

NON-COMMUTATIVE EXTENSION OF LUSIN'S THEOREM

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1. Introduction. Many important theorems in measure theory have been extended to operator algebras by many authors, especially, Dixmier [1], Dye and Segal. Considered as non-commutative extensions, those are interesting themselves and provide powerful tools in the further investigations of operator algebras. The purpose of this paper is to extend Lusin's theorem which is an important tool in measure theory into general operator algebra.

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2. Notations and Definitions. Let M be a W^* -algebra, namely, C^* -algebra with a dual structure as a Banach space, M_* be the predual of M , that is, the Banach space of all bounded normal functionals on M , and M_*^+ , the positive part of M_* , that is, the set of all functionals φ in M_* such that $\varphi(x^*x) \geq 0$ for all $x \in M$. We may consider the s^* -topology, that is, the topology defined by a family of semi-norms $\{\alpha_\varphi, \alpha_\varphi^*; \varphi \in M_*^+, \text{ where } \alpha_\varphi(x) = \varphi(x^*x)^{1/2}, \text{ and } \alpha_\varphi^*(x) = \varphi(xx^*)^{1/2} \text{ for all } x \in M\}$, and the s -topology is that defined by a family of semi-norms $\{\alpha_\varphi; \varphi \in M_*^+\}$. In [4, p. 1.64] Sakai shows that whenever M is represented as a weakly closed algebra of operators on some Hilbert space, the weak*-topology of M coincides with the weak operator topology on the bounded sets of M . It follows from this that the s^* -topology coincides with the strong *-operator topology on bounded sets of M , and the s -topology coincides with the strong operator topology on bounded sets of M .

3. Main theorems. The following theorem corresponds to the Egoroff theorem in the Lebesgue integration.

THEOREM 1. (Density theorem) *Let M be a W^* -algebra and M_* the predual of M , moreover, let φ be any positive functional in M_* . Let N be any set in S (the unit sphere of M), which is adherent to an element a in S in the s^* -topology. Then for any positive number ε , and a projection e in M , there exist a projection e_0 in M and a sequence $\{a_i\}_{i=1}^\infty \subset N$ such that $e \geq e_0$, $\varphi(e - e_0) < \varepsilon$ and $\lim_{i \rightarrow \infty} \|a_i e_0 - a e_0\| = 0$. In particular, for any sequence*

$\{a_n\}_{n=1}^\infty$ in S , which converges to a in s^* -topology, there exist a projection $e_0 \in M$ and a countable subsequence $\{a_{n_i}\}_{i=1}^\infty$ of $\{a_n\}_{n=1}^\infty$ such that $\varphi(e - e_0) < \varepsilon$, $e \geq e_0$, and $\|a_{n_i}e_0 - ae_0\| \rightarrow 0$ (as $i \rightarrow \infty$).

As a non-commutative extension of Lusin's theorem in the usual Lebesgue integration, we have

THEOREM 2. *Let N be a C^* -algebra with the identity 1 acting on some Hilbert space, M the weak closure of N , a be any element in M and e any projection in M . Then, for any positive functional φ in M_* and any positive numbers ε and δ (< 1), there exist a projection f in M , $f \leq e$, b in N such that $\varphi(e - f) < \varepsilon$, and $af = bf$ and $\|b\| \leq (1 + \delta)\|af\|$. [5].*

Moreover, if a is an hermitian element, then b may be chosen hermitian element such that $\|b\| \leq 2(1 + \delta)\|af\|$ and $\|b\| \leq \|a\| + \delta$, moreover, if a is a unitary operator, then b may be chosen unitary such that $\|b - 1\| < \|a - 1\| + \delta$.

