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NEARLY NORMAL OPERATORS

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1. We say that a bounded linear operator T on a Hilbert space H is nearly normal if $T \leftrightarrow T^*T$ where the symbol $\leftrightarrow$ denotes commutativity. $R(T)$ denotes the smallest von Neumann algebra containing T and $R(T)'$ its commutant. The terminology of von Neumann algebras will be found to conform with [2]. In [4], N. Suzuki proved that $R(V)$ is of type I if V is an isometry.

The purpose of this note is to prove that $R(T)$ is also of type I if T is nearly normal. Clearly, isometries are nearly normal.

2. For our object, the following Lemma 1 and Lemma 4 are essential.

LEMMA 1. *If T is a nearly normal operator on H and if E is the projection from H on $\mathfrak{N}_T = \{x \in H\ ;\ Tx=0\}$, then $E \in R(T) \cap R(T)'$.*

PROOF. Clearly, $E \in R(T)$ and $\mathfrak{N}_T$ is invariant under T. Hence we have only to prove $\mathfrak{N}_T$ is invariant under T^*. For any $x \in \mathfrak{N}_T$, we have $\|TT^*x\|^2 = (TT^*x, TT^*x) = ((T^*T)T^*x, T^*x) = (T^*(T^*T)x, T^*x) = 0$ by the definition of nearly normal operators; hence $T^*x \in \mathfrak{N}_T$.

The following lemma is a modification of the results of A. Brown [1].

LEMMA 2. *If T is a nearly normal operator on H such that $\mathfrak{N}_T = (0)$, then, in the polar decomposition $T = V(T^*T)^{1/2}$ of T, V is an isometry and $V \leftrightarrow (T^*T)^{1/2}$.*

PROOF. $\mathfrak{N}_T = (0)$ means that the closure of $(T^*T)^{1/2}H$ is equal to H and this implies V is an isometry. On the other hand $T \leftrightarrow T^*T$ implies $\{(T^*T)^{1/2}V - V(T^*T)^{1/2}\}(T^*T)^{1/2} = 0$. Hence $V \leftrightarrow (T^*T)^{1/2}$.

If V is an isometry, then we have easily $V^m\mathfrak{N}_{V^*} \perp V^n\mathfrak{N}_{V^*}$ for all non-negative integers $m, n, m \neq n$; hence we have the following lemma.

LEMMA 3. *If V is an isometry on H and if E is the projection from H on $\bigoplus_{n=0}^{\infty} V^n\mathfrak{N}_{V^*}$, then $E \in R(V) \cap R(V)'$ and $V|(I-E)H$ is unitary, where $V|(I-E)H$ denotes the restriction of V on its reducing subspace $(I-E)H$*

and I denotes the identity operator on H.

PROOF. Clearly, $\bigoplus_{n=0}^{\infty} V^n \mathfrak{N}_{V^*}$ is a reducing subspace of V and hence $E \in R(V)'$. On the other hand, by the simple calculation, we have easily $\{\bigoplus_{n=0}^{\infty} V^n \mathfrak{N}_{V^*}\}^{\perp} = \{x \in H; V^n V^{*n} x = x \text{ for all } n = 0, 1, 2, \cdots\}$. This implies $E \in R(V)$. Therefore $E \in R(V) \cap R(V)'$. The last assertion is clear.

REMARK. This kind of the decomposition of isometries is already known. Indeed, it is a special case of the canonical decomposition of contractions [3].

LEMMA 4. *Let T be a nearly normal operator on H with the polar decomposition $T = V(T^*T)^{1/2}$ such that $\mathfrak{N}_T = (0)$. If E is a projection from H on $\bigoplus_{n=0}^{\infty} V^n \mathfrak{N}_{V^*}$, then $E \in R(T) \cap R(T)'$ and $T|(I-E)H$ is normal.*

PROOF. Lemma 2 guarantees that V is an isometry; hence $E \in R(V) \cap R(V)'$ by Lemma 3. Since $R(T) = R(V, (T^*T)^{1/2})$, $E \in R(T)$. On the other hand, by Lemma 2, $V \leftrightarrow T$. Since $E \in R(V) \cap R(V)'$, $E \leftrightarrow T$ and this means $E \in R(T)'$. Thus $E \in R(T) \cap R(T)'$. The last assertion is clear by Lemma 2 and Lemma 3.

