A STRUCTURE THEOREM OF AUTOMORPHISMS OF VON NEUMANN ALGEBRAS

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In a recent few years automorphisms and derivations of $C^*$-algebras, especially of von Neumann algebras, have been investigated by various authors [1],[6],[7],[8],[11] and etc. We also are concerned with them in this paper. Our main purpose is to establish a structure theorem of (not necessarily $^*$-preserving) automorphisms of von Neumann algebras, which we may call the polar decomposition theorem. It asserts that any automorphism of a von Neumann algebra is composed of an inner automorphism defined by an invertible positive operator and a $^*$-automorphism in the unique way. This fact seems to assure us that the study of automorphisms may be reduced, in a sense, to that of $^*$-preserving ones. For example, it can be said that the property that an automorphism is outer is due to its $^*$-preserving part. Also we know, as its immediate consequence, that any automorphism of a von Neumann algebra is $\sigma$-strongly (that is, in the strongest operator-topology), as well as $\sigma$-weakly, bicontinuous. These arguments can be applied to those of isomorphisms between von Neumann algebras. Particularly we can give an answer to a problem proposed by S. Sakai in his lecture note [10], deciding that any isomorphism between von Neumann algebras is $\sigma$-strongly bicontinuous.

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1. Preliminaries. It is assumed that all $C^*$-algebras under consideration have identities. Automorphisms, isomorphisms and representations of $C^*$-algebras are said to be $^*$-automorphisms, $^*$-isomorphisms and $^*$-representations, respectively, if they are $^*$-preserving.

Let $H$ be a Hilbert space and $B(H)$ the $C^*$-algebra of all bounded linear operators on $H$. If an operator $a \in B(H)$ is invertible, $\rho_a(x) = axa^{-1}$ for $x \in B(H)$ is clearly an automorphism of $B(H)$. When $A$ is a $C^*$-algebra acting on $H$, an automorphism $\rho$ of $A$ is said to be spatial if there is an invertible operator $a \in B(H)$ such that $\rho = \rho_a|A$, the restriction of $\rho_a$ on $A$. A spatial automorphism $\rho$ of $A$ is said to be inner if the above $a$ can be chosen in $A$ and to be outer if it is not inner. An automorphism $\rho$ of $A$ is said to be weakly inner ($\pi$-inner in [8]) if for any faithful $^*$-representation $\pi$ of $A$ there...
is an inner automorphism $\sigma$ of the weak closure $\pi(A)$ in $B(K)$ such that $\pi \circ \rho \circ \pi^{-1} = \sigma | \pi(A)$, where $K$ denotes the representation space of $\pi$.

The following theorem asserts that all spatial $*$-automorphisms are defined by unitaries:

**Theorem 1.** Let $A$ be a $C^*$-algebra on a Hilbert space $H$, $a$ an invertible bounded linear operator on $H$ such as $\rho_a(A) \subseteq A$. If $\rho_a | A$ is a $*$-automorphism of $A$, then there is a unitary $u \in B(H)$ such as $\rho_u | A = \rho_a | A$.

Moreover, if $A$ is a von Neumann algebra and if $a$ belongs to $A$, then $u$ can be chosen in $A$.

**Proof.** By the hypotheses,

$$axa^{-1} = \rho_a(x) = [\rho_a(x^*)]^* = (a^*)^{-1}xa^*;$$

and hence,

$$a^*ax = xa^*a,$$

for all $x \in A$. Then $a^*a$ is contained in the commutator $A'$ of $A$. Let $a = uh$ be the polar decomposition of $a$, where $u$ is a unitary and $h = (a^*a)^{1/2}$ an invertible operator which is in $A'$. Therefore

$$\rho_u(x) = ah^{-1}xha^{-1} = axa^{-1} = \rho_a(x)$$

for all $x \in A$.

The second assertion is seen by remarking that, under the additional conditions, $u$ is contained in $A$ as well as $h$ is in the center of $A$. q.e.d.