4. Proof of Theorem 1. It is sufficient to prove only the case $a = 0$ and $e = 1$.

By the assumption, here is a net $\{a_\theta\}_{\theta \in \mathfrak{e}}$ in N which is convergent to 0 for s^* -topology, and $b_\theta = a_\theta^* a_\theta$ converges to 0 for s -topology and $\|b_\theta\| \leq 1$. Then we can choose a family of projections $\{e_\theta\}_{\theta \in \mathfrak{e}}$ in M such that $\lim_{\theta \in \mathfrak{e}} e_\theta = 1$ for s -topology and $\|b_\theta e_\theta\| \leq 1$ for each θ . In fact, let χ be the characteristic function of the interval $(-1, 1)$, and we define $e_\theta = \chi(b_\theta)$ for each θ . Then we have $b_\theta^2 \geq 1 - e_\theta$, and we see that the left member of the inequality converges to 0, for s -topology, so e_θ converges to 1, for s -topology. It is immediate that $\|b_\theta e_\theta\| \leq 1$ for each θ . Then, for a given $\varepsilon > 0$, there exists an index θ_1 such that $\varphi(1 - e_{\theta_1}) < \varepsilon/2$, and $\|b_{\theta_1} e_{\theta_1}\| \leq 1$, so that

$$\|a_{\theta_1} e_{\theta_1}\| = \|b_{\theta_1} e_{\theta_1}\|^{1/2} \leq 1.$$

Consider the family $\{a_\theta e_{\theta_1}; \theta \geq \theta_1\}$, then $a_\theta e_{\theta_1}$ converges to 0 for s^* -topology. Again denoting as $b'_\theta = e_{\theta_1} b_\theta e_{\theta_1}$ ($\in e_{\theta_1} M e_{\theta_1}$) and by the same way (but for 2^{-2}), we can choose a projection e_{θ_2} in $e_{\theta_1} M e_{\theta_1}$ such that $\varphi(e_{\theta_1} - e_{\theta_2}) < \varepsilon/2^2$, $\|b'_{\theta_2} e_{\theta_2}\| \leq 2^{-2}$, so that $\|a_{\theta_2} e_{\theta_2}\| = \|e_{\theta_2} b'_{\theta_2} e_{\theta_2}\|^{1/2} < 1/2$.

By the mathematical induction, we can choose a decreasing sequence of projections $\{e_{\theta_i}\}$ ($\theta_i \uparrow$) in M such that

$$\varphi(e_{\theta_{i-1}} - e_{\theta_i}) < \varepsilon/2^i \quad (e_{\theta_0} = 1) \text{ for each } i,$$

and

$$\|a_{\theta_i}e_{\theta_i}\| \leq 1/i.$$

Putting $e_0 = \inf_i e_{\theta_i}$, we have $\varphi(1 - e_0) = \varphi(\sup_i (1 - e_{\theta_i})) = \sup_i \varphi(1 - e_{\theta_i}) = \sum_{i=1}^{\infty} \varphi(e_{\theta_{i-1}} - e_{\theta_i}) = \sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon$, and $\|a_{\theta_i}e_{\theta_i}\| \leq 1/i$ for each i , hence $\lim_{i \rightarrow \infty} \|a_{\theta_i}e_{\theta_i}\| = 0$. This completes the proof.

For the proof of theorem 2, we need some lemmas.

LEMMA 1. *Let N be a C^* -algebra with the identity acting on some Hilbert space and M the weak closure of N (Note that M is a W^* -algebra.). Let φ be any functional in M_*^+ . Then for any element $a \in M$, any projection e in M and any positive numbers ε, δ , we can choose $a_1 \in N$ and a projection $f \in M$ such that $f \leq e$, $\|(a - a_1)f\| < \delta$, $\|a_1\| \leq \|ae\|$ and $\varphi(e - f) < \varepsilon$.*

PROOF. Since the unit sphere of N is adherent to the element ae in the s^* -topology by [2, theorem 1] (We may assume $\|ae\| = 1$ without loss of generality.), it is clear from theorem 1.

5. Proof of Theorem 2.

Case (i). General case. We may consider the case $\|ae\| = 1$ without loss of generality.