THEOREM. *If T is a nearly normal operator on H such that $\mathfrak{N}_T = (0)$ and if $H = \bigoplus_{n=0}^{\infty} V^n \mathfrak{N}_{V^*}$ where V is the isometry in the polar decomposition $T = V(T^*T)^{1/2}$ of T, then $R(T)$ is of type I and $R(T) \cap R(T)' = R((T^*T)^{1/2})$.*

PROOF. Since V is an isometry by Lemma 2, $\dim V^n \mathfrak{N}_{V^*} = \dim \mathfrak{N}_{V^*}$ for all $n = 1, 2, \cdots$. Hence, for each subspace $V^n \mathfrak{N}_{V^*}$, there exists an isometric mapping U_n from $V^n \mathfrak{N}_{V^*}$ onto $\mathfrak{N}_{V^*}$. Let U be the direct sum $\bigoplus_{n=0}^{\infty} U_n$ and let ϕ be the mapping defined as follows; $\phi(T) = UTU^*$ for all $T \in B(H)$, where $B(H)$ denotes the full operator algebra on H. Then ϕ is clearly a spatial isomorphism from $B(H)$ on $B(\bigoplus_{n=0}^{\infty} M_n)$, where $M_n = \mathfrak{N}_{V^*}$ for all $n = 0, 1, 2, \cdots$.

For each non-negative integers n, the projection E_n from H on $V^n \mathfrak{N}_{V^*}$ belongs to $R(V)$ because $E_0 = I - VV^*$, $E_n = V^n V^{*n} - V^{n+1} V^{*n+1}$, $n = 1, 2, \cdots$. This means that each subspace $V^n \mathfrak{N}_{V^*}$ is a reducing subspace for all $S \in R(V)'$. And clearly,

$$\phi(V) = \begin{pmatrix} 0 & & & & 0 \\ I & 0 & & & \\ 0 & I & 0 & & \\ \cdot & 0 & I & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \text{ where } I \text{ denotes the identity operator on } \mathfrak{N}_{V^*}.$$

Therefore, by the simple calculation, we have

$$\phi(R(V)') = \left\{ \begin{pmatrix} S & & & & 0 \\ & S & & & \\ & & S & & \\ & & & \cdot & \\ 0 & & & & \cdot \\ & & & & & \cdot \end{pmatrix} ;\ S \in B(\mathfrak{N}_{V^*}) \right\}.$$

Since $(T^*T)^{1/2} \in R(V)'$ by Lemma 2 and $R(T)=R(V, (T^*T)^{1/2})$, we have

$$\phi(R(T)) = \phi(R(V)' \cap R((T^*T)^{1/2})') = \left\{ \begin{pmatrix} S & & & & 0 \\ & S & & & \\ & & S & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & & \cdot \end{pmatrix} ;\ S \in R((T^*T)^{1/2} | \mathfrak{N}_{V^*})' \right\};$$

hence,

$$\phi(R(T) \cap R(T)') = \left\{ \begin{pmatrix} S & & & & 0 \\ & S & & & \\ & & S & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & & \cdot \end{pmatrix} ;\ S \in R((T^*T)^{1/2} | \mathfrak{N}_{V^*}) \right\}.$$

Next, we define the mapping ψ as follows:

$$\psi\left(\begin{pmatrix} S & & & & 0 \\ & S & & & \\ & & S & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & & \cdot \end{pmatrix} \right) = S \text{ for all } \begin{pmatrix} S & & & & 0 \\ & S & & & \\ & & S & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & & \cdot \end{pmatrix} \in \phi(R(T)').$$

Then ψ is clearly an algebraic isomorphism from $\phi(R(T)')$ on $R((T^*T)^{1/2} | \mathfrak{N}_{V^*})'$. Since $R((T^*T)^{1/2} | \mathfrak{N}_{V^*})$ is abelian, $R((T^*T)^{1/2} | \mathfrak{N}_{V^*})'$ is of type I; and hence $\phi(R(T)')$ is also of type I. Thus $R(T)'$ is of type I and $R(T) \cap R(T)' = R((T^*T)^{1/2})$, because ϕ is a spatial isomorphism from $B(H)$ on $B(\bigoplus_{n=0}^{\infty} M_n)$. Therefore $R(T)$ is of type I.

COROLLARY. *If T is a nearly normal operator on H, then $R(T)$ is of type I.*

PROOF. By Lemma 1 and Lemma 4, we have $T = 0 \oplus T_1 \oplus T_2$ and $R(T) = \{\lambda I\} \oplus R(T_1) \oplus R(T_2)$, where T_1 is normal and T_2 satisfies the conditions of Theorem. Hence $R(T_1)$ is abelian and $R(T_2)$ is of type I by Theorem. Therefore $R(T)$ is of type I.

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