**Lemma 2.** Let $H$ be a Hilbert space, $a \in B(H)$ an invertible operator. Then, $S_p(\cdot)$ denoting the spectrum,

$$S_p(\rho_a) \subseteq S_p(a)S_p(a)^{-1} = \{\lambda \mu^{-1}; \lambda, \mu \in S_p(a)\}.$$

**Proof.** For each $b \in B(H)$ let us consider the operators $l_b x = bx$ and $r_b x = xb$ for $x \in B(H)$. The mapping $b \mapsto l_b$ is an isomorphism of $B(H)$ into the Banach algebra of all bounded linear operators on $B(H)$ and the mapping $b \mapsto r_b$ is an anti-isomorphism. Then $S_p(l_b) \subseteq S_p(b)$ and $S_p(r_b) \subseteq S_p(b)$. Since $(r_b)^{-1} = r_{b^{-1}}$ and $l_b(r_b)^{-1} = (r_b)^{-1}l_{b^{-1}} = \rho_a$, we have

$$S_p(\rho_a) = S_p(l_{a^{-1}}r_a) \subseteq S_p(l_a)S_p(r_a)^{-1} \subseteq S_p(a)S_p(a)^{-1}. \quad \text{q.e.d.}$$

If a linear operator $\delta$ on a $C^*$-algebra $A$ satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for each $x, y \in A$, it is called a derivation of $A$. That derivations are necessarily continuous is known. It is clear that, for $a \in B(H)$, $\text{ad}(a) = ax - xa, x \in B(H)$.
is a derivation of $B(H)$. When a $C^*$-algebra $A$ acts on $H$, a derivation $\delta$ of $A$ is said to be spatial if there is an operator $a \in B(H)$ such as $\delta = ad_a|A$. The meanings of inner derivations, outer derivations and weakly inner derivations should be taken similarly as for those of automorphisms. A result of S. Sakai [11] (see [6], [8] also) asserting that each derivation of a von Neumann algebra is necessarily inner is remarkable and will play an important role in the following discussions. Using the method of the proof of Lemma 2, we can show

**Lemma 3.** Let $H$ be a Hilbert space and $a$ a bounded linear operator on $H$. Then,

$$S_p(ad_a) \subseteq S_p(a) - S_p(a) = \{\lambda - \mu ; \lambda, \mu \in S_p(a)\}.$$  

A derivation $\delta$ on a $C^*$-algebra $A$ is said to be skew-adjoint if it satisfies $\delta(x^*) = -\delta(x)^*$ for all $x \in A$.

**Lemma 4.** Let $a$ be a bounded linear operator on a Hilbert space $H$. If $a$ is self-adjoint, then $ad_a$ is skew-adjoint; and conversely if $ad_a$ is skew-adjoint, then there is a self-adjoint operator $b \in B(H)$ such as $ad_a = ada$. Moreover if $a$ is in a $C^*$-algebra $A$ on $H$, then $b$ mentioned above can be chosen in $A$.

**Proof.** The first half is easily verified; its converse is seen by, for a given $a$, putting $b = [a^* + a]/2$, together with the last assertion. q.e.d.

2. Automorphisms. We denote by $\log$ the principal analytic continuation of the log onto the domain $C \setminus \{\infty < t \leq 0\}$ obtained by excluding the negative half-axis $\{\infty < t \leq 0\}$ from the complex plane $C$.

**Theorem 5 (cf.[4]).** Let $H$ be a Hilbert space, $S$ a closed subspace of $B(H)$. If $a \in B(H)$ has its spectrum in $\{\lambda; \text{Re}(\lambda) > 0\}$ and satisfies $\rho_a(S) \subseteq S$, then $ad(\log a)(S) \subseteq S$ and $\rho_{\exp(i\log a)}(S) = S$ for any real $t$.

**Proof.** Let us put $b = \log a$ and $\sigma_t = \rho_{\exp(it)}$ for each real $t$, then the mapping $t \rightarrow \sigma_t$ becomes a norm-continuous one-parameter group of operators of $B(H)$ into itself. Since $[\exp(tb) - 1]/t$ converges to $b$ uniformly as $t$ converges to $0$, we have, $t$ denoting the identity automorphism of $B(H)$,

$$\left\| \left[ \frac{\sigma_t - t}{t} \right] x \right\| = \left\| \frac{[\exp(tb)x \exp(-tb)] - x}{t} - bx + xb \right\| \//\|x\|$$