We can take a positive functional φ_0 in M_*^+ such that $\varphi_0((ae)^*(ae))^{1/2} \geq 1 - \delta$ and $\varphi_0(1) = 1$. Put $\bar{\varphi} = \varphi + \varphi_0$. Then, by lemma 1, we can choose a_1 in N and a projection e_1 in M such that $\|(a - a_1)e_1\| < \delta/2$, $\|a_1\| \leq \|ae\| = 1$ and $\bar{\varphi}(e - e_1) < \min(\varepsilon/2, \delta^2/2)$, and $e_1 \leq e$. For $(a - a_1)e_1$ in M , again by the same lemma, there exist a projection e_2 in M $e_2 \leq e_1$ and $a_2 \in N$ such that

$$\bar{\varphi}(e_1 - e_2) < \min(\varepsilon/2^2, \delta^2/2^2),$$

$$\|(a - a_1)e_1 - a_2\| = \|(a - a_1 - a_2)e_2\| < \delta/2^2$$

and

$$\|a_2\| \leq \|(a - a_1)e_1\|.$$

By the mathematical induction, we can choose a decreasing sequence of projections $\{e_i\}_{i=1}^{\infty}$ in M and $\{a_i\}_{i=1}^{\infty}$ in N such that

$$\|(a - \sum_{j=1}^{k-1} a_j)e_{k-1} - a_k\| < \delta/2^k,$$

$$\|a_k\| \leq \|(a - \sum_{j=1}^{k-1} a_j)e_{k-1}\|,$$

and $\overline{\varphi}(e_{k-1} - e_k) < \min(\varepsilon/2^k, \delta^2/2^k)$ (where $e_0 = e$).

Since $\|a_k\| \leq \|(a - \sum_{j=1}^{k-1} a_j)e_{k-1}\| < \delta/2^{k-1}$ ($k \geq 2$), we have $\sum_{k=1}^{\infty} \|a_k\| \leq 1 + \sum_{k=2}^{\infty} \delta/2^{k-1} < 1 + \delta < \infty$. If we define b as $\sum_{j=1}^{\infty} a_j$ (in N), then $b \in N$ and $\|b\| \leq 1 + \delta$.

Putting $f = \inf_i e_i$, f is a projection in M and $\|(a - \sum_{j=1}^k a_j)f\| < \delta/2^k$ for all k .

Hence $\|(a-b)f\| = 0$, that is, $af = bf$.

As $\overline{\varphi}(e-f) < \min(\varepsilon, \delta^2) < \varepsilon$, we have $\varphi(e-f) < \varepsilon$ and $\|af\| \geq (1-3\delta)\|b\|$; in fact, we have $\|af\| \geq \varphi_0((af)^*(af))^{1/2} \geq \varphi_0((ae)^*(ae))^{1/2} - \varphi_0(\{ae(e-f)\}^*\{ae(e-f)\})^{1/2}$. Since $\varphi_0(e-f) \leq \delta^2$ and $\varphi_0(\{ae(e-f)\}^*\{ae(e-f)\})^{1/2} \leq \|ae\|\delta = \delta$, we have $\|af\| \geq 1-2\delta$. Noting that $\|ae\|=1$ and $1+\delta \geq \|b\|$, we obtain $\|af\| \geq 1-2\delta \geq (1-3\delta)\|b\|$.

Case (ii): a is an hermitian element of M . Firstly we can choose an hermitian element a_1 and a projection e_1 in M such that $e_1 \leq e$, $\|(a-a_1)e_1\| < \delta/2^2$, $\|a_1\| \leq \|a\|$, $\|a_1\| \leq 2\|ae\|$, and $\overline{\varphi}(e-e_1) < \min(\varepsilon/2, \delta^2/2)$.

Case (ii, a) $2\|ae\| \geq \|a\|$: As $\{x; x \in N, \|x\| \leq \|a\|, x \text{ is hermitian}\}$ is adherent to the element a for s -topology, there exists a net $\{a_\theta\}_{\theta \in \mathfrak{o}}$ such that a_θ converges to a for s -topology. Hence $a_\theta e$ converges to ae for s^* -topology. By theorem 1, there exist a projection $e_1 (e_1 \leq e)$, and an hermitian element a_1 in N such that $\|(a-a_1)e_1\| < \delta/2^2$, $\|a_1\| \leq \|a\| \leq 2\|ae\|$, and $\overline{\varphi}(e-e_1) < \min(\varepsilon/2, \delta^2/2)$.

