state of B such that $\pi_\psi | \overline{R_\varphi(I)}$, the restriction of π_ψ to $\overline{R_\varphi(I)}$, is a non-zero representation of a C^*-algebra $\overline{R_\varphi(I)}$.

For any element x in $A \underset{\alpha}{\otimes} B$ and a bounded linear functional ϕ of $\pi_\psi(B)$ we have by Theorem 1

$$< \pi_\psi(R_\varphi(x)), \phi > = < R_\varphi(x), {}^t\pi_\psi(\phi) > = < x, \varphi \otimes {}^t\pi_\psi(\phi) >$$
$$= < x, {}^t\pi_\varphi(\omega_\xi) \otimes {}^t\pi_\psi(\phi) > = < x, {}^t(\pi_\varphi \otimes \pi_\psi)(\omega_\xi \otimes \phi) > = < \pi_\varphi^{**} \otimes \pi_\psi(x), \omega_\xi \otimes \phi > \cdots (*)$$

Since $\pi_\psi|R_\varphi(I)\neq 0$, the above relation implies that $\pi_\varphi\otimes\pi_\psi|I\neq 0$, so that $\pi_\varphi\otimes\pi_\psi$ becomes an irreducible representation of I and as I is a CCR ideal,

$$\pi_\varphi\otimes\pi_\psi(I)=C(H_{\pi_\varphi}\otimes H_{\pi_\varphi})=C(H_{\pi_\varphi})\underset{\alpha}{\otimes}C(H_{\pi_\psi}).$$

Hence $\pi_\psi(B)\supset C(H_{\pi_\varphi})$ by Lemma 3. Therefore the equality (*) also implies that the polar of $\pi_\psi(R_\varphi(I))$ and that of $C(H_{\pi_\varphi})$ in the conjugate space of $\pi_\psi(B)$ coincide each other, which shows $\pi_\psi(\overline{R_\varphi(I)})=\pi_\psi(\overline{R_\varphi(I)})=C(H_{\pi\varphi})$. As every irreducible representations of $\overline{R_\varphi(I)}$ arise in this way, $\overline{R_\varphi(I)}$ must be CCR ideal in B.

Next, let I be a GCR ideal in $A\underset{\alpha}{\otimes}B$ and $\{I_\lambda\}_{0\leq\lambda\leq\lambda_0}$ a canonical composition series for I. Denote by $\{J_\Lambda\}_{0\leq\Lambda\leq\Lambda_0}$ the derived composition series of $\overline{R_\varphi(I)}$ by R_φ(cf. Lemma 2) and let ψ be a pure state of B such that $\pi_\psi|J_\Lambda=0$ and $\pi_\psi|J_{\Lambda+1}\neq 0$ for some index Λ. The representation π_ψ can be considered naturally as an irreducible representation of a C^*-algebra $J_{\Lambda+1}/J_\Lambda$. Consider the corresponding family of the ideal I's with $\overline{R_\varphi(I_\lambda)}=J_{\Lambda+1}$. There exists the smallest ideal I_λ among them and in this case λ is not a limit ordinal. Thus $\lambda=\lambda_1+1$ and clearly $\overline{R_\varphi(I_{\lambda_1})}=J_\Lambda$.

Now for a vector state ω_η in H_{π_φ} the equality (*) turns out to be

$$<\pi_\psi(R_\varphi(x)),\omega_\eta>=<\pi_\varphi\otimes\pi_\psi(x),\omega_\xi\otimes\omega_\eta>=<\pi_\varphi\otimes\pi_\psi(x),\omega_{\xi\otimes\eta}>.$$

Hence $\pi_\psi|J_\Lambda=0$ implies that the restriction of a vector state $\omega_{\xi\otimes\eta}$ to $\pi_\varphi\otimes\pi_\psi(I_{\lambda_1})$ is zero, which implies $\pi_\varphi\otimes\pi_\psi|I_{\lambda_1}=0$. On the other hand, $\pi_\varphi\otimes\pi_\psi|I\neq 0$ because $\pi_\psi|R_\varphi(I_{\lambda_1+1})\neq 0$. Therefore $\pi_\varphi\otimes\pi_\psi$ can be considered to be an irreducible representation of $I_{\lambda_1+1}/I_{\lambda_1}$ which is a CCR algebra by assumption. Then the same argument as in the proof of the first half part of the theorem shows that

$$\pi_\psi(J_{\Lambda+1}/J_\Lambda)=\pi_\psi(J_{\Lambda+1})=\pi_\varphi(\overline{R_\varphi(I_{\lambda_1+1})})=C(H_{\pi_\psi})$$

and since each irreducible representation of $J_{\Lambda+1}/J_\Lambda$ arises in this way $J_{\Lambda+1}/J_\Lambda$ is a CCR algebra. This completes the proof. Similar arguments show that the results also hold for L_φ.

