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ON DUAL INTEGRAL EQUATIONS AS CONVOLUTION TRANSFORMS

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1. Introduction. C. Fox [1] introduced the following dual integral equations

$$\int_0^\infty H\left(ux\left|\begin{matrix}\alpha_i, a_i\\ \beta_i, a_i\end{matrix}\right.; n\right) f(u)du = g(x), \qquad 0<x<1, \tag{1}$$

and

$$\int_0^\infty H\left(ux\left|\begin{matrix}\lambda_i, a_i\\ \mu_i, a_i\end{matrix}\right.; n\right) f(u)du = h(x), \qquad x>1, \tag{2}$$

where $g(x)$ and $h(x)$ are given and $f(x)$ is to be determined.

This class involves some important equations in mathematical phisics as special cases.

The H functions of order n used in equation (1) are of the form

$$H\left(x\left|\begin{matrix}\alpha_i, a_i\\ \beta_i, a_i\end{matrix}\right.; n\right) = H\left(x\left|\begin{matrix}\alpha_1, a_1\\ \beta_1, a_1\end{matrix} : \begin{matrix}\alpha_2, a_2\\ \beta_2, a_2\end{matrix} : \cdots : \begin{matrix}\alpha_n, a_n\\ \beta_n, a_n\end{matrix}\right.\right) \tag{3}$$

$$= \frac{1}{2\pi i}\int_C \prod_{i=1}^n \left\{\frac{\Gamma(\alpha_i + sa_i)}{\Gamma(\beta_i - sa_i)}\right\} x^{-s}\, ds \tag{4}$$

where $a_i > 0$, α_i, β_i are all real, $i = 1, 2, \cdots, n$, the contour C along which the integral of (4) is taken is the straight line parallel to the imaginary axis in the complex s-plane and all the poles of the integrand of (4) are simple and lie to the left of the line $\sigma = \sigma_0 > -\alpha_i/a_i \quad i = 1, 2, \cdots, n \quad (s = \sigma + i\tau)$. The integral (4), taken along the line, converges if $2\sigma_0 \sum_{i=1}^n a_i < \sum_{i=1}^n (\beta_i - \alpha_i)$ and converges absolutely if $2\sigma_0 \sum_{i=1}^n a_i < \sum_{i=1}^n (\beta_i - \alpha_i) - 1$.

Under the assumptions that the H function of (2) satisfies these conditions with α_i replaced by λ_i and β_i replaced by μ_i $(i = 1, 2, \cdots, n)$ and a common value of σ_0 can be found for both the H functions, he found a formal solution of (1) and (2) with the help of fractional integration.

More generally, R. K. Saxena [3] discussed the following dual integral equations

$$(5)\qquad \int_0^\infty H^{m,n}_{p+n,q+m}\left(ux\left|\begin{matrix} a_j, A_j \\ b_k, B_k\end{matrix}\right.\right) f(u)du = g(x), \qquad 0<x<1,$$

$$(6)\qquad \int^\infty H^{m,n}_{p+n,q+m}\left(ux\left|\begin{matrix} c_j, A_j \\ d_k, B_k\end{matrix}\right.\right) f(u)du = h(x), \qquad x>1,$$

where

$$(7)\qquad H^{m,n}_{p+n,q+m}\left(x\left|\begin{matrix} a_j, A_j \\ b_k, B_k\end{matrix}\right.\right) = H^{m,n}_{p+n,q+m}\left(x\left|\begin{matrix} a_1, A_1 \\ b_1, B_1\end{matrix} : \begin{matrix} a_2, A_2 \\ b_2, B_2\end{matrix} : \cdots : \begin{matrix} a_{n+p}, A_{n+p} \\ b_{m+q}, B_{m+q}\end{matrix}\right.\right)$$

$$(8)\qquad = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(a_j-A_js)}{\prod_{j=1}^{q}\Gamma(b_{m+j}-B_{m+j}s)\prod_{j=1}^{p}\Gamma(a_{n+j}+A_{n+j}s)}x^{-s}ds,$$

the integral of (8) converges or converges absolutely under some conditions. The method he employed to solve the dual integral equations (5), (6) is the same as that of Fox, that is, to use fractional integration so as to reduce to transforms with a common H kernel.

