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OPERATING FUNCTIONS ON SOME SUBSPACES OF l_p

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1. Let $L^2(0, 2\pi)$ be the set of all square integrable functions defined on $(0, 2\pi)$ and continued by periodicity. We set

$$A_{\beta,\delta}(f) = \left[\int_0^1 \frac{dt}{t^{2-\beta/2+\delta}}\left\{\int_0^{2\pi} |f(x+t) - f(x-t)|^2\, dx\right\}^{\beta/2}\right]^{1/\beta}$$

for $f \in L^2(0, 2\pi)$, where $1 \leqq \beta \leqq 2$ and $3\beta/2 - 1 > \delta > \beta/2 - 1$.

We define a space $A_{\beta,\delta}$ by

$$A_{\beta,\delta} = \{f\colon A_{\beta,\delta}(f) < \infty\}\,.$$

If $f \in A_{\beta,\delta}$ and $f_a(x) = f(x-a)$, then $A_{\beta,\delta}(f_a) = A_{\beta,\delta}(f)$, if c is a constant, then $A_{\beta,\delta}(cf) = |c|A_{\beta,\delta}(f)$ and if $f,\ g \in A_{\beta,\delta}$, then $A_{\beta,\delta}(f+g) \leqq A_{\beta,\delta}(f) + A_{\beta,\delta}(g)$ by Minkowski's inequality.

We shall characterize the complex valued function φ of a complex variable which operates in $A_{\beta,\delta}$ i.e. $\varphi(f) \in A_{\beta,\delta}$ for all $f \in A_{\beta,\delta}$, where $\varphi(f)(x) = \varphi(f(x))$.

2. Let the Fourier series of $f \in L^2(0, 2\pi)$ be

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}\,.$$

For β and δ which satisfy the above conditions, we set

$$B_{\beta,\delta}(f) = \left\{\sum_{n=1}^{\infty} n^{-\beta/2+\delta}\left(\sum_{|k|>n} |c_k|^2\right)^{\beta/2}\right\}^{1/\beta}$$

$$C_{\beta,\delta}(f) = \left\{\sum_{n=1}^{\infty} n^{-3\beta/2+\delta}\left(\sum_{|k|\leqq n} |c_k|^2 k^2\right)^{\beta/2}\right\}^{1/\beta}\,.$$

We can prove a following theorem by the same method as Prof. G. Sunouchi in [2].

THEOREM 1. *For* $1 \leqq \beta \leqq 2$ *and* $\beta/2-1<\delta<3\beta/2-1$, *the finiteness of* $A_{\beta,\delta}(f)$, $B_{\beta,\delta}(f)$ *and* $C_{\beta,\delta}(f)$ *are equivalent each other.*

In the proof of Theorem 1, we use the fact that the convergency of $B_{\beta,\delta}(f)$ and $C_{\beta,\delta}(f)$ are equivalent to the convergency of $B'_{\beta,\delta}(f)$ and $C'_{\beta,\delta}(f)$ respectively where

$$B'_{\beta,\delta}(f) = \left\{\sum_{n=1}^{\infty} 2^{n(1-\beta/2+\delta)} \Big(\sum_{|k|>2^n} |c_k|^2\Big)^{\beta/2}\right\}^{1/\beta}$$

$$C'_{\beta,\delta}(f) = \left\{\sum_{n=1}^{\infty} 2^{n(1-3\beta/2+\delta)} \Big(\sum_{|k|\leqq 2^n} |c_k|^2 k^2\Big)^{\beta/2}\right\}^{1/\beta}.$$

For the sake of simplicity, we omit the proof.

3. THEOREM 2. *Let* β *and* δ *be numbers satisfying the above conditions.*

(i) *For* $1-\beta+\delta>0$, φ *operates in* $A_{\beta,\delta}$ *if and only if* φ *satisfies locally the Lipschitz condition.*
(ii) *For* $1-\beta+\delta=0$, *if* φ *operates in* $A_{\beta,\delta}$, *then* φ *satisfies locally the Lipschitz condition. Moreover if* $\beta=1$, *the condition is necessary and sufficient.*
(iii) *For* $1-\beta+\delta<0$, φ *operates in* $A_{\beta,\delta}$ *if and only if* φ *satisfies the Lipschitz condition.*

Dr. S. Igari [1] proved the cases of $\beta=1$ and $\delta=0$ in (ii) and $\beta=2$. Our method of proof is inspired by Igari's paper.

