ON CONFORMAL KILLING TENSOR IN A RIEMANNIAN SPACE

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0. Let $M^n$ be an $n$ dimensional Riemannian space. A vector field $v^a$ is called a Killing vector if it satisfies the Killing's equation:

$$\nabla_a v_b + \nabla_b v_a = 0,$$

where $\nabla_a$ means the operator of the covariant derivation with respect to the Riemannian connection. A Killing tensor $v_{bc}$ is, by definition, a skew symmetric tensor satisfying the Killing-Yano's equation:

$$(0.1) \quad \nabla_a v_{bc} + \nabla_b v_{ac} = 0.$$ 

In recent papers we discussed the integrability condition of the equation (0.1) and determined such tensors completely in the Euclidean space and the sphere.

A conformal Killing vector $u_b$ is a vector field satisfying

$$(0.2) \quad \nabla_a u_b + \nabla_b u_a = 2\rho g_{ab},$$

where $\rho$ is a scalar function and $g_{ab}$ the Riemannian metric. As for a generalization of such a vector it is not suitable to define a conformal Killing tensor as a skew symmetric tensor field $u_{bc}$ satisfying

$$\nabla_a u_{bc} + \nabla_b u_{ac} = 2\rho_c g_{ab},$$

where $\rho_c$ is a certain vector field. Because we can easily show that a conformal Killing tensor in this sense is a Killing tensor, i.e., we have $\rho_c = 0$. Thus this definition of conformal Killing tensor is meaningless.

In this paper we shall define a conformal Killing tensor in another way and generalize some results about a conformal Killing vector to the conformal Killing tensor. The definition which we shall adopt is suggested by the following fact. A parallel vector field in the Euclidean space $E^{n+1}$ induces a

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1) We adopt the identification of a vector field with a 1-form by virtue of the Riemannian metric.
2) S. Tachibana [1], S. Tachibana and T. Kashiwada [2].
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conformal Killing vector on the sphere $S^n$ of constant curvature. Thus a tensor field on $S^n$ induced from a parallel tensor field in $E^{n+1}$ is to be a model of conformal Killing tensor.

We shall concern only with tensor of degree 2 and the general case will be discussed in Kashiwada's forthcoming paper [6].

1. Preliminaries. Consider an $n$ dimensional Riemannian space $M^n$ whose Riemannian metric is given by $g_{bc}$ with respect to local coordinates $\{x^a\}^3$.

Let $R_{abc}^d$ be the Riemannian curvature tensor. Then Ricci's identity for any tensor $u_{ab}^c$ is given by

$$\nabla_a \nabla_b u_{cd}^e - \nabla_b \nabla_a u_{cd}^e = - R_{abc}^f u_{fd}^e - R_{abf}^e u_{cf}^e + R_{abf}^e u_{cd}^f.$$  

Especially we obtain the following formula for any skew symmetric tensor $u_{bc}$,

\begin{equation}
2 \nabla_b \nabla_c u^{ab} - \nabla_c \nabla_b u^{ab} = R_{bce} u^{de} + R_{bce} u^{ae} = R_{ce} - R_{bc} \delta^b_a = 0,
\end{equation}

where $R_{ce} = R_{bce}^b$ is the Ricci tensor.

The conformal curvature tensor $C_{abc}^d$ is defined by

$$C_{abc}^d = R_{abc}^d + \frac{1}{n-2} (R_{ac} \delta_b^d - R_{bc} \delta_a^d + g_{ac} R^d_b - g_{bc} R^d_a)$$

$$- \frac{R}{(n-1)(n-2)} (g_{ac} \delta_b^d - g_{bc} \delta_a^d),$$

where $R$ denotes the scalar curvature.

If the tensor $C_{abc}^d$ vanishes identically, then $M^n (n > 3)$ is called to be conformally flat.

A space of constant curvature $(n > 2)$ is a Riemannian space satisfying

$$R_{abc}^d = k (g_{bc} \delta_a^d - g_{ac} \delta_b^d)$$

and then $k$ is a constant given by $k = R/n(n-1)$.

A space of constant curvature is necessarily conformally flat.