An immediate consequence of Theorem 3 is the only if parts of the assertions (c) and (d). A direct proof of the if parts of (c) and (d) are found in [18] and [19]. The proof of the assertion (e) goes as follows; if either A or B is an $NGCR$ algebra there exist no non-zero GCR ideals in $A\underset{\alpha}{\otimes}B$ by Theorem 1 and 3, hence $A\underset{\alpha}{\otimes}B$ is an $NGCR$ algebra. The only if part of the assertion (e) is an easy consequnce of the if part of (d). Direct proofs of the if parts of the assertions (a) and (b) are due to Wulfsohn [19]. But, before going into our final discussions we shall quote here the definitions of a C^*-algebra with continuous trace and a GTC algebra following Dixmier [4].

Let $Tr\pi(a)$ be the trace of the operator $\pi(a)$ for an irreducible representation π of a C^*-algebra A and a positive element a in A. Since $Tr\pi(a)$ only depends on the unitary equivalent class of π, $Tr\pi(a)$ is considered a function on $\widehat{A}$, the dual space of A. In the following we shall identify an irreducible representation π with $\hat{\pi}$, the unitary equivalence class to which π belongs. Let $\mathfrak{p}$ be the set of all positive elements a in A such that $Tr\pi(a)$ is a finite continuous function on $\widehat{A}$, then there exists a self-adjoint two-sided ideal $\mathfrak{m}(A)$ in A whose positive part coincides with $\mathfrak{p}$. Put $J(A)=\overline{\mathfrak{m}(A)}$, the closure of $\mathfrak{m}(A)$. A is called a C^*-algebra with continuous trace if $J(A)=A$. On the other hand, generally we can find a composition series $\{I_\lambda\}_{0\leqq\lambda\leqq\lambda_0}$ of a closed ideal I_{λ_0} in A such as $J(A/I_{\lambda_0})=0$ and $J(A/I_\lambda)=I_{\lambda+1}/I_\lambda$. If $I_{\lambda_0}=A$, we call A a GTC algebra and $\{I_\lambda\}$ the canonical composition series of A. We shall use the following characterization of GTC algebras by [4; proposition 12 and 13], that is, *a C^*-algebra A is a GTC algebra if and only if there exists a well ordered ascending series $\{U_\lambda\}$ of open sets in $\widehat{A}$, beginning with null set and ending with $\widehat{A}$, and such that if λ is a limit ordinal U_λ is the union of the preceding U's and each point in $U_{\lambda+1}-U_\lambda$ admits a fundamental system of closed neighbrohoods in $\widehat{A}-U_\lambda$.*

LEMMA 4. *Let φ be a pure state of A (resp. B). Then $R_\varphi(J(A\underset{\alpha}{\otimes}B))\subset J(B)$ (resp. $L_\varphi(J(A\underset{\alpha}{\otimes}B))\subset J(A)$).*

PROOF. It is sufficient to show that $R_\varphi(\mathfrak{m}(A\underset{\alpha}{\otimes}B)^+)\subset\mathfrak{m}(B)^+$ where the sign "+" indicates positive parts of algebras. We choose a complete orthonormal basis $\{\xi_i\}$ in H_{π_φ} so that $\varphi={}^t\pi_\varphi(\omega_{\xi_0})$. Let ψ be a pure state of B and $\{\eta_k\}$ be a complete orthonormal basis in H_{π_ψ}. Note that $\{\xi_i\otimes\eta_k\}$ is a complete orthonormal basis in $H_{\pi_\varphi}\otimes H_{\pi_\psi}$. For an arbitrary element a in $\mathfrak{m}(A\underset{\alpha}{\otimes}B)^+$,