$$= \left\| \left[ \exp(tb) - 1 \right] x \frac{1 - \exp(tb)}{t} \exp(-tb) + x \left[ \frac{1 - \exp(tb)}{t} + b \right] \exp(-tb) \right\| \//\|x\|$$
as $t \to 0$. Therefore we have $\sigma_t = \exp(tadb)$, in particular $\rho_a = \exp(adb)$ (see [3], p.283, Theorem 9.4.2). Since $S_p(\rho_a) \subseteq S_p(a)S_p(a)^{-1} \subseteq \mathbb{C} \setminus \{-\infty < t \leq 0\}$ and $S_p(adb) \subseteq S_p(b) - S_p(b) \subseteq \{\lambda; \ |\text{Im}(\lambda)| < \pi\}$, we know $\log \rho_a = adb$ by Lorch's theorem which determines the period of $\exp([9])$. The theorem of Runge enables us to choose a sequence of polynomials which converges to $\log$ uniformly in the wider sense, hence $\log \rho_a$ is a uniform limit of polynomials of $\rho_a$. Thus, by the assumption, $adb(S) \subseteq S$ and also $\rho_{exp(tdb)}(S) \subseteq S$ for any $t$. The latter gives $\rho_{exp(tdb)^{-1}}(S) = \rho_{exp(-t)}(S) \subseteq S$, so that $\rho_{exp(tdb)}(S) = S$. q.e.d.

**Theorem 6.** Let $H, K$ be Hilbert spaces, $M$ a von Neumann algebra on $H$, $A$ a C*-algebra on $K$ and $\pi$ an isomorphism of $M$ onto $A$. If $a \in B(K)$ with $S_p(a) \subseteq \{\lambda; \ \text{Re}(\lambda) > 0\}$ satisfies $\rho_a(A) \subseteq A$, then there is an invertible operator $b$ in $M$ such as $\pi \circ \exp(ab) = \rho_a b$. 

In particular, if $\pi$ is a *-isomorphism of $M$ onto $A$ and if $a \in B(K)$ is an invertible positive operator, we can choose $b$ to be invertible and positive.

**Proof.** By Theorem 5, $t \to \tau_t = \pi^{-1} \circ \rho_{exp(tdb)} \circ \pi$ becomes a norm-continuous one-parameter group of automorphisms of $M$. Moreover, by the similar calculation to that in the proof of Theorem 5,

\[
\left\| \frac{\tau_t - \tau_0}{t} (x) - \pi^{-1}[\text{ad}(\log a)(\pi(x))] \right\|/\|x\| \to 0,
\]

as $t \to 0$. Since the mapping $M \ni x \mapsto \pi^{-1}[\text{ad}(\log a)(\pi(x))]$ becomes a derivation of $M$, by S. Sakai's result, there is an operator $c$ in $M$ such as $\pi^{-1}[\text{ad}(\log a)(\pi(x))] = \text{ad}(c(x))$ for all $x \in M$. Let us put $\tau_t' = \rho_{\exp(tadb)} \circ M$ for each real $t$. Then, by $\left\| \frac{\tau_t' - \tau_0}{t} - \text{ad}c \right\| M \to 0$ as $t \to 0$, we have $\tau_t' = \exp(tadc) \circ M = \tau_t$, especially

\[
\rho_b | M = \pi^{-1} \circ \rho_a \circ \pi,
\]

where $b = \exp c$, which is of course in $M$.

If $\pi$ is a *-isomorphism and $a$ an invertible positive operator, $M \ni x \mapsto \pi^{-1}[\text{ad}(\log a)(\pi(x))]$ becomes a skew-adjoint derivation of $M$, then again by
Sakai’s result and by Lemma 4, we can choose a self-adjoint operator \( c \in M \) such as \( \pi^{-1}\{\mathrm{ad}(\log_a(\pi(x)))\} = \mathrm{ad}(x) \) for all \( x \in M \). Putting \( b = \exp c \in M \) as above, we have \( \rho_b | M = \pi^{-1} \circ \rho_c \circ \pi \).

As a direct corollary, we have a sufficient condition for a spatial automorphism of a von Neumann algebra to be inner.

**Corollary 7.** Let \( H \) be a Hilbert space, \( M \) a von Neumann algebra on \( H \). If \( a \in B(H) \) has its spectrum in \( \{ \lambda; \Re(\lambda) > 0 \} \) and satisfies \( \rho_a(M) \subseteq M \), then the spatial automorphism \( \rho_a | M \) defined by \( a \) is inner.

