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THE OPERATIONS ρ_R^k ON THE GROUP $\widetilde{K}_R(CP^n)$

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The operations ρ_C^k and ρ_R^k play significant roles in K-theory. Their definitions and the actions on the K-groups of diverse complexes are described in [2].

In this note, we calculate the action of ρ_R^k on the reduced K-rings of a complex projective space CP^n. The method used here is the same as the one which is employed by J. F. Adams in the calculation of ρ_R^k on the ring $\widetilde{K}_R(S^{4n})$ [2. (5. 18)].

Preliminaries. Let CP^n be the (complex) n-dimensional complex projective space and $\widetilde{K}_C(CP^n)$ (resp. $\widetilde{K}_R(CP^n)$) be its complex (resp. real) (reduced) K-rings. We write

$$c : \widetilde{K}_R(CP^n) \longrightarrow \widetilde{K}_C(CP^n),$$
$$r : \widetilde{K}_C(CP^n) \longrightarrow \widetilde{K}_R(CP^n),$$
$$t : \widetilde{K}_C(CP^n) \longrightarrow \widetilde{K}_C(CP^n)$$

for the homomorphisms induced by complexification, realification and complex conjugation. As is well-known ([1], Lemma 3.9), we have

$$cr = 1 + t\,,$$
$$rc = 2\,.$$

The ring $\widetilde{K}_C(CP^n)$ is generated by one generator μ which satisfies the relation $\mu^{n+1}=0$ ([1], Theorem 7.2). The ring $\widetilde{K}_R(CP^n)$ is generated by one generator $\omega=r\mu$ which satisfies the following relations:

$$\omega^{2w+1} = 0\,, \qquad \text{if} \quad n = 4w\,,$$

$$2(\omega^{2w+1}) = 0\,, \quad \omega^{2w+2} = 0\,, \qquad \text{if} \quad n = 4w+1\,,$$

$$\omega^{2w+2}=0\,,\qquad \text{if } n=4w+2\,,\ n=4w+3$$

([3], Theorem 2.2, (i)).

Let

$$ch\colon \widetilde{K}_C(X) \longrightarrow \widetilde{H}^*(X,Q)$$

denote the Chern character. Note that

$$ch\colon \widetilde{K}_C(CP^n/CP^m) \longrightarrow \widetilde{H}^*(CP^n/CP^m)$$

is a ring monomorphism ([1], p. 621). For the generator $\mu \in \widetilde{K}_C(CP^n)$, $ch\,\mu = e^{-y}-1 \pmod{y^{n+1}}$, where $y \in H^2(CP^n, Q)$ is the generator. Therefore, we have

$$\begin{aligned} ch\cdot c\omega &= ch\cdot cr\,\mu = ch(1+t)\,\mu = ch\{\mu^2/(1+\mu)\} \\ &= (2\sinh y/2)^2 \pmod{y^{n+1}}, \end{aligned}$$

and

$$ch_{2t}\cdot c\omega = 2\cdot y^{2t}/(2t)\,!\,. \tag{1}$$

Next consider the stunted projective space CP^n/CP^1. Since $H^2(CP^n/CP^1) = 0$, every real vector bundle over CP^n/CP^1 has the vanishing 2-dimensional Stiefel-Whitney class and therefore every element in $\widetilde{K}_R(CP^n/CP^1)$ is considered as a linear combination of $Spin(8m)$-bundles. By the exactness of the sequence

$$0 \longrightarrow \widetilde{K}_R(CP^n/CP^1) \xrightarrow{\ j^*\ } \widetilde{K}_R(CP^n) \xrightarrow{\ i^*\ } \widetilde{K}_R(CP^1) \longrightarrow 0\,,$$

$\widetilde{K}_R(CP^n/CP^1)$ is additively generated by $\omega_1, \omega_2, \omega_3, \cdots$ such that the equalities

$$\begin{aligned} j^*\omega_1 &= 2\omega \\ j^*\omega_s &= \omega^s, \qquad s=2,3,\cdots \end{aligned} \tag{2}$$

hold ([3], Theorem 2.2, (iii)).

Determination of $\rho_R^k\omega$. Let Q_k be the additive group of fractions of the form p/k^q, where p and q are integers.

THEOREM. *For the generator* $\omega \in \widetilde{K}_R(CP^n)$, *we have*

$$\rho_R^k \omega = 1 + a_1\omega + a_2\omega^2 + \cdots,$$

in $1+\widetilde{K}_R(CP^n)\otimes Q_k$, *where* a_t $(t=1, 2, \cdots)$ *is given by the following formula*:

$$a_t = (k^2-1)\cdots(k^2-(2t-1)^2)/(2^{2t}(2t+1)!)\,.$$

PROOF. By [2, (5.2)],

$$\mathrm{Log}\, sh\,\omega_1 = \sum_{t=1}^{\infty} (1/2)\,\alpha_{2t}\, ch_{2t}\, c\omega_1\,.$$