The object of this paper is to obtain the solution for more general class of dual integral equations. This class involves the cases of Fox and Saxena as special ones.

From the point of view that the dual integral equations (1), (2) and (5), (6) can be reduced to the dual convolution transform after exponential change of variables, we consider the followings:

$$(9)\qquad \int_{-\infty}^\infty G(x-t)\varphi(t)dt = f(x), \qquad x>\lambda,$$

$$(10)\qquad \int_{-\infty}^\infty H(x-t)\varphi(t)dt = g(x), \qquad x<\lambda,$$

where $G(t)$ and $H(t)$ are generated by certain meromorphic functions and λ is a constant, the explicit forms of these functions are defined in the following section.

2. Definition of kernels. Let us define four admissible entire functions $E_i(s)$ $(i = 1, 2, 3, 4)$ (see [6]):

$$E_i(s) = e^{\delta_i s} \prod_{k=1}^{\infty} (1 - s/a_{i,k}) e^{s/a_{i,k}},$$

where δ_i, $a_{i,k}$ are real numbers such that $\sum_{k=1}^{\infty} a_{i,k}^{-2} < \infty$ $(i = 1, 2, 3, 4;\quad k = 1, 2, \cdots)$ and $a_{1,k} < 0$, $a_{2,k} > 0$, $a_{3,k} < 0$, $a_{4,k} > 0$ $(k = 1, 2, \cdots)$. These assumptions for the sign of zeros of $E_i(s)$ may be possible to weaken slightly.

The corresponding intervals ([6]) $I_i (i = 1, 2, 3, 4)$ are intervals containing no zero of E_i(s) except perhaps at an end point, that is, if we define

$$\alpha_1^* = \max a_{1,k},\ \alpha_2^* = \min a_{2,k},\ \alpha_3^* = \max a_{3,k},\ \alpha_4^* = \min a_{4,k},$$

then

$$I_1 = (\alpha_1^*, +\infty),\ I_2 = (-\infty, \alpha_2^*),\quad I_3 = (\alpha_3^*, +\infty),\ I_4 = (-\infty, \alpha_4^*).$$

Let (a, b) be an interval included in the corresponding intervals I_1, I_2, I_3, and I_4, and let

$$\int_{-\infty}^{\infty} \left| \frac{E_2(\sigma + i\tau)}{E_1(\sigma + i\tau)} \right| d\tau < \infty \qquad \text{for every } \sigma \text{ in } a < \sigma < b,$$

$$\int_{-\infty}^{\infty} \left| \frac{E_4(\sigma + i\tau)}{E_3(\sigma + i\tau)} \right| d\tau < \infty \qquad \text{for every } \sigma \text{ in } a < \sigma < b,$$

$$\lim_{|\tau| \to \infty} \frac{E_2(\sigma + i\tau)}{E_1(\sigma + i\tau)} = 0 \qquad \text{uniformly in } a < \sigma < b$$

$$\lim_{|\tau| \to \infty} \frac{E_4(\sigma + i\tau)}{E_3(\sigma + i\tau)} = 0 \qquad \text{uniformly in } a < \sigma < b.$$

Then there exist two functions $G(t)$ and $H(t)$ such that

$$E_2(s)/E_1(s) = \int_{-\infty}^{\infty} e^{-st} G(t) dt \qquad a < \sigma < b,$$

$$E_4(s)/E_3(s) = \int_{-\infty}^{\infty} e^{-st} H(t) dt \qquad a < \sigma < b,$$

the integrals converging absolutely.

In fact, the classical inversion formula ([7]) gives $G(t)$ and $H(t)$ explicitly,

$$G(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{E_2(s)}{E_1(s)}e^{st}ds,\quad -\infty<t<\infty,\quad a<c<b, \tag{11}$$

$$H(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{E_4(s)}{E_3(s)}e^{st}ds,\quad -\infty<t<\infty,\quad a<c<b, \tag{12}$$

the integrals converging absolutely.