LEMMA 1. *If* $f\in A_{\beta,\delta}$ *and* $f(x)\sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$, *then we have*

$$\sum_{n=-\infty}^{\infty} |c_n|^\beta |n|^\delta < \infty .$$

PROOF. Since $2/\beta>1$, by Hölder's inequality we have

$$\sum_{n=-\infty}^{\infty} |c_n|^\beta |n|^\delta = \sum_{k=1}^{\infty} \sum_{|n|=2^{k-1}}^{2^k-1} |c_n|^\beta |n|^\delta$$

$$\leqq \sum_{k=1}^{\infty} 2^{k\delta} \sum_{|n|=2^{k-1}}^{2^k-1} |c_n|^\beta$$

$$\leqq \sum_{k=1}^{\infty} 2^{k\delta} \left(\sum_{|n|=2^{k-1}}^{2^k-1} |c_n|^2 \right)^{\beta/2} \left(\sum_{|n|=2^{k-1}}^{2^k-1} 1 \right)^{1-\beta/2}$$

$$\leqq 2^\delta \sum_{k=0}^{\infty} 2^{k(1-\beta/2+\delta)} \left(\sum_{|n|=2^k}^{\infty} |c_n|^2 \right)^{\beta/2}$$

$$\leqq C\{B'_{\beta,\delta}(f)\}^\beta .$$

where C is a constant. By Theorem 1 the proof is complete.

PROOF OF SUFFICIENCY OF THEOREM 2. In (i) and (ii), we may show that $f \in A_{\beta,\delta}$ is bounded. If $\beta = 1$ in (i) and (ii), $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ by Lemma 1 and hence f is bounded. If $\beta \neq 1$ in (i), we have

$$\sum_{n=-\infty}^{\infty} |c_n| \leqq \left(\sum_{n=-\infty}^{\infty} |c_n|^\beta |n|^\delta \right)^{1/\beta} \left(\sum_{n=-\infty}^{\infty} |n|^{-\delta/(\beta-1)} \right)^{1-1/\beta}$$

by Hölder's inequality. The right side is convergent by Lemma 1 and then f is bounded.

In (iii) the sufficiency of the condition is clear.

$M_{\beta,\delta}$, M_β, etc. will denote constants depending on only the indices, not always the same in each occurrence.

LEMMA 2. *Let $\eta(x)$ be a continuous function which is equal to* 1 *on* $[-a, a]$, *equal to zero outside of* $(-a-\varepsilon, a+\varepsilon)$ *and linear otherwise where* $0 < a < \pi/4$, $0 < \varepsilon < 1/2$. *Then*

$$A_{\beta,\delta}(\eta) \leqq \begin{cases} M_{\beta,\delta}/\varepsilon^{(1-\beta+\delta)/\beta} & \text{if } 1-\beta+\delta > 0 \\ M_{\beta,\delta}\{\log(1/\varepsilon)\}^{1/\beta} & \text{if } 1-\beta+\delta = 0 . \end{cases}$$

PROOF. If $0 \leqq t \leqq \varepsilon/2$, then we have

$$|\eta(x+t) - \eta(x-t)| \leqq 2t/\varepsilon$$

for $-a-\varepsilon-t \leqq x \leqq -a+t$ and $a-t \leqq x \leqq a+\varepsilon+t$. If $\varepsilon/2 \leqq t \leqq 1$, then we have

$$|\eta(x+t)-\eta(x-t)|\leqq 1$$

for $-a-\varepsilon-t\leqq x\leqq -a-t$ and $a-t\leqq x\leqq a+\varepsilon+t$. And we have $|\eta(x+t)-\eta(x-t)|=0$ otherwise. Therefore

$$A^{\beta}_{\beta,\delta}(\eta)\leqq\int_0^{\varepsilon/2}\frac{dt}{t^{2-\beta/2+\delta}}\left\{\left(\frac{2t}{\varepsilon}\right)^2 2(2t+\varepsilon)\right\}^{\beta/2}+\int_{\varepsilon/2}^1\frac{dt}{t^{2-\beta/2+\delta}}\{2(2t+\varepsilon)\}^{\beta/2}$$

$$\leqq M_\beta\left\{\frac{1}{\varepsilon^{\beta/2}}\int_0^{\varepsilon/2}\frac{dt}{t^{2-3\beta/2+\delta}}+\int_{\varepsilon/2}^1\frac{dt}{t^{2-\beta+\delta}}\right\}.$$