Case (ii, b) $\|a\| \geq 2\|ae\|$: As $\{x; x \in N, \|x\| \leq \|c_0\|, x \text{ is hermitian}\}$ is adherent to the element c_0 for s -topology (where $c_0 = eae + (1-e)ae + ea(1-e)$ and note that $c_0 e = ae$), by [2] and our lemma 1, there are an hermitian element $a_1 \in N$ and a projection $e_1 (e_1 \leq e)$ in M such that $\|(a-a_1)e_1\| < \delta/2^2$, $\|a_1\| \leq \|c_0\|$, $\|c_0\| \leq 2\|ae\|$, and $\overline{\varphi}(e-e_1) < \min(\varepsilon/2, \delta^2/2)$. Hence we can choose an hermitian element a_1 (in N) and a projection e_1 in M such that $\|(a-a_1)e_1\| < \delta/2^2$, $\|a_1\| \leq \|a\|$ and $\|a_1\| \leq 2\|ae\|$.

Putting $c_1 = e_1(a-a_1)e_1 + (1-e_1)(a-a_1)e_1 + e_1(a-a_1)(1-e_1)$, c_1 becomes an hermitian element of M such that $(a-a_1)e_1 = c_1 e_1$ and $\|c_1\| \leq 2\|(a-a_1)e_1\|$. For c_1 , by the same reason, there exist an hermitian element a_2 in N , and a projection e_2 in M , $e_2 \leq e_1$ such that $\|a_2\| \leq \|c_1\|$, $\|(c_1-a_2)e_2\| \leq \delta/2^3$ and $\overline{\varphi}(e_1-e_2) < \min(\varepsilon/2^2, \delta^2/2^2)$. Since $(c_1-a_2)e_2 = (c_1-a_2)e_1e_2 = c_1e_1e_2 - a_2e_2 = (a-a_1-a_2)e_2$, we have that $\|(a-a_1-a_2)e_2\| < \delta/2^3$, hence by the same way as in case (i), we can choose a decreasing sequence of projections $\{e_i\}_{i=1}^{\infty}$ in M , a sequence of hermitian elements $\{a_i\}_{i=1}^{\infty}$ in N and a sequence of hermitian elements $\{c_i\}_{i=1}^{\infty}$ in M such that

$$\|(c_{i-1} - a_i)e_i\| < \delta/2^{i+1},$$

and

$$\|a_i\| \leq \|c_{i-1}\| \leq 2\|(a - \sum_{j=1}^{i-1} a_j)e_{i-1}\|,$$

where

$$c_{i-1} = e_{i-1}(a - \sum_{j=1}^{i-1} a_j)e_{i-1} + (1 - e_{i-1})(a - \sum_{j=1}^{i-1} a_j)e_{i-1} + e_{i-1}(a - \sum_{j=1}^{i-1} a_j)(1 - e_{i-1})$$

and

$$\bar{\varphi}(e_{i-1} - e_i) < \min(\varepsilon/2^i, \delta^2/2^i), \quad (e_0 = e).$$

As $\|a_i\| \leq 2\|(a - \sum_{j=1}^{i-1} a_j)e_{i-1}\| < \delta/2^i (i \geq 2)$, we have $\sum_{i=1}^{\infty} \|a_i\| \leq \min(\|a\|, 2\|ae\|) + \delta$.

Now we define b as $\sum_{j=1}^{\infty} a_j$ (in N), then $b \in N$, and $\|b\| \leq \min(\|a\|, 2\|ae\|) + \delta$.

