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SOME REMARKS ON ANALYTIC CONTINUATIONS

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1. The purpose of the present paper is to prove some theorems concerning continuations of analytic functions across simple open arcs. Here, a simple open arc means a topological image of the open interval $\{t;\ 0 < t < 1\}$.

Let D_1 and D_2 be Jordan domains in the z-plane having no point in common and I be a simple open arc lying on the non-empty common boundary of D_1 and D_2. Then there arises the following

PROBLEM. Given two analytic functions f_1 and f_2 in D_1 and D_2 respectively, we set $f = f_1$ in D_1 and $f = f_2$ in D_2. Under what condition do there exist an open subset I^* of I and an analytic function $F(z)$ in $D_1 \cup I^* \cup D_2$ such that $F(z) = f_j(z)$ for $z \in D_j$ $(j = 1, 2)$? In other words, under what conditions on f and I can f be extended analytically to an open subset I^* of I?

This problem was investigated by some authors, e.g., Carleman [5], Wolf [14], Meier [8] and from cluster-sets-theoretic viewpoint, Bagemihl [3] gave an answer to this problem under the restriction of I being an open interval on a straight line. Recently, Noshiro [9] gave an improvement of Bagemihl's theorem [3] (cf. also [10]).

First in §2 we shall prove an analogous theorem to Bagemihl-Noshiro's in the case where I is an open locally rectifiable arc. Instead of the condition (c) in Theorem 6 in [9] we shall give a global restriction to f. In §3 we assume that I is a simple open smooth arc. We give an answer to the problem under the condition that f_j belongs to the Hardy class $H_p(D_j)$ for $p > 1$ $(j=1, 2)$. In §4 we assume that I is a simple open analytic arc. Under the weaker condition that f_j is in the class $H_1(D_j)$ $(j=1, 2)$, we shall give another answer to the problem. Finally in §5 we shall state some remarks on null-sets for the class H_p, $p \geqq 1$, as applications of two theorems in §3 and in §4.

2. By an open locally rectifiable arc I we mean a simple open arc such that every point of I has a neighbourhood which is a rectifiable subarc of I.

We remark that a rectifiable simple arc must be a topological image of the closed interval $\{t\,;\,0 \leqq t \leqq 1\}$. We also remark that a rectifiable arc has a tangent at every point except for a set of linear measure zero. Here a subset E of an open locally rectifiable arc I is said to be of linear measure zero if for any $\varepsilon > 0$, there exists a countable number of open subarcs $\{I_n\}$ of I such that $\bigcup I_n \supset E$ and $\sum mI_n < \varepsilon$, where m denotes the linear measure (the length).

An analytic function f in a plane domain D is said to be in the class $S(D)$ provided that the subharmonic function $\log^+|f| = \max(\log|f|, 0)$ admits a harmonic majorant in D which is quasi-bounded, i.e., the limiting function of a monotone non-decreasing sequence of non-negative bounded harmonic functions in D (cf. e.g., [15]).

An analytic function f in D is said to be in the Hardy class $H_p(D)$ $(0 < p < \infty)$ if the subharmonic function $|f|^p$ admits a harmonic majorant in D.

Both classes $S(D)$ and $H_p(D)$ have local property, i.e., if f is in the class $X(D)$, then f is in $X(D')$ for any subdomain $D' \subset D$. Furtermore, $H_p \subset H_q \subset S$, for $p \geqq q$.

We state the definition of another class $E_1(D)$. Let D be a Jordan domain with the rectifiable boundary and $z = z(w)$ be a one-to-one conformal map of the disc U: $|w| < 1$ onto D. An analytic function $f(z)$ defined in D is said to belong to the class $E_1(D)$ if the function $f(z(w))\,z'(w)$ is in the class $H_1(U)$. It is shown that this definition is independent of the choice of a map $z(w)$ (cf. [6]).

The following lemma will play a fundamental rôle.