But we have $-1<2-(3\beta/2)+\delta<1$ from the conditions of β and δ. If $1-\beta+\delta>0$, we have easily

$$A^{\beta}_{\beta,\delta}(\eta)\leqq M_{\beta,\delta}/\varepsilon^{1-\beta+\delta}.$$

If $1-\beta+\delta=0$, we have

$$A^{\beta}_{\beta,\delta}(\eta)\leqq M_{\beta,\delta}\log(1/\varepsilon).$$

LEMMA 3. *Let $f(x)$ be a function which is equal to* 1 *on* $[-a,a]$, *equal to zero outside of* $(-a-\varepsilon,a+\varepsilon)$ *and allowed to take arbitrary value otherwise, where* $0<\varepsilon<a/2<\pi/8$. *Then*

$$A_{\beta,\delta}(f)\geqq\begin{cases}M_{\beta,\delta,a}/\varepsilon^{(1-\beta+\delta)/\beta} & \textit{if}\ 1-\beta+\delta>0\\ \{\log(a/\varepsilon)\}^{1/\beta} & \textit{if}\ 1-\beta+\delta=0.\end{cases}$$

PROOF. If $\varepsilon/2\leqq t\leqq a$, we have $f(x+t)=1$ and $f(x-t)=0$ for $-a-t\leqq x\leqq -a-\varepsilon+t$. Therefore

$$A^{\beta}_{\beta,\delta}(f)\geqq\int_{\varepsilon/2}^{a}\frac{dt}{t^{2-\beta/2+\delta}}\left\{\int_{-a-t}^{-a-\varepsilon+t}dx\right\}^{\beta/2}\geqq\int_{\varepsilon}^{a}\frac{(2t-\varepsilon)^{\beta/2}}{t^{2-\beta/2+\delta}}dt.$$

But $(2t-\varepsilon)^{\beta/2}\geqq t^{\beta/2}$ when $\varepsilon\leqq t\leqq a$. Therefore we have

$$A^{\beta}_{\beta,\delta}(f)\geqq\int_{\varepsilon}^{a}\frac{dt}{t^{2-\beta+\delta}}\geqq\begin{cases}M_{\beta,\delta,a}/\varepsilon^{1-\beta+\delta} & \text{if}\ 1-\beta+\delta>0\\ \log(a/\varepsilon) & \text{if}\ 1-\beta+\delta=0.\end{cases}$$

LEMMA 4. *Let $\eta(x)$ be the same function as it in Lemma* 2. *Then for $f\in A_{\beta,\delta}$ we have*

$$A_{\beta,\delta}(\eta f) \leqq A_{\beta,\delta}(f) + M_{\beta,\delta}(f)/\varepsilon^{(1-\beta/2+\delta)/\beta}\,.\,{}^{*)}$$

PROOF. By Minkowski's inequality we have

$$\begin{aligned} A_{\beta,\delta}(\eta f) &= \Big[\int_0^1 \frac{dt}{t^{2-\beta/2+\delta}}\Big(\int_{-\pi}^{\pi} |\eta(x+t)\{f(x+t)-f(x-t)\} \\ &\qquad + f(x-t)\{\eta(x+t)-\eta(x-t)\}|^2 dx\Big)^{\beta/2}\Big]^{1/\beta} \\ &\leqq \Big[\int_0^1 \frac{dt}{t^{2-\beta/2+\delta}}\Big\{\int_{-\pi}^{\pi} |f(x+t)-f(x-t)|^2 dx\Big\}^{\beta/2}\Big]^{1/\beta} \\ &\qquad + \Big[\int_0^1 \frac{dt}{t^{2-\beta/2+\delta}}\Big\{\int_{-\pi}^{\pi} |f(x-t)|^2 |\eta(x+t)-\eta(x-t)|^2 dx\Big\}^{\beta/2}\Big]^{1/\beta} \\ &= A_{\beta,\delta}(f) + I^{1/\beta} \quad \text{say.} \end{aligned}$$