2. Conformal Killing tensor. We shall call a skew symmetric tensor $u_{cd}$ a conformal Killing tensor if there exists a vector field $\rho_c$ such that

3) Indices $a, b, \cdots$ run over $1, \cdots, n$. Throughout this paper we assume that $n > 3$. 
We call \( \rho_c \) the associated vector of \( u_{cd} \). And if \( \rho_c \) vanishes identically, then \( u_{cd} \) is called a Killing tensor.\(^4\)

First we shall seek for differential equations of second order satisfied by \( u_{cd} \).

Transvecting (2.1) with \( g^{bc} \), we have

\[
\nabla^b u_{bd} = (n-1) \rho_d,
\]

where \( \nabla^b = g^{bc} \nabla_c \). Taking account of (1.1) it follows that

\[
\nabla^c \nabla^d u_{bc} = 0, \quad \nabla^c \rho_c = 0.
\]

In the following we shall write \( \rho_{ab} \) instead of \( \nabla_a \rho_b \) for brevity.

Operating \( \nabla_a \) to (2.1) we get

\[
\nabla_a \nabla_b u_{cd} + \nabla_a \nabla_d u_{bc} = 2 \rho_{ad} g_{bc} - \rho_{ac} g_{bd} - \rho_{ab} g_{cd}.
\]

By interchanging indices \( a, b, c \) as \( a \to b \to c \to a \) in this equation we obtain the following two equations:

\[
\nabla_b \nabla_a u_{cd} + \nabla_b \nabla_c u_{ad} = 2 \rho_{bd} g_{ac} - \rho_{bc} g_{ad} - \rho_{ab} g_{cd},
\]

\[
\nabla_c \nabla_a u_{bd} + \nabla_c \nabla_d u_{ab} = 2 \rho_{cd} g_{ab} - \rho_{ca} g_{bd} - \rho_{cb} g_{ad}.
\]

If we form (2.4)+(2.5)-(2.6), then it follows that

\[
2 \nabla_a \nabla_b u_{cd} - 2 R_{cba} u_{de} - R_{bad} u_{ce} - R_{acd} u_{be} - R_{bcd} u_{ae} = 2 (\rho_{ad} g_{bc} + \rho_{bd} g_{ac} - \rho_{cd} g_{ab}) + (\rho_{bc} - \rho_{cb}) g_{ad} + (\rho_{ca} - \rho_{ac}) g_{bd} - (\rho_{ab} + \rho_{ba}) g_{cd}.
\]

We shall deform (2.7) into another form. By \( b \to c \to d \to b \) in (2.7) we have

\[
2 \nabla_a \nabla_b u_{cd} - 2 R_{dca} u_{be} - R_{cad} u_{de} - R_{ab} u_{ce} + R_{cd} u_{ae} = 2 (\rho_{ab} g_{cd} + \rho_{bc} g_{da} - \rho_{ac} g_{bd}) + (\rho_{bd} - \rho_{dc}) g_{ab} + (\rho_{da} - \rho_{ad}) g_{cb} - (\rho_{ac} + \rho_{ca}) g_{db},
\]

4) S. Tachibana, [1].
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\[(2.9) \quad 2\nabla_a \nabla_a u_{bc} - 2R^{\delta a} c e u_{de} - R^{\delta a} c e u_{de} - R^{\delta a} c e u_{ae} \]
\[= 2(\rho_{ac} g_{db} + \rho_{ac} g_{ba} - \rho_{bc} g_{ad}) + (\rho_{bd} - \rho_{db}) g_{ac} + (\rho_{ba} - \rho_{ab}) g_{ac} \]
\[- (\rho_{ad} + \rho_{da}) g_{bc}. \]

Adding (2.8) and (2.9) to (2.7) side by side we can get

\[(2.10) \quad 2\nabla_a \nabla_a u_{cd} + \rho_{ac} g_{de} u_{cd} = (\rho_{bd} - \rho_{db}) g_{ac} + (\rho_{de} - \rho_{dc}) g_{ab} + (\rho_{cb} - \rho_{bc}) g_{ad} + 2\rho_{ad} g_{bc} \]
\[- 2(\rho_{ad} + \rho_{da}) g_{bc}. \]

where we have used the following equations which follows from (2.1):

\[\nabla_a \nabla_b u_{cd} + \nabla_a \nabla_c u_{bd} + \nabla_a \nabla_d u_{bc} = 3(\nabla_a \nabla_b u_{cd} + \rho_{ac} g_{bd} - \rho_{ad} g_{bc}). \]

Next we shall obtain algebraic relations between components of \( u_{cd} \) and the curvature tensor, ((2.14) below).