Now we state the polar decomposition theorem for automorphisms of von Neumann algebras. It will be extended to \( C^* \)-algebras in a natural sense using the concept “weakly inner automorphisms”.

**Theorem 8.** Let \( H \) be a Hilbert space, \( M \) a von Neumann algebra on \( H \), \( \rho \) an automorphism of \( M \). Then there are inner automorphisms \( \rho_1, \rho_1' \) defined by invertible positive operators in \( M \) and *-automorphisms \( \rho_2, \rho_2' \) such as \( \rho = \rho_2 \circ \rho_1 = \rho_1' \circ \rho_2' \). Suppose that \( \rho_1, \rho_1' \) are inner automorphisms defined by invertible positive operators in \( M \) and \( \rho_2, \rho_2' \) *-automorphisms such as \( \rho = \rho_2 \circ \rho_1 = \rho_1' \circ \rho_2' \), then \( \rho_1 = \rho_1, \rho_2 = \rho_2 \) and \( \rho_1' = \rho_1', \rho_2' = \rho_2' \).

**Proof.** Let \( \alpha \) be the greatest atomic representation of \( M \), that is, the direct sum of all irreducible *-representations of \( M \), \( L \) its representation space. It is well-known that \( \alpha \) is a *-isomorphism. Hence \( \alpha = \alpha \circ \rho \circ \alpha^{-1} \) becomes an automorphism of \( A = \alpha(M) \). Because for each maximal left ideal \( I \) in \( M \) the quotient Banach space \( M/I \) becomes a Hilbert space ([12]; also [5]) and \( \rho \) carries \( I \) to a maximal left ideal \( \rho(I) \), we can define an invertible linear operator \( s_I \) of \( M/I \) to \( M/\rho(I) \) by

\[
s_I(x + I) = \rho(x) + \rho(I).
\]

Here \( \|s_I\| \leq \|\rho\| \) and \( \|(s_I)^{-1}\| \leq \|\rho^{-1}\| \) hold. We can consider that \( L \) is a direct sum of \( \{M/I\} \) with suitable multiplicities and define an invertible bounded linear operator \( s \) as the direct sum of \( s_I \)'s with corresponding multiplicities. Then the formula \( \sigma = \rho \circ A \) can be observed. Let \( s = v h \) be the polar decomposition, where \( v \) is a unitary and \( h = (s^* s)^{1/2} \) an invertible positive operator. Then for any \( b \in A \),

\[
h^{-1/2} b h^{1/2} = (s^* s)^{-1/2} b (s^* s) = s^{-1}(s^{-1})^* s^* s = s^{-1}(s b s^{-1})^* s = \sigma^{-1}(\sigma(b^*))
\]

is contained in \( A \), therefore \( \rho_{h^\ast}(A) \subseteq A \); and hence by Theorem 5,
also for any \( b \in A \),

\[ vbv^{-1} = sh^{-1}bh^{-1} = \sigma(\rho_h(b)) \in A, \]

in other words, \( \rho_h(A) \subseteq A \) (cf. Lemma 4.6 in [4]). We define here

\[ \sigma_1 = \rho_h \circ A, \; \sigma_2 = \rho_v \circ A; \; \rho_1 = \alpha^{-1} \circ \sigma_1 \circ \alpha, \; \rho_2 = \alpha^{-1} \circ \sigma_2 \circ \alpha. \]

Then, by Theorem 6, \( \rho_1 \) is an inner automorphism which is defined by an invertible positive operator in \( M \); and trivially \( \rho_1 \) is a \(*\)-automorphism. Because of \( \sigma = \sigma_2 \circ \sigma_1 \), we have

\[ \rho = \alpha^{-1} \circ \sigma \circ \alpha = (\alpha^{-1} \circ \sigma_2 \circ \alpha) \circ (\alpha^{-1} \circ \sigma_1 \circ \alpha) = \rho_2 \circ \rho_1. \]

An analogous argument using the polar decomposition \( s^* = w^*k \) of \( s^* \), where \( k = (ss^*)^{1/2} \) and \( w \) a unitary, leads us to \( \rho = \rho_1 \circ \rho_2 \).