By the same method in Lemma 2 we have

$$\begin{aligned} I &\leqq \int_0^{\varepsilon/2} \frac{dt}{t^{2-\beta/2+\delta}}\Big\{\int_{-\pi}^{\pi} |f(x-t)|^2 \Big(\frac{2t}{\varepsilon}\Big)^2 dx\Big\}^{\beta/2} + \int_{\varepsilon/2}^1 \frac{dt}{t^{2-\beta/2+\delta}}\Big\{\int_{-\pi}^{\pi} |f(x-t)|^2 dx\Big\}^{\beta/2} \\ &= M_\beta \|f\|_2^\beta \Big\{\frac{1}{\varepsilon^\beta}\int_0^{\varepsilon/2} \frac{dt}{t^{2-3\beta/2+\delta}} + \int_{\varepsilon/2}^1 \frac{dt}{t^{2-\beta/2+\delta}}\Big\}. \end{aligned}$$

We note $1-(\beta/2)+\delta > 0$ and $1-(3\beta/2)+\delta < 0$, then

$$I \leqq M_{\beta,\delta}(f)/\varepsilon^{1-\beta/2+\delta}\,.$$

Therefore we have

$$A_{\beta,\delta}(\eta f) \leqq A_{\beta,\delta}(f) + M_{\beta,\delta}(f)/\varepsilon^{(1-\beta/2+\delta)/\beta}\,.$$

PROOF OF NECESSITY OF THEOREM 2. Let $\xi(x)$ be a continuous function which is equal to 1 on $[-1, 1]$, equal to zero outside of $(-3/2, 3/2)$ and linear otherwise. For $k = 1, 2, \cdots$, we set

$$\xi_k(x) = \xi\{(x-2^{-k})\,2^{k+4}\}$$

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*) $M_{\beta,\delta}(f)$ denotes a constant depending on β, δ and f.

$$\eta_k(x) = \xi\{(x-2^{-k})\, 2^{k+3}\}$$

$$I_k = \{x\,;\, \xi_k(x) = 1\} = [-2^{-k-4} + 2^{-k}, 2^{-k-4}+2^{-k}]\,.$$

For $f \in A_{\beta,\delta}$ and $z \in \boldsymbol{C}$ (the field of complex numbers) we set

$$\Phi_z(f)(x) = \varphi\{f(x) + z\} - \varphi(z)\,.$$

Then $\Phi_z(f) \in A_{\beta,\delta}$ since φ operates in $A_{\beta,\delta}$.

Firstly we shall show the necessity of (i) and (ii). Our proof is divided into four parts.

(I) *For every $z \in \boldsymbol{C}$, there exist two positive constants α_z and M_z, and an interval I_z such that $A_{\beta,\delta}\{\Phi_z(f)\} \leqq M_z$ if $A_{\beta,\delta}(f) \leqq \alpha_z$ and the support of f is in I_z.*

PROOF. Suppose that the statement is false. Then there exists a sequence of functions f_k such that

$$A_{\beta,\delta}(f_k) \leqq 1/k^2\,,\ \operatorname{supp} f_k \subset I_k$$

and

$$A_{\beta,\delta}\{\Phi_z(f_k)\} \geqq k\, 2^{k(1-\beta/2+\delta)/\beta}\,.$$

Since the supports of f_k are disjoint each other, there exists $f = \sum_{k=1}^{\infty} f_k$ and we have

$$A_{\beta,\delta}(f) \leqq \sum_{k=1}^{\infty} A_{\beta,\delta}(f_k) \leqq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty\,.$$

But

$$\xi_k \Phi_z(f) = \Phi_z(f_k)\,,$$

and hence by Lemma 4

$$A_{\beta,\delta}\{\Phi_z(f_k)\} = A_{\beta,\delta}\{\xi_k \Phi_z(f)\}$$
$$\leqq A_{\beta,\delta}\{\Phi_z(f)\} + M_{\beta,\delta}\{\Phi_z(f)\}\, 2^{k(1-\beta/2+\delta)/\beta}\,.$$

When k is large enough, the inequality contradicts the condition of $A_{\beta,\delta}\{\Phi_z(f_k)\}$.