For the definition of α_{2t}, see [2, §2]. By the naturalities of ch and c, we have

$$\begin{aligned} j^*\mathrm{Log}\, sh\,\omega_1 &= \sum (1/2)\,\alpha_{2t}\, ch_{2t}\cdot cj^*\omega_1 \\ &= \sum (1/2)\,\alpha_{2t}\, ch_{2t}\cdot c(2\omega) \qquad \text{(by (2))} \\ &= 2\sum \alpha_{2t}(y^{2t}/(2t)!) \qquad \text{(by (1))} \\ &= 2\,\mathrm{Log}((\sinh y/2)/(y/2)) \qquad \text{(by [2, (2.1)])}. \end{aligned}$$

Therefore,

$$j^*sh\,\omega_1 = ((\sinh y/2)/(y/2))^2\,. \tag{3}$$

We define

$$\Psi_H^k:\ \sum_{s\geqq 0} H^{2s}(X, Q) \longrightarrow \sum_{s\geqq 0} H^{2s}(X, Q)$$

by

$$\Psi_H^k(x) = k^s x\,, \quad \text{for} \quad x \in H^{2s}(X, Q)\,.$$

This is a ring homomorphism and we have

$$j^*\Psi_H^k sh\,\omega_1 = ((\sinh ky/2)/(ky/2))^2\,. \tag{4}$$

By [2, (5.6)], we have

$$j^* ch\cdot c\rho_R^k\omega_1 = j^*\{(\Psi_H^k sh\,\omega_1)/(sh\,\omega_1)\}$$
$$= ((\sinh ky/2)/(k\sinh y/2))^2 \quad \text{(by (3), (4))}.$$

Since

$$j^* ch\cdot c\rho_R^k\omega_1 = ch\cdot c\rho_R^k(2\omega)$$
$$= (ch\cdot c\rho_R^k\omega)^2,$$

we have

$$ch\cdot c\rho_R^k\omega = (\sinh ky/2)/(k\sinh y/2). \tag{5}$$

Recall the formula in the elementary calculus

$$\sinh nx = \sum_{r=0}^{\infty}(n/(2r+1)!)\left[\prod_{t=1}^{r}(n^2-(2t-1)^2)\right]\sinh^{2r+1}x.$$

Here, $\Pi[\quad]$ means 1, when $r=0$. If n is odd, the right hand side has the finite summands. But when n is even it is a infinite series. Therefore we have

$$ch\cdot c\rho_R^k\omega = \sum_{t=0}^{\infty}(1/(2t+1)!)\left[\prod_{u=1}^{t}(n^2-(2u-1)^2)\right]\sinh^{2t}y/2.$$

In the case $n \not\equiv 1 \pmod 4$ the theorem follows since $ch\cdot c$ is a monomorphism on $\widetilde{K}_R(CP^n)$. In the case $n=4w+1$, consider the exact sequence

$$0 \longrightarrow \widetilde{K}_R(S^{8w+4}) \longrightarrow \widetilde{K}_R(CP^{4w+2}) \xrightarrow{i^*} \widetilde{K}_R(CP^{4w+1}) \longrightarrow 0.$$

For the generator ω' of the ring $\widetilde{K}_R(CP^{4w+2})$, we have

$$i^*(\omega'^s) = \omega^s, \quad s = 1, 2, \cdots, 2w+1.$$

Since $ch\cdot c$ is a monomorphism on $\widetilde{K}_R(CP^{4w+2})$, by the naturalities of $ch\cdot c$ and ρ_R^k, the coefficient of ω^{2w+1} in $\rho_R^k\omega$ is a mod 2 reduction of the coefficient of ω'^{2w+1} in $\rho_R^k\omega'$ and the theorem is valid.

Determination of $\rho_R^k\omega^s$ ($s\geq 2$). Since $ch\cdot c$ is the ring homomorphism, we have

$$ch\cdot c\omega^s = 2^{2s}\sinh^{2s}y/2$$
$$= 2\left\{\sum_{r=0}^{s-1}(-1)^r\binom{2s}{r}\cosh(2s-2r)y/2+(-1)^s(1/2)\binom{2s}{s}\right\}$$

and

$$ch_{2t}\cdot c\omega^s = 2\left\{\sum_{r=0}^{s-1}(-1)^r\binom{2s}{r}(s-r)^{2t}/(2t)!\right\}y^{2t}, \qquad t=1, 2, \cdots .$$

Therefore

$$\begin{aligned}\operatorname{Log} sh\,\omega^s &= \sum_t (1/2)\,\alpha_{2t}\,ch_{2t}\,c\omega^s \\ &= \sum_t \alpha_{2t}\left\{\sum_{r=0}^{s-1}(-1)^r\binom{2s}{r}(s-r)^{2t}/(2t)!\right\}y^{2t} \\ &= \sum_{r=0}^{s-1}(-1)^r\binom{2s}{r}\left\{\sum_t \alpha_{2t}(s-r)^{2t}y^{2t}/(2t)!\right\} \\ &= \sum_{r=0}^{s-1}(-1)^r\binom{2s}{r}\operatorname{Log}\{(\sinh(s-r)y/2)/((s-r)y/2)\} .\end{aligned}$$

Therefore,

$$sh\omega^s = \prod_{r=0}^{s-1}\{(\sinh(s-r)y/2)/((s-r)y/2)\}^{(-1)^r\binom{2s}{r}} .$$