$$Tr(\pi_\varphi\otimes\pi_\psi(a))=\sum_{i,k}(\pi_\varphi\otimes\pi_\psi(a)\xi_i\otimes\eta_k,\ \xi_i\otimes\eta_k)<\infty$$

by assumptions. Therefore,

$$\begin{aligned}Tr(\pi_\varphi\otimes\pi_\psi(a))&=\sum_i\sum_k(\pi_\varphi\otimes\pi_\psi(a)\xi_i\otimes\eta_k,\ \xi_i\otimes\eta_k)\\&=\sum_i\sum_k<\pi_\varphi\otimes\pi_\psi(a),\ \omega_{\xi_i\otimes\eta_k}>=\sum_i\sum_k<\pi_\varphi\otimes\pi_\psi(a),\ \omega_{\xi_i}\otimes\omega_{\eta_k}>\end{aligned}$$

$$=\sum_i\sum_k < a,\ {}^t\pi_\varphi(\omega_{\xi_i})\ \otimes {}^t\pi_\psi(\omega_{\eta_k}) > = \sum_i\sum_k < R_{{}^t\pi_\varphi(\omega_{\xi_i})}(a),\ {}^t\pi_\psi(\omega_{\eta_k}) >$$

$$=\sum_i\sum_k (\pi_\psi(R_{{}^t\pi_\varphi(\omega_{\xi_i})}(a))\eta_k,\ \eta_k) = \sum_i Tr(\pi_\psi(R_{{}^t\pi_\varphi(\omega_{\xi_i})}(a))).$$

Hence,

$$Tr(\pi_\psi(R_\varphi(a))) = Tr(\pi_\varphi\otimes\pi_\psi(a)) - \sum_{i\neq 0} Tr(\pi_\psi(R_{{}^t\pi_\varphi(\omega_{\xi_i})}(a))).$$

Now the first term of right hand is continuous in $\pi_\varphi\otimes\pi_\psi \in \widehat{A\otimes_\alpha B}$, the dual space of $A\otimes_\alpha B$, hence continuous in π_ψ fixing π_φ, while the second term is the sum of positive lower semi-continuous functions (cf. [3; Proposition 5]) with variable π_ψ hence itself lower semi-continuous. Thus $Tr(\pi_\psi(R_\varphi(a)))$ is a finite upper semi-continuous function on $\hat{B}$ and as $Tr(\pi_\psi(R_\varphi(a)))$ is always lower semi-continuous in $\hat{B}$ ([3]) this concludes the proof. The proof for L_φ is almost the same.

Since R_φ and L_ψ are onto mappings Lemma 4 contains the only if part of the assertion (a).

At last, suppose that $A\otimes_\alpha B$ is a GTC-algebra. Take a pure state φ of A and let $\{J_\Lambda\}_{0\leq\Lambda\leq\Lambda_0}$ be a composition series of B derived by Lemma 2 from the canonical composition series $\{I_\lambda\}_{0\leq\lambda\leq\lambda_0}$ of $A\otimes_\alpha B$. The duals of all J's form a well ordered ascending series of open sets in $\hat{B}$, beginning with null set and ending with $\hat{B}$, and such that if Λ is a limit ordinal $\hat{J}_\Lambda$ is the union of the preceding $\hat{J}$'s. Take an element π_0 in $\hat{J}_{\Lambda+1}-\hat{J}_\Lambda$. The arguments in the proof of the last half part of Theorem 3 show that there exists an index λ satisfying $\overline{R_\varphi(I_{\lambda+1})} = J_{\Lambda+1}$ and $\overline{R_\varphi(I_\lambda)} = J_\Lambda$ and that $\pi_\varphi\otimes\pi_0$ belongs to $\hat{I}_{\lambda+1}-\hat{I}_\lambda$.