LEMMA. *Let D_1, D_2, I, f_1, f_2 and f be as in the problem. Assume that the boundaries of D_1 and D_2 are rectifiable and $D_1 \cup I \cup D_2$ is a Jordan domain (with the rectifiable boundary). Let E be a subset of I of linear measure zero. For every $\zeta \in I - E$, let L^j_ζ be a simple arc in D_j terminating at ζ $(j=1,2)$. Suppose that*

(*) $\lim_{\substack{z\to\zeta \\ z\in L^1_\zeta}} f_1(z) = \lim_{\substack{z\to\zeta \\ z\in L^2_\zeta}} f_2(z)$ $(\neq\infty)$ *at every point ζ of $I-E$;*

(**) *f_j is in the class $E_1(D_j)$ $(j = 1, 2)$.*

Then f can be extended analytically to the whole I in the sense stated in the problem.

PROOF. By the condition (**) we obtain

$$\frac{1}{2\pi i}\int_{\partial D_j} \frac{f_j^*(\zeta)}{\zeta - z}\,d\zeta = f_j(z) \quad \text{if} \quad z \in D_j\,,$$

$$= 0 \qquad \text{if} \quad z \notin \overline{D}_j\,,$$

where $f_j^*(\zeta)$ is the non-tangential limit of f_j at the point $\zeta \in \partial D_j$ except for a set of linear measure zero, the integration is taken to the positive sense, and bar means the closure ($j = 1, 2$) (cf. chap. 10, §5 in [6]). By (*) and by Bagemihl's ambiguous-point theorem [2], we have $f_1^*(\zeta) = f_2^*(\zeta)$ except for a set of linear measure zero in I. Now we set

$$F(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f^*(\zeta)}{\zeta - z}\, d\zeta , \qquad z \in D ,$$

where $D = D_1 \cup I \cup D_2$ and $f^*(\zeta) = f_j^*(\zeta)$, $\zeta \in \partial D_j$ ($j = 1, 2$). Then F is analytic in D and

$$\begin{aligned} F(z) &= \frac{1}{2\pi i} \int_{\partial D_1} \frac{f_1^*(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_{\partial D_2} \frac{f_2^*(\zeta)}{\zeta - z}\, d\zeta \\ &= f_j(z) \quad \text{if} \quad z \in D_j \ (j = 1, 2) . \end{aligned}$$

This completes the proof of the lemma.

Now we are ready to prove

THEOREM 1. *Let D_1, D_2, I, f_1, f_2 and f be as in the problem. Assume that I is an open locally rectifiable arc and let E be a subset of I of linear measure zero. For every $\zeta \in I - E$, let L_ζ^j be a simple arc in D_j terminating at ζ ($j = 1, 2$). Suppose that*

(a) $\lim_{z \to \zeta,\, z \in L^1_\zeta} f_1(z) = \lim_{z \to \zeta,\, z \in L^2_\zeta} f_2(z) = \omega_\zeta \ (\neq \infty)$ *for every point ζ of $I - E$;*

(b) *the function $\varphi(\zeta) = \omega_\zeta$ defined on $I - E$ is bounded in some neighbourhood of every point ζ of $I - E$;*

(c) *f_j is in the class $S(D_j)$ ($j = 1, 2$).*

Then there exists a closed set e relative to I such that e is a subset of E and f can be extended to be analytic in the open set $I - e$.

PROOF. First we prove:

For any point $\zeta_0 \in I - E$ there exists a Jordan domain D_0 with the rectifiable boundary such that

(1) $\zeta_0 \in D_0$ and the open set $D_1 \cup D_2$ has no exterior point belonging to D_0;

(2) $D_0^j = D_0 \cap D_j$ is a Jordan domain with the rectifiable boundary ($j = 1, 2$);

(3) $\overline{D_0^1} \cap I = \overline{D_0^2} \cap I = J_0$ is a rectifiable arc containing ζ_0 in its interior;

(4) f is bounded in $D_0^1 \cup D_0^2$.