(II) *φ is bounded on every compact set.*

PROOF. We can choose $a>0$ and $\varepsilon>0$ of a function $\eta(x)$ in Lemma 1 such that supp $\eta\subset I_z$. If $A_{\beta,\delta}(z'\eta)\leqq\alpha_z$, i.e. $|z'|\leqq\alpha_z/A_{\beta,\delta}(\eta)$, then by (I) we have $M_z\geqq A_{\beta,\delta}\{\Phi_z(z'\eta)\}$. By supp $\eta\subset I_z$, we have $\Phi_z(z'\eta)(x)=\varphi\{z'\eta(x)+z\}-\varphi(z)$, and hence

$$\Phi_z(z'\eta)(x)=\begin{cases}\varphi(z'+z)-\varphi(z) & \text{if} \quad \eta(x)=1\\ 0 & \text{if} \quad \eta(x)=0.\end{cases}$$

Therefore we can write

$$\Phi_z(z'\eta)(x)=f(x)\{\varphi(z'+z)-\varphi(z)\}$$

where $f(x)=1$ if $\eta(x)=1$ and $f(x)=0$ if $\eta(x)=0$. By Lemma 3

$$A_{\beta,\delta}\{\Phi_z(z'\eta)\}\geqq|\varphi(z'+z)-\varphi(z)|\begin{cases}M_{\beta,\delta,a}/\varepsilon^{(1-\beta+\delta)/\beta} & \text{if} \quad 1-\beta+\delta>0\\ \{\log(a/\varepsilon)\}^{1/\beta} & \text{if} \quad 1-\beta+\delta=0.\end{cases}$$

Consequently $\varphi(z+z')$ is bounded for $|z'|\leqq\alpha_z/A_{\beta,\delta}(\eta)$, and hence it is bounded on every compact set.

(III) *For every* $z\in\boldsymbol{C}$, *there exist two positive constants* α'_z *and* M'_z *and an interval* I'_z *such that* $A_{\beta,\delta}\{\Phi_{z+z'}(f)\}\leqq M'_z$ *if* $A_{\beta,\delta}(f)\leqq\alpha'_z$, supp $f\subset I_z$ *and* $|z'|\leqq\alpha'_z$.

PROOF. Conversely suppose that there exist two sequences of functions f_k and complex numbers z_k such that

$$A_{\beta,\delta}(f_k)\leqq 1/k^2\,,\ \operatorname{supp} f_k\subset I_k,\ |z_k|\leqq 1/k^2\,A_{\beta,\delta}(\eta_k)$$

and

$$A_{\beta,\delta}\{\Phi_{z+z'}(f_k)\}\geqq k\,2^{k(1-\beta/2+\delta)/\beta}.$$

We set $f=\sum_{k=1}^{\infty}f_k+\sum_{k=1}^{\infty}z_k\eta_k$. Then

$$A_{\beta,\delta}(f)\leqq\sum_{k=1}^{\infty}A_{\beta,\delta}(f_k)+\sum_{k=1}^{\infty}|z_k|A_{\beta,\delta}(\eta_k)$$

$$\leqq\sum_{k=1}^{\infty}\frac{1}{k^2}+\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty.$$

Therfore we have $f\in A_{\beta,\delta}$. Now

$$
\begin{aligned}
\xi_k \Phi_z(f) &= \xi_k \Phi_z(f_k + z_k) \\
&= \xi_k \{\varphi(f_k + z_k + z) - \varphi(z)\} \\
&= \xi_k \Phi_{z+z_k}(f_k) + \xi_k \{\varphi(z_k + z) - \varphi(z)\} \\
&= \Phi_{z+z_k}(f_k) + \xi_k \{\varphi(z_k + z) - \varphi(z)\}
\end{aligned}
$$