Now a GTC-algebra is a GCR algebra, hence a C^*-algebra of type I. Thus by (d) and [18; Theorem 4] we can identify the topological space $\widehat{A\otimes_\alpha B}$ with $\hat{A}\times\hat{B}$ (the identification goes through $\pi_1\otimes\pi_2 = (\pi_1, \pi_2)$). Let U be an arbitrary neighborhood of π_0 in $\hat{B}$. Denoting by V an open neighborhood of π_φ in $\hat{A}$ we get a neighborhood $V\times U$ of $\pi_\varphi\otimes\pi_0$. Hence there exists an open neighborhood $W(\subset V\times U)$ of $\pi_\varphi\otimes\pi_0$ such that

$$\overline{W\cap(\hat{A}\times\hat{B}-\hat{I}_\lambda)} \subset V\times U \cap (\hat{A}\times\hat{B}-\hat{I}_\lambda)$$

by the above cited result in [4]. For this neighborhood W we can find open neighborhoods V_1 and U_1 of π_φ and π_0, satisfying $V_1\times U \subset W$. Then one easily sees that

$$\overline{U_1 \cap (\hat{B} - \hat{J}_A)} \subset U \cap (\hat{B} - \hat{J}_A),$$

that is, π_0 admits in $\hat{B} - \hat{J}_A$ a fundamental system of closed neighborhoods. Therefore B is a GTC-algebra and similarly A is a GTC-algebra, too. Thus all proofs of Theorem 2 are completed.

2. Here we consider at first the types of general tensor products $A \underset{\beta}{\otimes} B$ of C^*-algebras A and B by a compatible norm β. A cross-norm β in $A \odot B$ is called a compatible norm if the completion of the normed $*$-algebra $A \odot B$ by β-norm becomes a C^*-algebra. That is, β is the norm satisfying the following conditions

$$\left\| \left(\sum_{i=1}^{n} a_i \otimes b_i \right) \left(\sum_{j=1}^{m} c_j \otimes d_j \right) \right\| \leqq \left\| \sum_{i=1}^{n} a_i \otimes b_i \right\| \quad \left\| \sum_{j=1}^{m} c_j \otimes d_j \right\|,$$

$$\left\| \left(\sum_{i=1}^{n} a_i^* \otimes b_i^* \right) \left(\sum_{i=1}^{n} a_i \otimes b_i \right) \right\| = \left\| \sum_{i=1}^{n} a_i \otimes b_i \right\|^2,$$

and $\|a \otimes b\| = \|a\| \|b\|$.

Among these norms Turumaru's α-norm is the smallest one (Takesaki [14; Theorem 2]). Hence a product functional $\varphi \otimes \psi$ is always continuous on $A \underset{\beta}{\otimes} B$. On the other hand, there exists the largest compatiale norm ν as shown in Guichardet [8]. $A \underset{\nu}{\otimes} B$ is nothing but the enveloping C^*-algebra of an involutive Banach algebra $A \underset{\gamma}{\otimes} B$, tensor product by γ-norm in the sense of Schatten [13] (cf. [11]). Following [14], we say that a C^*-algebra A has the property (T) if α-norm is the unique compatible norm in $A \odot B$ for any C^*-algebra B. A GCR algebra has the property (T) by [14; Theorem 3], hence there are no problems for compatible norms in the corresponding if parts of the assertions (a), (b), (c) and (d) in Theorem 2. On the other hand, suppose that $A \underset{\beta}{\otimes} B$ is a GCR algebra. Since $\beta \geqq \alpha$, $A \underset{\alpha}{\otimes} B$ becomes a homomorphic image of a GCR algebra, $A \underset{\beta}{\otimes} B$, hence itself a GCR algebra. Therefore both A and B are GCR algebras by the assertion (d) of Theorem 2 and a posteriori $\beta = \alpha$. As all classes of C^*-algebras in the assertions (a), (b) and (c) are GCR algebras, the above result shows that there are no distinguished points in the corresponding only if parts in (a), (b) and (c) for a compatible norm from those in Theorem 2. As to the assertion (e) the only if part holds for any compatible norm β. In fact, suppose that $A \underset{\beta}{\otimes} B$ is an $NGCR$ algebra and that both A and B have non-zero GCR ideals I and J respectively. Then the closure of $I \odot J$ in $A \underset{\beta}{\otimes} B$ is a non-zero GCR ideal by the above consideration, a contradiction. Hence either A or B

is an *NGCR* algebra. However the author does not know whether or not $A \underset{\beta}{\otimes} B$ is an *NGCR* algebra when A or B is an *NGCR* algebra.