First by subtraction (2.7) from (2.10) we can get

\[(2.11) \quad R^{\delta a c e} u_{de} + R^{\delta a b d} u_{ce} + R^{\delta a c e} u_{be} + R^{\delta a b d} u_{ae} \]
\[= \sigma_{bd} g_{ca} + \sigma_{ca} g_{bd} - \sigma_{cd} g_{ab} - \sigma_{ab} g_{cd}, \]

where we have put

\[\sigma_{bd} = \rho_{bd} + \rho_{db}. \]

Transvecting (2.11) with \( g^{ab} \) and making use of

\[R_{\delta b c e} u^{be} + R_{\delta c d e} u^{be} = 0, \]

we obtain

\[(2.12) \quad \sigma_{bd} = \frac{1}{(n-2)} (R^{\delta a c e} u_{de} + R^{\delta a b d} u_{ce}). \]

We substitute (2.12) into (2.11) and put

\[(2.13) \quad T^{\delta a c e} = (n-2)R^{\delta a c e} - R^{\delta a} g_{ca} + R^{\delta a} g_{ba}, \]

so it follows that
(2.14) \[ (T_{bc a} \delta_{\alpha} + T_{ad c} \delta_{\beta} + T_{a c d} \delta_{\gamma} + T_{c d a} \delta_{\alpha}) u_{fe} = 0. \]

Now we shall show the following

**THEOREM 1.**\(^5\) If there exists (locally) a conformal Killing tensor which takes any preassigned (skew symmetric) value at any point of an \( n > 3 \) dimensional Riemannian space, then the space is conformally flat.

**PROOF.** Under the assumption as the skew symmetric parts of coefficients of \( u_{fe} \) in (2.14) vanish, we have

\[ T_{b c a} \delta_{\alpha} + T_{a d c} \delta_{\beta} + T_{a c d} \delta_{\gamma} + T_{c d a} \delta_{\alpha} \]

\[ = T_{b c a} \delta_{\alpha} + T_{a d c} \delta_{\beta} + T_{a c d} \delta_{\gamma} + T_{c d a} \delta_{\alpha}. \]

Contracting \( d \) and \( f \) in this equation we get

\[ T_{b c a \alpha} \left( -R_{ab} + \frac{R}{n-1} g_{ab} \right) \delta_{\alpha} + \left( R_{ac} - \frac{R}{n-1} g_{ac} \right) \delta_{\alpha}. \]

Substituting this into (2.13) it follows that \( C_{b c a} \delta_{\alpha} = 0. \) **Q.E.D.**

**3. A sufficient condition to be a conformal Killing tensor.** Let \( u_{cd} \) be a conformal Killing tensor. Then we can get

(3.1) \[ \nabla^a \nabla_a u_{cd} - R_c u_{de} - R^e_{de} u_{be} = - (n - 3) \rho_{cd} - \rho_{dc}, \]

by transvection (2.7) with \( g^{ab} \). Taking the skew symmetric part of (3.1), we have the following equations:

(3.2) \[ 2 \nabla^a \nabla_a u_{cd} - R_c u_{de} + R_d u_{ce} - R_{de} u_{be} = (n - 4) (\rho_{de} - \rho_{cd}). \]

In this section we shall show that a skew symmetric tensor \( u_{cd} \) satisfying (3.1) or (3.2) is a conformal Killing tensor provided that \( M^n \) is compact. To this purpose we prepare an integral formula about a tensor field.