and hence by Lemma 4

$$
\begin{aligned}
A_{\beta,\delta}\{\Phi_{z+z_k}(f_k)\} &\leqq A_{\beta,\delta}\{\xi_k \Phi_z(f)\} + |\varphi(z_k+z) - \varphi(z)| A_{\beta,\delta}(\xi_k) \\
&\leqq A_{\beta,\delta}\{\Phi_z(f)\} + M_{\beta,\delta}\{\Phi_z(f)\} 2^{k(1-\beta/2+\delta)/\beta} \\
&\quad + M_{\beta,\delta} |\varphi(z_k+z) - \varphi(z)| \begin{cases} 2^{k(1-\beta+\delta)/\beta} & \text{if } 1-\beta+\delta > 0 \\ (\log 2^k)^{1/\beta} & \text{if } 1-\beta+\delta = 0 . \end{cases}
\end{aligned}
$$

By (II) $|\varphi(z+z_k) - \varphi(z)|$ is bounded. This implies the contradiction.

(IV) *For every $z \in \boldsymbol{C}$, φ satisfies the Lipschitz condition in a neighbourhood of z.*

PROOF. We can choose $a > 0$ of $\eta(x)$ in Lemma 2 such that $\operatorname{supp} \eta \subset I'_z$ for all $\varepsilon \in (0, a/2)$. Let the function $\eta(x)$ denote by $\eta_\varepsilon(x)$. We note that the number a depends on only z. If $|z'| \leqq \alpha'_z$ and

$$
|z' - z''| \leqq \alpha'_z / M_{\beta,\delta} \begin{cases} (2/a)^{(1-\beta+\delta)/\beta} & \text{if } 1-\beta+\delta > 0 \\ \{\log(2/a)\}^{1/\beta} & \text{if } 1-\beta+\delta = 0 , \end{cases}
$$

then we can choose ε such that

$$
0 < \varepsilon < a/2 \quad \text{and} \quad \alpha'_z = |z' - z''| M_{\beta,\delta} \begin{cases} 1/\varepsilon^{(1-\beta+\delta)/\beta} & \text{if } 1-\beta+\delta > 0 \\ \{\log(2/\varepsilon)\}^{1/\beta} & \text{if } 1-\beta+\delta = 0 . \end{cases}
$$

Let $\eta_\varepsilon(x)$ for this ε denote by $\eta(x)$. By Lemma 2 we have

$$
\begin{aligned}
A_{\beta,\delta}\{(z' - z'')\eta\} &= |z' - z''| A_{\beta,\delta}(\eta) \\
&\leqq |z' - z''| M_{\beta,\delta} \begin{cases} 1/\varepsilon^{(1-\beta+\delta)/\beta} & \text{if } 1-\beta+\delta > 0 \\ \{\log(1/\varepsilon)\}^{1/\beta} & \text{if } 1-\beta+\delta = 0 \end{cases} \\
&= \alpha'_z
\end{aligned}
$$

and hence by (III)

$$M_z' \geqq A_{\beta,\delta}[\Phi_{z+z'}\{(z''-z')\eta\}] = A_{\beta,\delta}[\varphi\{(z''-z')\,\eta+z+z'\} - \varphi(z+z')] .$$

But we have

$$\varphi\{(z''-z')\,\eta(x) + z+z'\} - \varphi(z+z') = \begin{cases} \varphi(z''+z)-\varphi(z+z') & \text{if} \quad \eta(x) = 1 \\ 0 & \text{if} \quad \eta(x) = 0 \end{cases}$$

and therefore we write

$$\varphi\{(z''-z')\,\eta(x)+z+z'\} - \varphi(z+z') = f(x)\{\varphi(z''+z) - \varphi(z+z')\}$$

where $f(x) = 1$ if $\eta(x) = 1$ and $f(x) = 0$ if $\eta(x) = 0$. Therefore by Lemma 3

$$M_z' \geqq |\varphi(z''+z) - \varphi(z+z')|\,A_{\beta,\delta}(f)$$

$$\geqq |\varphi(z''+z) - \varphi(z+z')| \begin{cases} M_{\beta,\delta,a}/\varepsilon^{(1-\beta+\delta)/\beta} & \text{if} \quad 1-\beta+\delta > 0 \\ \{\log(a/\varepsilon)\}^{1/\beta} & \text{if} \quad 1-\beta+\delta = 0 \end{cases}$$

$$\geqq \frac{|\varphi(z''+z)-\varphi(z+z'')|}{|z'-z''|} M_{\beta,\delta,a}\,\alpha_z'.$$

Constants in the above inequality are independent of z' and z'', and hence φ satisfies the Lipschitz condition in a neighbourhood of z.