Next, let K_1, K_2 and K be the largest *GCR* ideals in A, B and $A \underset{\alpha}{\otimes} B$ respectively. It is plausible that generally $K_1 \underset{\alpha}{\otimes} K_2 = K$, which will clarify the reason of the assertions (d) and (e) in Theorem 2. Unfortunately, we get only the following partial result.

THEOREM 4. *Let A and B be C^*-algebras. If either A or B has the property (T), then $K_1 \underset{\alpha}{\otimes} K_2 = K$ in $A \underset{\alpha}{\otimes} B$.*

PROOF. Assume that A has the property (T), i. e. $\alpha = \nu$ in any C^*-tensor product $A \underset{\nu}{\otimes} B$. It is shown in [8] that the kernel of the canonical homomorphism $A \underset{\nu}{\otimes} B$ to $A \underset{\nu}{\otimes} B/K_2$ induced by the quotient homomorphism $B \rightarrow B/K_2$ is $A \underset{\nu}{\otimes} K_2$. If $K_2 \neq B$, $A \underset{\nu}{\otimes} B/K_2 = A \underset{\alpha}{\otimes} B/K_2$ is an *NGCR* algebra by the assertion (e) in Theorem 2 and $A \underset{\nu}{\otimes} K_2 \supset K$. The case $K_2 = B$ also implies that $A \underset{\nu}{\otimes} K_2 = A \underset{\alpha}{\otimes} B \supset K$. Next, consider the homomorphism $A \underset{\nu}{\otimes} K_2 \rightarrow A/K_1 \underset{\nu}{\otimes} K_2$ induced by the quotient homomorphism $A \rightarrow A/K_1$. Then similar arguments as above show that its kernel $K_1 \underset{\nu}{\otimes} K_2$ (cf. [8]) contains K. As a *GCR* algebra has the property (T), $K_1 \underset{\nu}{\otimes} K_2$ coincides with $K_1 \underset{\alpha}{\otimes} K_2$ and $K_1 \underset{\alpha}{\otimes} K_2 \supset K$, while the converse inclusion is seen from the assertion (d) in Theorem 2.

As for $J(A)$, $J(B)$ and $J(A \underset{\alpha}{\otimes} B)$ it can be shown that if $K_1 \underset{\alpha}{\otimes} K_2 = K$ holds in $A \underset{\alpha}{\otimes} B$ we have $J(A) \underset{\alpha}{\otimes} J(B) = J(A \underset{\alpha}{\otimes} B)$. We omit the proof.

3. Let π be an irreducible representation of $A \underset{\alpha}{\otimes} B$ on H_π. As it is known, π induces canonically factor representations π_1 and π_2 of A and B on H_π such that $\pi(a \otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$ (cf. [7]) and there exists a natural isomorphism between the algebraic generation by $\pi_1(A)$ and $\pi_2(B)$ and $\pi_1(A) \odot \pi_2(B)$ (cf. for example [1; Chap. 1 §2, Exercise 6]). Hence transposing the norm in the algebraic generation by $\pi_1(A)$ and $\pi_2(B)$ to $\pi_1(A) \odot \pi_2(B)$ we get the tensor product $\pi_1(A) \underset{\beta}{\otimes} \pi_2(B)$ which is necessarily isomorphic to $\pi(A \underset{\alpha}{\otimes} B)$. These represenattions π_1 and π_2 are called restrictons of π to A and B. Now let I be a proper closed ideal in $A \underset{\alpha}{\otimes} B$ and π be an irreducible representation of $A \underset{\alpha}{\otimes} B$ vanishing on I. Denote by φ and ψ pure states of $\pi_1(A)$ and $\pi_2(B)$ and by θ_π the canonical isomorphism between $\pi(A \underset{\alpha}{\otimes} B)$ to $\pi_1(A) \underset{\beta}{\otimes} \pi_2(B)$, then an easy calculation shows that ${}^t\pi \cdot {}^t\theta_\pi(\varphi \otimes \psi) = {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi)$ and we get the following result.

"*For each proper closed ideal I in $A \underset{\alpha}{\otimes} B$, we can find a pure state ${}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi)$ vanishing on I*".