Define a tensor \( A_{bcd} \) by

(3.3) \[ A_{bcd} = \nabla_b u_{cd} + \nabla_c u_{bd} - 2 \rho_d g_{bc} + \rho_b g_{cd} + \rho_c g_{bd} \]

for a skew symmetric tensor \( u_{cd} \), where \( \rho_c \) is given by

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\(^5\) Analogous theorem is well known for a conformal Killing vector. As to a Killing tensor, see S. Tachibana [1].
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\[(n - 1)p_c = \nabla^b u_{bc}.\]

Simple computations give us the following equations:

\[(3.4) \quad u^{cd} \nabla^b A_{bcd} = u^{cd}(\nabla^b \nabla \mu_{cd} - R_c^e u_{de} - R^b_{cd} u_{be} + (n - 3)p_{cd} + p_{dc}),\]

\[(3.5) \quad A_{bcd} A^{bcd} = 2 A_{bcd} \nabla^b u^{cd},\]

where we have used (1.1) and the relation:

\[
\nabla \nabla \mu^b_d = \nabla \nabla \nabla \mu^b_d + R_{bce} \mu^c_e - R_{bcd} \mu^b_e
\]

\[
= (n - 1)p_{cd} - R^e_{de} u_{de} - R^e_{cd} u_{be}.
\]

Substituting (3.4) and (3.5) into

\[
\nabla^b (A_{bcd} u^{cd}) = u^{cd} \nabla^b A_{bcd} + A_{bcd} \nabla^b u^{cd},
\]

we obtain the following

**THEOREM 2.** In a compact orientable Riemannian space \(M\), the following integral formula is valid for any skew symmetric tensor field \(u_{cd}\):

\[
\int_M [u^{cd}(\nabla^a \nabla u_{cd} - R^e_{de} u_{de} - R^b_{cd} u_{be} + (n - 3)p_{cd} + p_{dc}) + (1/2) A_{bcd} A^{bcd}] d\sigma = 0,
\]

where \(d\sigma\) means the volume element of \(M\) and \((n - 1)p_{cd} = \nabla^c \nabla^b u_{bd}.

Thus we have

**THEOREM 3.** In a compact Riemannian space a necessary and sufficient condition for a skew symmetric tensor field \(u_{cd}\) to be a conformal Killing tensor is (3.1) (or (3.2)).

**4. Conformal Killing tensor in a space of constant curvature.** For a conformal Killing tensor \(u_{cd}\) we have

\[(4.1) \quad \nabla_b u_{cd} + \nabla_c u_{bd} = 2 p_d g_{bc} - p_b g_{cd} - p_c g_{bd},\]

\[(4.2) \quad \nabla^b u_{bc} = (n - 1)p_c,\]

The following theorem is a trivial consequence of (4.3).

**Theorem 4.** In an Einstein space, the associated vector of a conformal Killing tensor is a Killing vector.

In the following we shall assume the space under consideration is a space of constant curvature.

Let \( \nu_c \) be a Killing vector. Then as is well known we have

\[
\nabla_a \nabla_b \nu_c + R_{abc} \nu^c = 0.
\]

Then by virtue of

\[
R_{abc} = k(g_{ac}g_{ab} - g_{ac}g_{ab}), \quad k = R/(n-1),
\]

the last equation turns to

\[
\nabla_a \nabla_b \nu_c = k(v_b g_{ac} - v_c g_{ab})
\]

and hence we obtain

(4.4) \[
\nabla_a \nabla_b \nu_c + \nabla_b \nabla_a \nu_c = k(-2v_c g_{ab} + v_b g_{ac} + v_a g_{bc}).
\]

This equation shows that \( \nabla_c \nu_c \) is a conformal Killing tensor.