Next we show the uniqueness of the decomposition. For our purpose it is sufficient to see that if \( h_1 \) and \( h_2 \) are invertible positive operators in \( M \) and if \( \rho_1 \circ \rho_2 \mid M = \rho_1 \circ \rho_2 \mid M \) is \(*\)-preserving then \( \rho_1 \circ \rho_2 \mid M \) is just the identity automorphism \( \iota \mid M \) of \( M \). Since

\[ h_2 h_1 x h_2^{-1} h_1^{-1} = [\rho_1 \rho_2 (x^*)]^* = h_2^{-1} h_1^{-1} x h_1 h_2 \]

for all \( x \in M \), \( h_2 h_1 \) is contained in the center \( Z \) of \( M \). Let \( h_2 h_1 = u \) be the polar decomposition of \( h_2 h_1 \), where \( u \) is a unitary and \( h \) an invertible positive operator. Then, first

\[ h^2 = h_2 h_1 \in Z, \text{ hence } h \in Z; \]

and secondly

\[ h_1 h_2 h_1^{-1} = u^* = u^{-1} = h_1^{-1} h_2^{-1} h, \]

then \( h_1^2 = h_2^{-2} h^2 \), hence \( h_1 = h_2^{-1} h \).

Therefore \( h_2 h_1 = h \in Z \), and

\[ \rho_1 \circ \rho_2 \mid M = \rho_1 \mid M = \iota \mid M. \]

q.e.d.

Since any spatial automorphism and any \(*\)-automorphism of a von
Neumann algebra are $\sigma$-weakly and $\sigma$-strongly bicontinuous (see [2]), we have

**Theorem 9.** If $\rho$ is an automorphism of a von Neumann algebra $M$ on a Hilbert space $H$, then $\rho$ is $\sigma$-weakly and $\sigma$-strongly bicontinuous.

**3. Isomorphisms.** The following theorem will be shown by the same method employed in Theorem 8:

**Theorem I.** Let $H, K$ be Hilbert spaces, $M, N$ von Neumann algebras on $H, K$, respectively, $\rho$ an isomorphism of $M$ onto $N$. Then there are inner automorphisms $\rho_1$ of $M$, $\rho_1'$ of $N$ which are defined by invertible positive operators in $M, N$, respectively, and $*$-isomorphisms $\rho_2, \rho_2'$ of $M$ onto $N$ such as $\rho = \rho_1 \rho_1' = \rho_1' \rho_1$. Furthermore this decomposition is unique.

It has been known that if an isomorphism $\rho$ of a von Neumann algebra $M$ onto an another $N$ is $\ast$-preserving then it is $\sigma$-weakly and $\sigma$-strongly bicontinuous (J. Dixmier [2]); and that an isomorphism of $M$ onto $N$ is $\sigma$-weakly bicontinuous and on a bounded part $\sigma$-strongly bicontinuous (S. Sakai [11]). Then it is natural to raise a question, as on p.1.52 in [11], whether an isomorphism is $\sigma$-strongly bicontinuous. The answer is immediate from Theorem I:

**Theorem II.** Let $H, K$ be Hilbert spaces, $M, N$ von Neumann algebras on $H, K$, respectively, $\rho$ an isomorphism of $M$ onto $N$. Then $\rho$ is $\sigma$-weakly and $\sigma$-strongly bicontinuous.

**4. Addendum.** When this paper had been almost prepered, the author had a chance to see a preprint of S. Sakai's article [14] in which it had been shown that any derivation on a simple $C^*$-algebra with an identity is inner. Using this and following along the same line of the arguments in section 2, we can state

**Theorem III.** Let $A$ be a simple $C^*$-algebra with an identity, $B$ a $C^*$-algebra, $\rho$ an isomorphism of $A$ onto $B$. Then there are inner automorphisms $\rho_1$ of $A$, $\rho_1'$ of $B$ which are defined by invertible positive operators in $A, B$, respectively, and $*$-isomorphisms $\rho_2, \rho_2'$ of $A$ onto $B$ such as $\rho = \rho_1 \rho_1' = \rho_1' \rho_1$. Furthermore this decomposition is unique.

In [13] the author found a similar result to our Theorem 5.

**References**


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