Thus we may consider the dual convolution transform (9), (10) in §1, where $f(x)$ and $g(x)$ are given and $\varphi(x)$ is to be determined.

3. The reduction of (9) and (10) to transforms with a common kernel. As usual, we reduce two given transforms to a transform with a common kernel, then the problem is reduced to inverting the sole convolution transform. To attain this object we suppose further that

$$a_{3,k}<a_{1,k}<0,\quad a_{4,k}>a_{2,k}>0$$

for all k and $\sum_k (a_{1,k}^{-1}-a_{3,k}^{-1})$, $\sum_k (a_{2,k}^{-1}-a_{4,k}^{-1})$ converge and that

$$\delta_1-\delta_3+\sum_k (a_{1,k}^{-1}-a_{3,k}^{-1})=\delta_2-\delta_4+\sum_k (a_{2,k}^{-1}-a_{4,k}^{-1}),$$

and we denote this common sum by Λ.

If one of the two series $\sum_k (a_{1,k}^{-1}-a_{3,k}^{-1})$ and $\sum_k (a_{2,k}^{-1}-a_{4,k}^{-1})$ diverge, then the function $h(x)$ of (17) cannot be defined generally. Moreover, when the two series $\delta_1-\delta_3+\sum_k (a_{1,k}^{-1}-a_{3,k}^{-1})$ and $\delta_2-\delta_4+\sum_k (a_{2,k}^{-1}-a_{4,k}^{-1})$ converge but have different sums, we may easily modify the functions $G^*(t)$ and $H^*(t)$ of (13), (14) for which the corresponding two series have a common sum, whence modified functions $G^*(t)$ and $H^*(t)$ are obtained from those of (13), (14) by translations through some distance. Thus, conversely, we may take the constant Λ so as to suit the convenience of the practical use.

Now, as in the preceding section, if

$$\int_{-\infty}^{\infty}\left|\frac{E_4(\sigma+i\tau)}{E_2(\sigma+i\tau)}\right|d\tau<\infty \qquad \text{for every } \sigma \text{ in } a<\sigma<b,$$

$$\int_{-\infty}^{\infty}\left|\frac{E_3(\sigma+i\tau)}{E_1(\sigma+i\tau)}\right|d\tau<\infty \qquad \text{for every } \sigma \text{ in } a<\sigma<b,$$

$$\lim_{|\tau|\to\infty} \frac{E_4(\sigma+i\tau)}{E_2(\sigma+i\tau)} = 0 \qquad \text{uniformly in } a<\sigma<b,$$

$$\lim_{|\tau|\to\infty} \frac{E_3(\sigma+i\tau)}{E_1(\sigma+i\tau)} = 0 \qquad \text{uniformly in } a<\sigma<b,$$

then

$$G^*(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{E_4(s)}{E_2(s)} e^{st} ds, \quad -\infty<t<\infty,\ a<c<b, \tag{13}$$

$$H^*(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{E_3(s)}{E_1(s)} e^{st} ds, \quad -\infty<t<\infty,\ a<c<b \tag{14}$$

are well-defined and

$$\int_{-\infty}^{\infty} G^*(t)e^{-st}dt = E_4(s)/E_2(s) \qquad a<\sigma<b,$$

$$\int_{-\infty}^{\infty} H^*(t)e^{-st}dt = E_3(s)/E_1(s) \qquad a<\sigma<b,$$

the integrals converging absolutely.

It is know ([5]) that $G^*(t)$, $H^*(t)$ are frequency functions and that

$$G^*(t) = \begin{cases} 0 & t \geqq \Lambda, \\ q(t)\exp[\alpha_2^* t] + O(\exp[(\alpha_2^* + \varepsilon)t]) & t\to-\infty, \end{cases} \tag{15}$$

$$H^*(t) = \begin{cases} p(t)\exp[\alpha_1^* t] + O(\exp[(\alpha_1^* - \varepsilon)t]) & t\to+\infty, \\ 0 & t \leqq \Lambda, \end{cases} \tag{16}$$

for some $\varepsilon>0$, where $p(t)$ and $q(t)$ are real polynomials.