Thus proof of necessity of (i) and (ii) is complete.

Nextly we shall show the necessity of (iii). The proof is divided in three steps.

(I) *For every interval $I \subset [-\pi, \pi]$ and every positive number a, there exists a finite sum E of intervals in I such that $a = A_{\beta,\delta}(\chi_E)$ where χ_E is the characteristic function of E.*

PROOF. We shall first show that

$$\sup_E A_{\beta,\delta}(\chi_E) = \infty$$

where E runs all the finite sums of intervals in I.

Suppose that for all finite sums E of intervals in I

$$A_{\beta,\delta}(\chi_E) \leqq K_{\beta,\delta} < \infty ,$$

where $K_{\beta,\delta}$ is a constant independent of E. If f is a step function such that $0 \leqq f \leqq 1$, then $f = \sum \alpha_i \chi_{E_i}$ where $\alpha_i \geqq 0$ and $\sum \alpha_i \leqq 1$. Therefore we have

$$A_{\beta,\delta}(f) \leqq \sum \alpha_i A_{\beta,\delta}(\chi_{E_i}) \leqq K_{\beta,\delta}$$

and hence for any bounded measurable function f such that $\operatorname{supp} f \subset I$, we have

$$A_{\beta,\delta}(f) \leqq K_{\beta,\delta} \| f \|_\infty .$$

Now we may set $I = (-\varepsilon, \varepsilon)$. Let $f(x) = e^{iNx}$ for $x \in I$ and $f(x) = 0$ otherwise. Then we have

$$\begin{aligned} K^\beta_{\beta,\delta} &\geqq A^\beta_{\beta,\delta}(f) \\ &\geqq \int_0^\varepsilon \frac{dt}{t^{2-\beta/2+\delta}} \left\{ \int_{-\varepsilon+t}^{\varepsilon-t} | e^{iNx} e^{iNt} - e^{iNx} e^{-iNt} |^2 \, dx \right\}^{\beta/2} \\ &\geqq \int_0^\varepsilon \frac{dt}{t^{2-\beta/2+\delta}} \{ (4 \sin^2 Nt) \, 2(\varepsilon - t) \}^{\beta/2} . \end{aligned}$$

If $0 < t < 1/N$ for $N > 2/\varepsilon$, then $Nt < 1$ and hence $\sin Nt > cNt$ (c is a constant). Therefore

$$\begin{aligned} K^\beta_{\beta,\delta} &\geqq A^\beta_{\beta,\delta}(f) \\ &\geqq M_\beta \int_0^{1/N} \frac{dt}{t^{2-\beta/2+\delta}} \left\{ (\sin^2 Nt) \left(\varepsilon - \frac{1}{N} \right) \right\}^{\beta/2} \\ &\geqq M_\beta \int_0^{1/N} \frac{dt}{t^{2-\beta/2+\delta}} \left(\varepsilon - \frac{\varepsilon}{2} \right)^{\beta/2} dt \\ &= M_{\beta,\delta} \varepsilon^{\beta/2} N^{1-\beta/2+\delta} . \end{aligned}$$

This contradicts $1 - (\beta/2) + \delta > 0$, when N is sufficiently large. Therefore we have

$$\sup_E A_{\beta,\delta}(\chi_E) = \infty ,$$

and hence there exists a finite sum E of intervals in I such that $a < A_{\beta,\delta}(\chi_E) < \infty$. Now we set

$$I(h) = A_{\beta,\delta}(\chi_{E\cap(-\pi,h)})$$

and then $I(h)$ is continuous, $I(-\pi) = 0$ and $I(\pi) > a$. Consequently there exists h' such that $I(h') = a$. $E\cap(-\pi, h')$ satisfies the condition of (I).