Further, putting $f = \inf_{i \geq 1} e_i$, f is a projection in M and as $(c_{i-1} - a_i)e_i = (a - \sum_{j=1}^i a_j)e_i$, we have $\|(a - \sum_{j=1}^i a_j)f\| < \delta/2^{i+1}$ for all i . It follows $\|(a - b)f\| = 0$, that is, $af = bf$. By the same way as in case (i), we have $\|af\| \geq \|ae\|(1 - 2\delta)$. Noting that $\|b\| \leq 2\|ae\| + \delta$, we have $\|af\| \geq \|ae\|(1 - 2\delta) \geq 2\|b\|(1 - 3\delta)$.

Case (iii): $a = u$ and u is a unitary element in M . We need the following lemma, which we prove by making use of an argument of Riesz-Nagy [3; p. 266 Theorem].

LEMMA 2. *Let M be a W^* -algebra, e be a projection in M , and moreover, w be a unitary in M such that $\|(1 - w)e\| < 1/8$, then we can choose a unitary v in M as follows:*

$$ve = we,$$

and

$$\|1 - v\| \leq 7\|(1 - w)e\|.$$

PROOF. Putting $wew^* = f$, we have $\|e - f\| = \|e - wew^*\| = \|e - we + we - wew^*\| = \|(1 - w)e + w\{(1 - w)e\}^*\| \leq 2\|(1 - w)e\| < 1/4$. Next, putting $a = 1 + (1 - e)(e - f)(1 - e)$ and $u = (1 - f)a^{-1/2}(1 - e)$, we have $u^*u = 1 - e$, $uu^* = 1 - f$. In fact, it follows from our hypothesis that $\|(1 - e)(e - f)(1 - e)\| < 1$, and

that consequently the hermitian element $a = 1 + (1 - e)(e - f)(1 - e)$ is strictly positive. Hence a^{-1} and $a^{-1/2} = (a^{-1})^{1/2}$ exist. Consider the elements $u = (1 - f)a^{-1/2}(1 - e)$. $u^* = (1 - e)a^{-1/2}(1 - f)$. Since we obviously have that $1 - e$ commutes with a , we also have $(1 - e)a^{-1/2} = a^{-1/2}(1 - e)$, and since furthermore $(1 - e)(1 - f)^2(1 - e) = (1 - e)(1 - f)(1 - e) = (1 - e) + (1 - e)(e - f)(1 - e) = (1 - e)a$, it follows that

$$\begin{aligned} u^*u &= (1 - e)a^{-1/2}(1 - f)^2a^{-1/2}(1 - e) = a^{-1/2}(1 - e)aa^{-1/2} \\ &= (1 - e)a^{-1/2}aa^{-1/2} = 1 - e. \end{aligned}$$

Moreover uu^* is a projection majorized by $1 - f$. If $\xi \perp (uu^*)\mathfrak{H}$ (we assume that N acts on some Hilbert space \mathfrak{H}), $\xi \in (1 - f)\mathfrak{H}$, that is $(\xi, u\eta) = 0$ for all $\eta \in \mathfrak{H}$. Then $u^*\xi = 0$; hence $(1 - e)(1 - f)\xi = a^{1/2}a^{-1/2}(1 - e)(1 - f)\xi = a^{1/2}(1 - e)a^{-1/2}(1 - f)\xi = a^{1/2}u^*\xi = 0$ and, consequently $(e - f)(1 - f)\xi = (1 - f)\xi$. In view of the hypothesis $\|e - f\| < 1$, this equation is possible only if $(1 - f)\xi = 0$, that is $uu^* = 1 - f$. Then we have

$$\begin{aligned} \|(1 - u)(1 - e)\| &= \|(1 - e) - (1 - f)a^{-1/2}(1 - e)\| \\ &= \|(1 - e) - (e - f)a^{-1/2}(1 - e) - a^{-1/2}(1 - e)\| \\ &= \|(1 - a^{-1/2})(1 - e) - (e - f)a^{-1/2}(1 - e)\| \\ &\leq \| (1 - a^{-1/2}) \| + \| a^{-1/2} \| \| e - f \|. \end{aligned}$$

Noting that if $|x| < 1/4$ (where x is a real number), then $|(x + 1)^{-1/2} - 1| \leq |x|$ and $(x + 1)^{-1/2} < 2$, we have $\|(1 - u)(1 - e)\| \leq \|e - f\| + 2\|e - f\| \leq 6\|(1 - w)e\|$.