Since (a,b) is included in $I_1\cap I_2\cap I_3\cap I_4$ it follows that the bilateral Laplace transforms of $G(t)$ and $G^*(t)$ and of $H(t)$ and $H^*(t)$ have common regions of absolute convergence and hence by the product theorem ([7]) that

$$\frac{E_4(s)}{E_1(s)} = \int_{-\infty}^{\infty} e^{-st} G*G^*(t)dt = \int_{-\infty}^{\infty} e^{-st} H*H^*(t)dt,$$

both integrals converging absolutely.

From this fact, using the uniqueness theorem ([6]) of bilateral Laplace transform,

$$G*G^*(x) = H*H^*(x)$$

for all x. Hence, if we take the convolutions of both sides of (1) and $G^*(x)$ and of (2) and $H^*(x)$ formally, we have

$$\begin{aligned} G^**f(x) &= \int_{-\infty}^{\infty} G^*(x-u)f(u)du \\ &= \int_{x-\Lambda}^{\infty} G^*(x-u)f(u)du \\ &= \int_{-\infty}^{\infty} \{G*G^*(x-t)\}\varphi(t)dt, \qquad x > \lambda + \Lambda, \end{aligned}$$

and

$$\begin{aligned} H^**g(x) &= \int_{-\infty}^{\infty} H^*(x-u)g(u)du \\ &= \int_{-\infty}^{x-\Lambda} H^*(x-u)g(u)du \\ &= \int_{-\infty}^{\infty} \{H*H^*(x-t)\}\varphi(t)dt, \qquad x < \lambda + \Lambda. \end{aligned}$$

Therefore, if we set

$$h(x) = \begin{cases} G^**f(x) & x > \lambda + \Lambda, \\ H^**g(x) & x < \lambda + \Lambda, \end{cases} \tag{17}$$

the dual convolution transforms (9) and (10) can be reduce to the transform with common kernel $G*G^*$,

$$h(x) = \int_{-\infty}^{\infty} \{G*G^*(x-t)\}\varphi(t)dt. \tag{18}$$

The conditions for justification of this procedure are given in the next section.

4. The solution of (9) and (10). Now we can state our first result as follows.

THEOREM 1. *If*

1. $G(t)$, $H(t)$ *are defined as in* §2,
2. Λ, $G^*(t)$, $H^*(t)$ *are defined as in* §3,
3. $$f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt, \qquad x > \lambda,$$

$$g(x) = \int_{-\infty}^{\infty} H(x-t)\varphi(t)dt, \qquad x < \lambda,$$

4. $\varphi(t)$ *is continuous,* $-\infty < t < \infty$ *and when* $(a,b) \subset I_1 \cap I_2 \cap I_3 \cap I_4$

$$\begin{aligned} \varphi(t) &= O(e^{\alpha t}) && t \to -\infty \\ &= O(e^{\beta t}) && t \to \infty, \quad a < \alpha < \beta < b, \end{aligned}$$

then the function $h(x)$ *of* (17) *can be defined and*

$$\varphi(x) = \frac{E_1(D)}{E_4(D)} h(x) \qquad -\infty < x < \infty,$$

i.e.,

$$\varphi(x) = \begin{cases} \dfrac{E_1(D)}{E_4(D)} \{G^* * f(x)\} & x \geqq \lambda + \Lambda, \\ \dfrac{E_1(D)}{E_4(D)} \{H^* * g(x)\} & x < \lambda + \Lambda, \end{cases}$$

where D *stands for differentiation and the operator* $E_1(D)/E_4(D)$ *must be interpreted as usual meaning,* [6].

PROOF. By (11) and (12) it is clear that

$$\begin{aligned} G(t) &= O(e^{\alpha' t}) && t \to \infty \\ &= O(e^{\beta' t}) && t \to -\infty, \end{aligned}$$

and

$$\begin{aligned} H(t) &= O(e^{\alpha' t}) && t \to \infty \\ &= O(e^{\beta' t}) && t \to -\infty, \end{aligned}$$

where $a < \alpha' < \alpha < \beta < \beta' < b$. Hence, from these facts together with (15) and (16) we see that

$$G{*}G^*(t) = H{*}H^*(t) = O(e^{\alpha' t}) \qquad t \to \infty$$
$$= O(e^{\beta' t}) \qquad t \to -\infty\,,$$

using Widder's lemma. [6; p.123]

Again using Widder's lemma we have the required equation (18). The change in order of here required is justified by Fubini's theorem. Thus obtained convolution transform (18) is easily inverted by Widder's theorem [6, Theorem 1] under our assumptions. The unicity of the solution $\varphi(x)$ under the assumption 4 is easily seen. Thus we have the desired result.