(II) *There exist two positive constants M and α, and an interval I such that if* $\operatorname{supp} f\subset I$ *and* $A_{\beta,\delta}(f) \leqq \alpha$, *then* $A_{\beta,\delta}\{\varphi(f)\} \leqq M$.

PROOF. Conversely suppose that there exists a sequence of functions f_k such that

$$\operatorname{supp} f_k\subset I_k\,, \quad A_{\beta,\delta}(f_k) \leqq 1/k^2$$

and

$$A_{\beta,\delta}\{\varphi(f_k)\} \geqq k\, 2^{k(1-\beta/2+\delta)/\beta}\,.$$

We set $f = \sum_{k=1}^{\infty} f_k$, and then

$$A_{\beta,\delta}(f) \leqq \sum_{k=1}^{\infty} A_{\beta,\delta}(f_k) \leqq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty\,.$$

Therefore $f \in A_{\beta,\delta}$.

Without loss of generality we may assume $\varphi(0) = 0$. Then we have $\xi_k\varphi(f)=\varphi(f_k)$. Therefore by Lemma 3 we have

$$A_{\beta,\delta}\{\varphi(f_k)\} = A_{\beta,\delta}\{\xi_k\varphi(f)\} \leqq A_{\beta,\delta}\{\varphi(f)\} + M_{\beta,\delta}\{\varphi(f)\}2^{k(1-\beta/2+\delta)/\beta}.$$

This contradicts $A_{\beta,\delta}(f) < \infty$ when k is large enough.

(III) *φ satisfies the Lipschitz condition.*

PROOF. For fixed $z, z' \in \boldsymbol{C}$, by (I) there exists a finite sum E of intervals in I such that $A_{\beta,\delta}(z\chi_E) = \alpha/2$. Let J be an interval in E. Then there exists a finite sum F of intervals J such that $A_{\beta,\delta}(z'\chi_F) = \alpha/2$. Therefore by (II) we have

$$2M \geqq A_{\beta,\delta}\{\varphi(z\chi_E + z'\chi_F) - \varphi(z\chi_E)\}\,.$$

Since

$$\varphi(z\chi_E + z'\chi_F) - \varphi(z\chi_E) = \{\varphi(z+z') - \varphi(z)\}\chi_F ,$$

we have

$$2M \geqq |\varphi(z+z') - \varphi(z)| A_{\beta,\delta}(\chi_F) = \frac{|\varphi(z+z')-\varphi(z)|}{|z'|} \frac{\alpha}{2} .$$

This shows that φ satisfies the Lipschitz condition. Thus the proof of Theorem 2 is complete.

REMARK. For $\beta > 1$ and $1-\beta+\delta = 0$, there exists an unbounded function f belonging to $A_{\beta,\delta}$.

We set

$$f(x) = \sum_{n=2}^{\infty} \frac{\cos nx}{n(\log n)^{(1+\varepsilon)/\beta}}$$

where $\varepsilon > 0$ and $1+\varepsilon < \beta$. It is well-known that $f \in L^2(-\pi, \pi)$ and $f(x) \to \infty$ as $x \to 0$. We shall show that this function is in $A_{\beta,\delta}$. By Theorem 1, it is sufficient to show $C_{\beta,\delta}(f) < \infty$. Now by hypothesis we can write

$$C_{\beta,\delta}^{\beta}(f) = \sum_{n=1}^{\infty} n^{-1-\beta/2} \left(\sum_{|k| \leqq n} |c_k|^2 k^2 \right)^{\beta/2} .$$

Therefore we have

$$\begin{aligned} C_{\beta,\delta}^{\beta}(f) &= \sum_{n=1}^{\infty} n^{-1-\beta/2} \left(\sum_{|k|=2}^{n} \frac{k^2}{k^2(\log k)^{2(1+\varepsilon)/\beta}} \right)^{\beta/2} \\ &\leqq M_\beta \sum_{n=1}^{\infty} \frac{1}{n^{1+\beta/2}} \left(\frac{n}{(\log n)^{2(1+\varepsilon)/\beta}} \right)^{\beta/2} \\ &= M_\beta \sum_{n=1}^{\infty} \frac{1}{n(\log n)^{1+\varepsilon}} < \infty . \end{aligned}$$

Hence, in this case, our necessary condition is not sufficient.

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