Putting $v = we + u(1 - e)$, v is a unitary in M such that $ve = we$. Combining the above estimations, we have inequalities

$$\begin{aligned} \|1 - v\| &= \|1 - we - u(1 - e)\| = \|e + 1 - e - we - u(1 - e)\| \\ &\leq \|(1 - w)e + (1 - u)(1 - e)\| \leq 7\|(1 - w)e\|. \end{aligned}$$

Thus the lemma follows.

LEMMA 3. Let N be a C^* -algebra with the identity acting on some Hilbert space, M be the weak closure of N (Observe that M becomes a W^* -algebra). Let φ be any positive functional in M_*^+ . Then for any unitary u in M , any projection e in M , and any positive number ε, δ , we can choose unitary v in N and a projection f in M , $f \leq e$ such that

$$\|(u - v)f\| < \delta, \|1 - v\| \leq \|1 - u\| \text{ and } \varphi(e - f) < \varepsilon.$$

PROOF. By [2], $\{v; v \text{ is unitary in } N, \|1 - v\| \leq \|1 - u\|\}$ is s^* -dense in

$\{w; w$ is unitary in $M, \|1-w\| \leq \|1-u\|\}$. Hence, it is clear from lemma 1.

LEMMA 4. *Let $\{u_i\}_{i=1}^\infty$ be a family of unitary operators on some Hilbert space such that $\sum_{i=1}^\infty \|1-u_i\| < \infty$, then $\lim_{n \rightarrow \infty} \prod_{i=1}^n u_i$ exists for uniform operator topology and the limit is also a unitary operator.*

PROOF. Since, for each pair of positive integers n and p , $\prod_{i=1}^{n+p} u_i - \prod_{i=1}^n u_i = \prod_{i=1}^n u_i (1 - u_{n+1} + u_{n+1} - u_{n+1}u_{n+2} + u_{n+1}u_{n+2} - \dots - \prod_{i=n+1}^{n+p} u_i)$ we have $\left\| \prod_{i=1}^n u_i - \prod_{i=1}^{n+p} u_i \right\| \leq \sum_{i=n+1}^{n+p} \|1-u_i\|$.

By the hypothesis, the left side of the above inequality converges to 0 as $n \rightarrow \infty$. Hence $\lim_{n \rightarrow \infty} \prod_{i=1}^n u_i$ exists for uniform operator topology and the limit is also a unitary. The lemma follows.

Proof of case (iii). We may assume $e=1$ without loss of generality. For any unitary element u in M , any positive functional φ in M_* , and any given $\varepsilon > 0$ and $1 > \delta > 0$, we can choose families of unitary elements $\{u_i\}_{i=1}^\infty$ in N , $\{v_i\}_{i=1}^\infty$ in M , and a decreasing sequence of projections $\{f_i\}_{i=1}^\infty$ in M such that

$$u f_i = \prod_{j=1}^i u_j v_j f_i, \quad \|(1-u_i^* v_{i-1}) f_i\| \leq \|(v_{i-1} - u_i) f_i\| < \delta/8 \cdot 2^{i-1}$$

$$\varphi(f_{i-1} - f_i) < \varepsilon/2^i \quad (f_0=1) \text{ for each } i,$$

and

$$\|1-u_i\| \leq \|1-v_{i-1}\| \leq 7\|(1-u_{i-1}^* v_{i-2}) f_{i-1}\| \quad (i \geq 2), \quad (v_0 = u).$$