The following result is also established, using Widder's theorem. [6, Theorem 2]

THEOREM 2. *If*

1. $G(t)$, $H(t)$ *are defined as in* §2,
2. Λ, $G^*(t)$, $H^*(t)$ *are defined as in* §3,
3. $$f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt, \qquad x > \lambda\,,$$
$$g(x) = \int_{-\infty}^{\infty} H(x-t)\varphi(t)dt, \qquad x < \lambda\,,$$
4. $\varphi(t)$ *is continuous,* $-\infty < t < \infty$, *and when* $(a, b) \subset I_1 \cap I_2 \cap I_3 \cap I_4$
$$\varphi(t) = O(e^{\alpha t}) \qquad t \to -\infty$$
$$= O(e^{\beta t}) \qquad t \to \infty\,, \qquad a < \alpha < \beta < b\,,$$
5. $$\frac{E_4(s)}{E_1(s)}\,\frac{E_4(1-s)}{E_1(1-s)} = 1 \qquad a < \sigma < b,\ 1 - b > \alpha_1^*,$$

then the function $h(x)$ *of* (17) *can be defined, and if in addition*

$$h(x) = O(e^{\gamma x}) \qquad x \to -\infty$$
$$= O(e^{\delta x}) \qquad x \to \infty\,, \qquad 1 - b < \gamma < \delta < 1 - a\,,$$

then

$$\varphi(x) = \int_{-\infty}^{\infty} \{G{*}G^*(t-x)\}\, e^{x-t}h(t)dt \qquad -\infty < x < \infty\,,$$

or equivalently

$$\varphi(x)=\int_{-\infty}^{\lambda+\Lambda}\{G*G^*(t-x)\}\ e^{x-t}\{H^{**}g(t)\}\,dt$$

$$+\int_{\lambda+\Lambda}^{\infty}\{G*G^*(t-x)\}e^{x-t}\{G^{**}f(t)\}\,dt.$$

5. Applications. To avoid unnecessary confusion we consider Fox's dual integral equtions (1) and (2) of order 1, that is,

$$(1)' \qquad \int^{\infty} H\left(ux\,\middle|\,\begin{matrix}\alpha, a\\ \beta, a\end{matrix}\,;\,1\right)f(u)du=g(x), \qquad 0<x<1,$$

$$(2)' \qquad \int_0^{\infty} H\left(ux\,\middle|\,\begin{matrix}\lambda, a\\ \mu, a\end{matrix}\,;\,1\right)f(u)du=h(x), \qquad x>1.$$

In this case, the Mellin transform of $H\left(x\,\middle|\,\begin{matrix}\alpha, a\\ \beta, a\end{matrix}\,;\,1\right)$ with variable $s(=\sigma+i\tau)$ converges if $2\sigma a<\beta-\alpha$ and converges absolutely if $2\sigma a<(\beta-\alpha)-1$, and its generating function is $\Gamma(\alpha+sa)/\Gamma(\beta-sa)$, and for the H function of (2)′ same results with α replaced by λ and β replaced by μ hold.

On the other hand, after an exponential change of variables, (1)′ and (2)′ become

$$(1)'' \qquad \int_{-\infty}^{\infty} H\left(e^{-(x-t)}\,\middle|\,\begin{matrix}\alpha, a\\ \beta, a\end{matrix}\,;\,1\right)f(e^t)e^t dt=g(e^{-x}), \qquad x>0,$$

$$(2)'' \qquad \int_{-\infty}^{\infty} H\left(e^{-(x-t)}\,\middle|\,\begin{matrix}\lambda, a\\ \mu, a\end{matrix}\,;\,1\right)f(e^t)e^t dt=h(e^{-x}), \qquad x<0,$$

which are of the form (9), (10) with $\varphi(t)=f(e^t)e^t$.