Now if \( u_{cd} \) is a conformal Killing tensor, then its associated vector \( \rho_c \) is a Killing vector and hence \( \nabla_c \rho_c \) is a conformal Killing tensor whose associated vector is given by \(-k\rho_c\). Thus we have

(4.4) \[
\nabla_b \nabla_c \rho_d + \nabla_c \nabla_b \rho_d = -k(2\rho_d g_{bc} - \rho_b g_{cd} - \rho_c g_{bd}).
\]

Let us assume that \( k \neq 0 \) (i.e., \( R \neq 0 \)). If we put

\[
\rho_{cd} = u_{cd} + (1/k) \nabla_c \rho_d,
\]

then by virtue of (4.1) and (4.4), it follows that

\[
\nabla_b \rho_{cd} + \nabla_c \rho_{bd} = 0,
\]

which means \( \rho_{cd} \) is a Killing tensor. Consequently a conformal Killing tensor.
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$u_{cd}$ is decomposed in the form:

$$u_{cd} = p_{cd} + q_{cd},$$

where $p_{cd}$ is a Killing tensor and $q_{cd} = (-1/k)\nabla_c \rho_d$ is a conformal Killing tensor. Hence we have

**Theorem 5.**

In a space $M^n(n > 2)$ of constant curvature with $k = R/n(n-1) \neq 0$, a conformal killing tensor $u_{cd}$ is uniquely decomposed in the form:

$$u_{cd} = p_{cd} + q_{cd},$$

where $p_{cd}$ is a Killing tensor and $q_{cd}$ is a closed conformal Killing tensor. In this case $q_{cd}$ is the form

$$q_{cd} = (-1/k)\nabla_c \rho_d$$

where $\rho_d$ is the associated vector of $u_{cd}$.

Conversely if $p_{cd}$ is a Killing tensor and $\rho_d$ is a Killing vector, then $u_{cd}$ given by (4.5) is a conformal Killing tensor.

The uniqueness of the decomposition follows from the following

**Lemma.** Under the assumption of Theorem 5, if a Killing tensor is closed, then it is a zero tensor.

**Proof of Lemma.** Let $u_{cd}$ be a closed Killing tensor. Then we have

$$\nabla_b u_{cd} + \nabla_c u_{db} + \nabla_d u_{bc} = 0,$$

$$\nabla_b u_{cd} + \nabla_c u_{db} = 0.$$

Hence we get $\nabla_b u_{cd} = 0$. Thus by virtue of Ricci's identity it follows that

$$R_{ae} u_{fe} + R_{ae} u_{bf} = 0.$$  

As the space is of constant curvature, we can obtain by a transvection with $g^{ab}$

$$(n - 2)ku_{cd} = 0.$$

**Example.** Let $E^{n+1}$ be the Euclidean space with orthogonal coordinates

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7) For a conformal Killing vector, see K. Yano and T. Nagano [5].
Consider the unit sphere $S^n$ and let $\{x^a\}$ be its local coordinates. Putting $B^a_\lambda = \partial y^\lambda / \partial x^a$, we see that the second fundamental tensor $H^a_\lambda$ is given by

$$H^a_\lambda = \nabla^a B^b_\lambda = \partial_y B^b_\lambda - B^b_\lambda (e^a) + B^b_{\alpha} (e^a), \quad \partial_y = \partial / \partial x^b.$$ 

As $S^n$ is totally umbilic we have $H^a_\lambda = g^a_\lambda N^\lambda$, where $N^\lambda$ means the unit normal vector: $N^\lambda = -y^\lambda$.

Let $v_{\mu\nu}$ be a parallel skew symmetric tensor field and define a tensor field $u^a_{bc}$ on $S^n$ by $u^a_{bc} = B^a_b B^c_c v_{\mu\nu}$. Operating $\nabla^a$ to this equation we have

$$\nabla^a u^a_{bc} = B^a_b \nabla^a v_{\mu\nu} B^c_c v_{\mu\nu} + v_{\mu\nu} (H^a_b B^c_c v_{\mu\nu} + B^a_b H^c_c v_{\mu\nu})$$

$$= v_{\mu\nu} (N^\mu B^\nu g_{ab} + N^\nu B^\mu g_{ac}).$$

If we put $\rho^a = v_{\mu\nu} N^\mu B^\nu$, then it follows that

$$\nabla^a u_{bc} = \rho^a g_{ab} - \rho^a g_{ac}.$$ 

Thus we get

$$\nabla^a u_{bc} + \nabla^b u_{ac} = 2 \rho^a g_{ab} - \rho^a g_{bc} - \rho^a g_{ac},$$

which shows $u^a_{bc}$ is a conformal Killing tensor on $S^n$.

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