Since Ψ_H^k is a ring homomorphism, we have

$$\Psi_H^k sh\,\omega^s = \prod_{r=0}^{s-1}\{(\sinh k(s-r)y/2)/(k(s-r)y/2)\}^{(-1)^r\binom{2s}{r}}$$

and therefore

$$ch\cdot c\rho_R^k\,\omega^s = \Psi_H^k sh\,\omega^s/sh\,\omega^s = \prod_{r=0}^{s-1}\{(\sinh k(s-r)y/2)/(k\sinh(s-r)y/2)\}^{(-1)^r\binom{2s}{r}}. \quad (6)$$

From this we can determine $\rho_R^k\omega^s$ as above. But for general s it is very complicated. For example, in the case $s=2$, (6) reduces to

$$ch\cdot c\rho_R^k\,\omega^2 = (\sinh ky/k\sinh y)((k\sinh y/2)/(\sinh ky/2))^4 . \quad (7)$$

Therefore

$$\begin{aligned}ch\cdot c\rho_R^k\,\omega^2 = {}&(1+((k^2-1)/2)\sinh^2 y/2 + ((k^2-1)(k^2-3^2)/4!)\sinh^4 y/2 + \cdots) \\ &\cdot(1+((k^2-1)/3!)\sinh^2 y/2+((k^2-1)(k^2-3^2)/5!)\sinh^4 y/2 + \cdots)^{-3}\end{aligned}$$

$$= 1-((k^4-1)/15)\sinh^4 y/2$$
$$+ (2(k^2-1)(10k^4+31k^2+31)/(3^3\cdot 5\cdot 7))\sinh^6 y/2 + \cdots .$$

So we have

THEOREM.

$$\rho_R^k\omega^2 = 1-((k^4-1)/240)\,\omega^2 + ((k^2-1)(10k^4+31k^2+31)/(2^5\cdot 3^3\cdot 5\cdot 7))\,\omega^3 + \cdots .$$

The following theorem is more convenient.

THEOREM.

$$\rho_R^k(\omega^2+4\omega) = 1+((k^2-1)/3!))\,\omega + ((k^2-1)(k^2-2^2)/5!)\,\omega^2 + ((k^2-1)(k^2-2^2)(k^2-3^2)/7!)\,\omega^3 + \cdots .$$

PROOF.

$$\begin{aligned} ch\cdot c\rho_R^k(\omega^2+4\omega) &= (ch\cdot c\rho_R^k\omega^2)(ch\cdot c\rho_R^k\omega)^4 \\ &= \sinh ky/k\sinh y \qquad \text{(by (5) and (7))} \qquad (8) \\ &= \sum_{r=0}^{k-1}(2^{2r}/(2r+1)!)\prod_{t=1}^{r}(k^2-t^2)\sinh^{2r} y/2 . \end{aligned}$$

This completes the proof.

The first term of $\rho_R^k\omega^s$. We have

$$\mathrm{Log}\{\Psi_H^k sh\,\omega^s/sh\,\omega^s\} = \sum_t \alpha_{2t}(k^{2t}-1)\left\{\sum_{r=0}^{s-1}(-1)^r\binom{2s}{r}(s-r)^{2t}/(2t)!\right\}y^{2t} .$$

Since for arbitrary s,

$$\begin{aligned} \sum_{r=0}^{s-1}(-1)^r\binom{2s}{r}(s-r)^{2t} &= 0 \qquad & t<s \\ &= (1/2)(2t)! \qquad & t=s , \end{aligned}$$

we can easily see that

THEOREM. *For arbitrary s, we have*

$$\rho_R^k \omega^s = 1+(1/2)(k^{2s}-1)\alpha_{2s}\omega^s + (\textit{higher order terms}).$$

For $\omega^{4w+1} \in \widetilde{K}_R(CP^{4w+1})$, we have

THEOREM.

$$\rho_R^k \omega^{4w+1} = 1 + \omega^{4w+1} \quad \textit{if} \quad k \equiv \pm 3 \bmod 8$$
$$= 1 \quad \textit{if} \quad k \equiv \pm 1 \bmod 8.$$

PROOF. We know that for the generator $\alpha \in \widetilde{K}_R(S^{8w+2}) = Z_2$

$$\rho_R^k \alpha = 1 + \alpha \quad \text{if} \quad k \equiv \pm 3 \bmod 8$$
$$= 1 \quad \text{if} \quad k \equiv \pm 1 \bmod 8.$$

By the exactness of

$$0 \longrightarrow \widetilde{K}_R(S^{8w+2}) \longrightarrow \widetilde{K}_R(CP^{4w+1}) \longrightarrow \widetilde{K}_R(CP^{4w}) \longrightarrow 0,$$

the theorem follows immediately.

REMARK. Let h_C be the canonical complex line bundle over CP^n, we have

$$ch \cdot c\,\rho_R^k r(h_C^\lambda) = (\sinh k\lambda y/2)/(\sinh \lambda y/2).$$

(8) is obtained from this formula as the special case $\lambda = 2$.

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