From these facts, it is easily seen that entire functions $E_i(s)$, defined in §2, are

$$E_1(s)=1/\Gamma(\alpha+sa),\quad E_2(s)=1/\Gamma(\beta-sa),$$

$$E_3(s)=1/\Gamma(\lambda+sa),\quad E_4(s)=1/\Gamma(\mu-sa)$$

and that

$$a_{1,k}=-\frac{\alpha+k}{a},\ a_{2,k}=\frac{\beta+k}{a},\ a_{3,k}=-\frac{\lambda+k}{a},\ a_{4,k}=\frac{\mu+k}{a}$$

$$(k=0,1,2,\cdots).$$

From a familiar infinite expansion of Gamma-function it is clear that

$$\Lambda = \delta_1 - \delta_3 + \sum_k (a_{1,k}^{-1} - a_{3,k}^{-1}) = \delta_2 - \delta_4 + \sum_k (a_{2,k}^{-1} - a_{4,k}^{-1}) = 0 .$$

For the existence of kernels $G(t)$, $H(t)$, $G^*(t)$, $H^*(t)$ and for the other necessary conditions it is enough to assume, for example, that

$$\alpha > 0,\ \beta > 0,\ \lambda > \alpha + 1,\ \mu > \beta + 1,\ b^* < 1 + \alpha/a,\ a^* > - \alpha/a,$$

where a^* denotes a and b^* denotes b in the preceding sections, whence

$$\alpha^* = - \alpha/a,\ \alpha_2^* = \beta/a,\ \alpha_3^* = - \lambda/a,\ \alpha_4^* = \mu/a ,$$

the other cases can be also discussed after some modifications, if necessary. After simple calculations, we know that

$$G^*(t) = \begin{cases} \dfrac{b}{\Gamma(\mu-\beta)} (1 - e^{bt})^{\mu-\beta-1} e^{b\beta t} & t < 0 \\ 0 & t > 0 , \end{cases}$$

$$H^*(t) = \begin{cases} 0 & t < 0 \\ \dfrac{b}{\Gamma(\lambda-\alpha)} (1 - e^{-bt})^{\lambda-\alpha-1} e^{-b\alpha t} & t > 0 , \end{cases}$$

where $b = 1/a$.

In order to reduce given dual equations (1)′, (2)′ to which have a common kernel, Fox used two operators of fractional integration, denoted by $\mathfrak{J}$ and $\mathfrak{K}$,

$$\mathfrak{J}[w(x)] = \frac{b}{\Gamma(\mu-\beta)} x^{-b\mu+b} \int_0^x (x^b - v^b)^{\mu-\beta-1} v^{b\beta-1} w(v) dv,\ 0 < x < 1,$$

and

$$\mathfrak{K}[w(x)] = \frac{b}{\Gamma(\lambda-\alpha)} x^{b\alpha} \int_x^\infty (v^b - x^b)^{\lambda-\alpha-1} v^{b-b\lambda-1} w(v) dv,\quad x > 1 .$$

These equations, after an exponential change of variables, can be denoted as

$$\mathfrak{J}[w(e^{-x})] = \frac{b}{\Gamma(\mu-\beta)} \int_x^\infty (1-e^{b(x-v)})^{\mu-\beta-1} e^{b\beta(x-v)}\, w(e^{-v})dv, \qquad x>0,$$

and

$$\mathfrak{R}[w(e^{-x})] = \frac{b}{\Gamma(\lambda-\alpha)} \int_{-\infty}^x (1-e^{b(v-x)})^{\lambda-\alpha-1} e^{b\alpha(v-x)}\, w(e^{-v})dv, \qquad x<0,$$

and we can regard these equations as convolution transforms with the kernel functions $G^*(t)$ and $H^*(t)$.

The bilateral Laplace transforms of $G^*(t)$ and $H^*(t)$ are $\Gamma(\beta-sa)/\Gamma(\mu-sa)$ and $\Gamma(\alpha+as)/\Gamma(\lambda+as)$. See also [5].