In fact, by lemma 3, there exists a unitary element u_1 in N , and a projection f_1 in M such that $\|1-u_1\| \leq \|1-u\|$, $\varphi(1-f_1) < \varepsilon/2$, and $\|(u-u_1) f_1\| < \delta/8$. For $u_1^{-1}u$ (unitary in M), by lemma 2, there exists a unitary v_1 in M such that $v_1 f_1 = u_1^{-1}u f_1$ and $\|1-v_1\| \leq 7\|(1-u_1^{-1}u) f_1\| < (7/8)\delta < \delta$. Then for unitary v_1 in M , and a projection f_1 in M , by lemma 3, there exist a unitary u_2 in N and a projection f_2 in M , $f_2 \leq f_1$ such that $\|1-u_2\| \leq \|1-v_1\|$, $\varphi(f_1-f_2) < \varepsilon/2^2$, and $\|(v_1-u_2) f_2\| < \delta/8 \cdot 2$. For $u_2^{-1}v_1$, (unitary in M), by lemma 2, we can choose a unitary v_2 in M such that $v_2 f_2 = u_2^{-1}v_1 f_2$ and $\|1-v_2\| \leq 7\|(1-u_2^{-1}v_1) f_2\| < \delta/2$. Thus, by mathematical induction, we can choose families of unitary elements $\{u_i\}_{i=1}^\infty \subset N$, $\{v_i\}_{i=1}^\infty \subset M$ and a decreasing sequence of projections $\{f_i\}_{i=1}^\infty$ in M as follows :

$$\begin{aligned}
 uf_i &= \prod_{j=1}^i u_j v_j f_i, \quad \|(1-u_i^* v_{i-1})f_i\| \leq \| (v_{i-1} - u_i) f_i \| < \delta/8 \cdot 2^{i-1}, \\
 \varphi(f_{i-1} - f_i) &< \varepsilon/2^i \quad (f_0=1) \text{ for each } i,
 \end{aligned}$$

and

$$\|1 - u_i\| \leq \|1 - v_{i-1}\| \leq 7\|1 - u_{i-1}^* v_{i-2}\| f_{i-1} \quad (i \geq 2).$$

Putting $f = \inf_i f_i$, we have that

$$uf = uf_n f = \prod_{i=1}^n u_i v_n f_n f = \prod_{i=1}^n u_i v_n f.$$

Moreover, as $\|1 - u_i\| \leq 7\|(1 - u_{i-1}^* v_{i-2})f_{i-1}\| < (7/8)(\delta/2^{i-2}) < \delta/2^{i-2}$, and $\sum_{i=1}^\infty \|1 - u_i\| \leq \|1 - u\| + 2\delta < \infty$, it follows from Lemma 4 that $\lim_{n \rightarrow \infty} \prod_{i=1}^n u_i$ exists for uniform operator topology (we denote it by \bar{u} , \bar{u} is a unitary in N). Moreover, we have

$$\begin{aligned}
 \|(uf - \bar{u}f)\| &\leq \left\| \left(\prod_{i=1}^n u_i \right) v_n f - \left(\prod_{i=1}^n u_i \right) f \right\| + \left\| \left(\prod_{i=1}^n u_i \right) f - \bar{u}f \right\| \\
 &< \delta/2^{n-1} + \left\| \prod_{i=1}^n u_i - \bar{u} \right\| \text{ for each } n,
 \end{aligned}$$

hence, $\|uf - \bar{u}f\| = 0$, that is, $uf = \bar{u}f$, and

$$\begin{aligned}
 \left\| 1 - \prod_{i=1}^n u_i \right\| &= \|1 - u_1 + u_1 - u_1 u_2 + u_1 u_2 - u_1 u_2 u_3 + \dots - \prod_{i=1}^n u_i\| \\
 &\leq \|1 - u_1\| + \|1 - u_2\| + \dots + \left\| \prod_{i=1}^{n-1} u_i (1 - u_n) \right\| \\
 &< \|1 - u_1\| + \|1 - u_2\| + \dots + \|1 - u_n\| + \dots \\
 &< \|1 - u\| + 2\delta,
 \end{aligned}$$

for all n . Therefore, $\|1 - \bar{u}\| < \|1 - u\| + 2\delta$. This completes the proof of theorem 2.

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