Combinning this with Theorem 1 we can show that the α-tensor product of simple C^*-algebras is also simple. In fact, let A and B be both simple C^*-algebras and I be a proper closed ideal in $A \underset{\alpha}{\otimes} B$. There exists a pure state $\varphi_0 \otimes \psi_0$ vanishing on I, and $<I, \varphi_0 \otimes \psi_0> = <R_{\varphi_0}(I), \psi_0> = 0$. Hence $\overline{R_{\varphi_0}(I)}$ is a proper closed ideal in B by Lemma 1, and $R_{\varphi_0}(I) = 0$. Therefore for an arbitrary pure state ψ of B,

$$<L_\psi(I), \varphi_0> = <I, \varphi_0 \otimes \psi> = <R_{\varphi_0}(I), \psi> = 0.$$

As A is also simple, this means $\overline{L_\psi(I)} = 0$ i.e. $L_\psi(I) = 0$. Hence $I = 0$ by Theorem 1.

This fact is proved at first in [14].

Now the question naturally arises; when does the family of all (product) pure states $\varphi \otimes \psi$ of $A \underset{\alpha}{\otimes} B$ separates closed ideals in $A \underset{\alpha}{\otimes} B$? This question means that to what extent a closed ideal in $A \underset{\alpha}{\otimes} B$ is determined by its components in A and B and that, in other words, to what extent the quotient algebras of $A \underset{\alpha}{\otimes} B$ are compatible with Fubini type theorem described in Theorem 1. So we set the following definition.

DEFINITION. *$A \underset{\alpha}{\otimes} B$ is called to satisfy the condition (F) if the family $\{\varphi \otimes \psi \mid \varphi \in P(A)$ and $\psi \in P(B)\}$ separates all closed ideals in $A \underset{\alpha}{\otimes} B$.*

Before going into the structure theorem of the algebra satisfying condition (F) we note that a similar argument as in the first part of this section shows that there exists a canonical isomorphism θ_π between the image $\pi(A \underset{\alpha}{\otimes} B)$ of a factor representation π of $A \underset{\alpha}{\otimes} B$ and $\pi_1(A) \underset{\beta}{\otimes} \pi_2(B)$ such as

$$\theta_\pi \circ \pi \left(\sum_{i=1}^{n} a_i \otimes b_i \right) = \theta_\pi \left(\sum_{i=1}^{n} \pi_1(a_i) \pi_2(b_i) \right) = \sum_{i=1}^{n} \pi_1(a_i) \otimes \pi_2(b_i)$$

where π_1 and π_2 are restrictions of π to A and B.

THEOREM 5. *The following statements for $A \underset{\alpha}{\otimes} B$ are equivalent;*

1. *$A \underset{\alpha}{\otimes} B$ satisfies the condition (F),*

2. *for each closed ideal I in A and J in B, the kernel of the homomorphism $A \underset{\alpha}{\otimes} B \to A/I \underset{\alpha}{\otimes} B/J$ canonically induced by the quotient homomorphisms $A \to A/I$ and $B \to B/J$ is given by $I \underset{\alpha}{\otimes} B + A \underset{\alpha}{\otimes} J$.*

3. *for any factor representation π of $A \underset{\alpha}{\otimes} B$, $\pi(A \underset{\alpha}{\otimes} B)$ is canonically isomorphic to the α-tensor product $\pi_1(A) \underset{\alpha}{\otimes} \pi_2(B)$ where π_1 and π_2 are restrictions of π to A and B.*

If either A or B is of type I, $A\otimes_{\alpha}B$ satisfies the condition (F) as seen from the condition 3 and [14; Theorem 3]. Hence Theorem 5 explains the back of Theorem 1 in [19]. Moreover, since the homomorphic images of the C^*-algebra which is an inductive limit of C^*-subalgebras of type I in the sense of Takeda [20] are also algebras of the same type, we can see by Theorem 5 in [14] and the condition 3 that $A\otimes_{\alpha}B$ satisfies the condition (F) if A or B is an inductive limit of C^*-subalgebras of type I.