From these facts it was natural that above two operators $\mathfrak{J}$ and $\mathfrak{R}$ was used to solve given Fox's dual equations.

Similarly, we can solve the Fox's dual equations of order n by operating $\mathfrak{J}$ and $\mathfrak{R}$ repeatedly under some conditions, however, from our point of view these repeated operations are merely to take convolutions of some $G^*(t)$ and $g(e^{-x})$ and of some $H^*(t)$ and $h(e^{-x})$ since the generating functions of given kernels are nothing but meromorphic functions.

Further, we can treat similarly Saxena's dual equations in §1 as a special case of ours.

However, one of their aims was to solve given equations by inspection. For this object we must assume much more conditions than that of Theorem 1. For the purpose of solving given dual equations by inspection, as an generalization of the dual integral equatisns (1)'', (2)'', let us consider the following equations:

$$\int_{-\infty}^\infty G(x-t)e^t\varphi(-t)dt = g(x), \qquad x>\lambda, \tag{19}$$

$$\int_{-\infty}^\infty H(x-t)e^t\varphi(-t)dt = h(x), \qquad x<\lambda, \tag{20}$$

where $G(t)$, $H(t)$ was defined as in §2.

We suppose that $\varphi(t)$ is continuous, $-\infty<t<\infty$, and

$$\begin{aligned} \varphi(t) &= O(e^{\alpha t}) && t\to\infty \\ &= O(e^{\beta t}) && t\to-\infty,\ a<\alpha<\beta<b. \end{aligned}$$

In the proof of Theorem 1, it was shown that

$$G*G^*(t) = O(e^{\alpha' t}) \qquad t\to\infty$$
$$= O(e^{\beta' t}) \qquad t\to-\infty\,,$$

where $a<\alpha'<\alpha<\beta<\beta'<b$.

Therefore, if $1-\beta<\beta'$, $\alpha'<1-\alpha$, then by the similar arguments in the preceding sections, using Widder's lemma [6; p.123], there exists

$$k(x)=\begin{cases} G^* * g(x) & x>\lambda+\Lambda\,,\\ H^* * h(x) & x<\lambda+\Lambda\,,\end{cases}$$

and is of the form

$$k(x)=\int_{-\infty}^{\infty}\{G*G^*(x-t)\}e^t\varphi(-t)dt\,, \tag{21}$$

where $G^*(t)$ and $H^*(t)$ was defined as in §2.

We denote by $\mathcal{L}$ the bilateral Laplace transform

$$\mathcal{L}_s k=\int_{-\infty}^{\infty}e^{-st}k(t)dt\,,$$

then from (21) we have

$$\mathcal{L}_s k=\frac{E_4(s)}{E_1(s)}\mathcal{L}_{1-s}\varphi \qquad \max(\alpha',1-\beta)<\Re s<\min(1-\alpha,\beta').$$

By a change of variable it is obvious that

$$\mathcal{L}_s\varphi=\frac{E_1(1-s)}{E_4(1-s)}\mathcal{L}_{1-s}k \qquad 1-\min(1-\alpha,\beta')<\Re s<1-\max(\alpha',1-\beta).$$

Thus we may replace the condition 5 in Theorem 2 by the existence of function $K^*(t)$ of which bilateral Laplace transform is $E_1(1-s)/E_4(1-s)$ and as an inversion formula of (21), accordingly of (19) and (20) we have

$$\varphi(x)=\int_{-\infty}^{\infty}K^*(x-t)e^tk(-t)dt \qquad -\infty<x<\infty\,.$$

In particular, for (1)″, (2)″, it is clear that we must take $H\left(t\,\middle|\,{\mu-a,\,a \atop \alpha+a,\,a};1\right)$ as $K^*(t)$ under some conditions, because in that case $\mathcal{L}_sK^*$ is $\Gamma(\mu-a+sa)/\Gamma(\alpha$

$+ a - sa)$ $(= E_1(1-s)/E_4(1-s))$.

Finally, it must be noted that in the present case and in more general case we may develop inversion theory in L^2 space similar to Saxena's. [2] [4]

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