PROOF OF THEOREM 5. The implication $1\rightarrow 2$. Let I and J be closed ideals in A and B and θ_1, θ_2 be the quotient homomorphism $A\rightarrow A/I$ and $B\rightarrow B/J$. Denote by $\theta_1\otimes\theta_2$ the canonical homomorphism $A\otimes_{\alpha}B\rightarrow A/I\otimes_{\alpha}B/J$ induced by them. Clearly the kernel of $\theta_1\otimes\theta_2$, $(\theta_1\otimes\theta_2)^{-1}(0)$, contains the ideal $I\otimes_{\alpha}B+A\otimes_{\alpha}J$. Suppose that $\varphi\otimes\psi\,|\,I\otimes_{\alpha}B+A\otimes_{\alpha}J=0$ where $\varphi\in P(A)$ and $\psi\in P(B)$. We have $\varphi\,|\,I=0$ and $\psi\,|\,J=0$, so that φ and ψ induce pure states φ' and ψ' on A/I and B/J. Then for an arbitrary element x in $(\theta_1\otimes\theta_2)^{-1}(0)$,

$$<x,\varphi\otimes\psi> \;=\; <x,{}^t\theta_1(\varphi')\otimes{}^t\theta_2(\psi')> \;=\; <x,{}^t(\theta_1\otimes\theta_2)(\varphi'\otimes\psi')>=0.$$

Therefore the condition 1 implies that $(\theta_1\otimes\theta_2)^{-1}(0)=I\otimes_{\alpha}B+A\otimes_{\alpha}J$.

The implication $2\rightarrow 3$. Let π be a factor representation of $A\otimes_{\alpha}B$ and π_1, π_2 its restrictions to A and B. As we said above, there exists a compatible norm β in $\pi_1(A)\odot\pi_2(B)$ such that $\pi(A\otimes_{\alpha}B)$ is isomorphic to $\pi_1(A)\otimes_{\beta}\pi_2(B)$ by θ_π. Let ρ be the canonical homomorphism from $\pi_1(A)\otimes_{\beta}\pi_2(B)$ to $\pi_1(A)\otimes_{\alpha}\pi_2(B)$. The composed homomorphism $\rho\circ\theta_\pi\circ\pi$ is nothing but the product homomorph'sm from $A\otimes_{\alpha}B$ to $\pi_1(A)\otimes_{\alpha}\pi_2(B)$ induced by π_1 and π_2. As the latter is a composition of the homomorphism $A\otimes_{\alpha}B\rightarrow A/\pi_1^{-1}(0)\otimes_{\alpha}B/\pi_2^{-1}(0)$ and the isomorphism between $A/\pi_1^{-1}(0)\otimes_{\alpha}B/\pi_2^{-1}(0)$ and $\pi_1(A)\otimes_{\alpha}\pi_2(B)$, the kernel of $\rho\circ\theta_\pi\circ\pi$ is $\pi_1^{-1}(0)\otimes_{\alpha}B+A\otimes_{\alpha}\pi_2^{-1}(0)$ by the assumption 2. On the other hand,

$$\pi_1^{-1}(0)\otimes_{\alpha}B+A\otimes_{\alpha}\pi_2^{-1}(0)\subset\pi^{-1}(0)\subset\text{the kernel of }\rho\circ\theta_\pi\circ\pi.$$

Hence $\pi^{1}(0)=\pi_1^{-1}(0)\otimes_{\alpha}B+A\otimes_{\alpha}\pi_2^{-1}(0)$, and ρ is an isomorphism, that is, $\beta=\alpha$.

The implication $3\rightarrow 1$. Let I and J be distinct closed ideals in $A\otimes_{\alpha}B$. We may assume that I is not contained in J. Take an irreducible representation π of $A\otimes_{\alpha}B$ such that $\pi(I)\neq 0$ and $\pi(J)=0$ and let π_1 and π_2 be its restrictions. We have $\theta_\pi\circ\pi(I)\neq 0$ in $\pi_1(A)\otimes_{\alpha}\pi_2(B)$ and we can find pure states φ and ψ of $\pi_1(A)$ and $\pi_2(B)$ such as $\varphi\otimes\psi\,|\,\theta_\pi\circ\pi(I)\neq 0$. Thus,

$${}^t\pi\circ{}^t\theta_\pi(\varphi\otimes\psi)\,|\,I={}^t\pi_1(\varphi)\otimes{}^t\pi_2(\psi)\,|\,I\neq 0\text{ and }{}^t\pi_1(\varphi)\otimes{}^t\pi_2(\psi)\,|\,J=0.$$

This completes the proof.

In the case that both A and B are separable C^*-algebras the situation is described also in the following manner. We shall show its outline.

Let $\Omega(A), \Omega(B)$ and $\Omega(A \underset{\alpha}{\otimes} B)$ be structure spaces of A, B and $A \underset{\alpha}{\otimes} B$ i.e. spaces of all primitive ideals with hull kernel topology. We shall define the mapping Φ from $\Omega(A \underset{\alpha}{\otimes} B)$ to the product space $\Omega(A) \times \Omega(B)$. Let P be a primitive ideal in $A \underset{\alpha}{\otimes} B$ and take an irreducible representation π of $A \underset{\alpha}{\otimes} B$ such as $\pi^{-1}(0) = P$. Its restrictions π_1 and π_2 are factor representations and both C^*-algebras $\pi_1(A)$ and $\pi_2(B)$ have no ideal divisors. Hence $\pi_1^{-1}(0) \in \Omega(A)$ and $\pi_2^{-1}(0) \in \Omega(B)$. Both ideals $\pi_1^{-1}(0)$ and $\pi_2^{-1}(0)$ do not depend on the choice of the representation π and we get the mapping Φ defined as $\Phi(P) = (\pi_1^{-1}(0), \pi_2^{-1}(0))$. Φ is a continuous onto mapping from $\Omega(A \underset{\alpha}{\otimes} B)$ to $\Omega(A) \times \Omega(B)$. On the other hand, let P and Q be primitive ideals in A and B and consider the irreducible representations π_1 and π_2 such as $\pi_1^{-1}(0) = P$ and $\pi_2^{-1}(0) = Q$. The representation $\pi_1 \otimes \pi_2$ is airreducible and $(\pi_1 \otimes \pi_2)^{-1}(0) \in \Omega(A \underset{\alpha}{\otimes} B)$ and again the kernel $(\pi_1 \otimes \pi_2)^{-1}(0)$ does not depend on the choice of π_1 and π_2, so that we can define the mapping Ψ from $\Omega(A) \times \Omega(B)$ into $\Omega(A \underset{\alpha}{\otimes} B)$ by $\Psi(P, Q) = (\pi_1 \otimes \pi_2)^{-1}(0)$. The mapping Ψ is also continuous.

Now the composed mapping $\Phi \cdot \Psi$ is the identity map in $\Omega(A) \times \Omega(B)$. While, *"the composed mapping $\Psi \circ \Phi$ is the identity map in $\Omega(A \underset{\alpha}{\otimes} B)$ if and only if $A \underset{\alpha}{\otimes} B$ satisfies the condition (F)."* In this case, $\Omega(A \underset{\alpha}{\otimes} B)$ is homomorphic with the product space $\Omega(A) \times \Omega(B)$.

The relation between the condition (F) and the property (T) is the following; if the homomorphic images of a C^*-algebra A having the property (T) have always the property (T), then $A \underset{\alpha}{\otimes} B$ satisfies the condition (F), by 3 of Theorem 5, for any C^*-algebra B. However it is not known whether the homomorphic images of a C^*-algbera having the property (T) have also the property (T) or not.

We note at last that if $A \underset{\alpha}{\otimes} B$ satisfies the condition (F) the equality $K_1 \underset{\alpha}{\otimes} K_2 = K$ holds where K_1, K_2 and K are the largest GCR ideals in A, B and $A \underset{\alpha}{\otimes} B$ used in Theorem 4. In fact, if a pure state $\varphi \otimes \psi$ ($\varphi \in P(A)$, $\psi \in P(B)$) vanishes on $K_1 \underset{\alpha}{\otimes} K_2$ we have $\varphi | K_1 = 0$ or $\psi | K_2 = 0$, and

$$< x, \varphi \otimes \psi > = < R_\varphi(x), \psi > = < L_\psi(x), \varphi > = 0$$

for every $x \in K$ because $R_\varphi(K) \subset K_2$ and $L_\psi(K) \subset K_1$ by Theorem 3. Since $K_1 \underset{\alpha}{\otimes} K_2 \subset K$, this implies the conclusion.

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