By the condition (b) there exists a rectifiable subarc J_0^1 of I such that J_0^1 has ζ_0 as its interior point, I has normals from the interior of D_1 to both terminal points of J_0^1 and such that $\varphi(\zeta)$ is bounded in J_0^1-E. Therefore there exists a Jordan domain $G_0^1\subset D_1$ with the rectifiable boundary such that $\bar{G}_0^1\cap I=J_0^1$. Let $z=z(w)$ be a one-to-one comformal map of the unit disc U onto G_0^1. Let J_0^* be the inverse image of J_0^1 by the natural extension $\zeta=z(\zeta^*)$ of $z=z(w)$ to the unit circle $|\zeta^*|=1$. Set $\zeta_0=z(\zeta_0^*)$. Evidently the composite function $F_1(w)=f_1(z(w))$ is in the class $S(U)$ and F_1 has the asymptotic value $\varphi(z(\zeta^*))$ at a.e. (almost every) point ζ^* in the interior I_0^* of J_0^*. By using Bagemihl's theorem [2] again we know that F_1 has radial limit $F_1(\zeta^*)=\varphi(z(\zeta^*))$ at a.e. point ζ^* of I_0^* since F_1 is of bounded type. Without loss of generality, we may assume that $|F_1(\zeta^*)|<1$ at a.e. point in I_0^* since the space $S(U)$ is linear. Let $h(w)$ be the least harmonic majorant of $\log^+|F_1(w)|$ in U, which is quasi-bounded, and hence is represented as the Poisson integral of its radial limits. Then $h(w)$ has radial limit $h(\zeta^*)=\log^+|F_1(\zeta^*)|=0$ at a.e. point ζ^* in the open arc I_0^*. Hence h can be continued harmonically across I_0^*. This shows that there exists an open disc d with the centre ζ_0^* such that h and consequently F_1 are bounded in $d\cap U$. Therefore there exists an open disc v_1 with the centre ζ_0 such that f_1 is bounded in $v_1\cap D_1$. Similarly we can choose a disc v_2 for f_2. Now we can make easily $D_0^j\subset v_1\cap D_j$ as we wanted $(j=1,2)$.

Next we remark that the derived function $z'(w)$ in the definition of the class $E_1(D)$ is in the class $H_1(U)$ since D has the rectifiable boundary. This shows that any bounded analytic function in D belongs to $E_1(D)$. Now we can apply the lemma to D_0^j and f_j since $f_j\in E_1(D_0^j)$ $(j=1,2)$. As a consequence we know that f can be extended analytically to I_0, the interior of J_0. Thus we have proved that f can be extended to be analytic to an open arc $I_\zeta\subset I$ corresponding to every point $\zeta\in I-E$. Set $e=I-\bigcup_{\zeta\in I-E} I_\zeta$. Then e satisfies the conditions of the theorem. This completes the proof of the theorem.

REMARK. Let A_∞ be the set of points of I at which at least one of f_1 and f_2 has ∞ as an asymptotic value. Then, instead of the condition (c) in our theorem, we can take

(c*) $\bar{A}_\infty\cap(I-E)=\emptyset$.

In fact, this condition implies that the function $F_1(w)=f_1(z(w))$ in our proof is bounded in $d\cap U$ since the technique in the proof of Bagemihl-Noshiro's theorem ([3], [9]) is available. The rest of the proof is the same as in ours. Furthermore, this shows that we can mix these two conditions, i.e., we can take the following (c**) instead of (c):

(c**) f_1 is in the class $S(D_1)$ and $\bar{A}_\infty^2\cap(I-E)=\emptyset$, where A^2_∞ is the set of points in I at which f_2 has ∞ as an asymptotic value.

The problem of finding relations between the conditions (c) and (c*) seems to be open.

3. By a simple open smooth arc we mean a simple open arc I such that at every point $\zeta \in I$ there exists a unique tangent vector T_ζ and such that the angle $\theta(\zeta)$ of the vector T_ζ to the positive real axis is a continuous function of $\zeta \in I$. Clearly a simple open smooth arc is locally rectifiable.

We obtain

THEOREM 2. *Let D_1, D_2, I, f_1, f_2 and f be as in the problem. Assume that I is a simple open smooth arc and let E be a subset of I of linear measure zero. For every $\zeta \in I-E$, let L^j_ζ be a simple arc in D_j terminating at ζ $(j=1,2)$. Suppose that*

(I) $\lim_{\substack{z\to\zeta \\ z\in L^1_\zeta}} f_1(z) = \lim_{\substack{z\to\zeta \\ z\in L^2_\zeta}} f_2(z)$ $(\neq\infty)$ *for every point ζ of $I-E$;*

(II) *f_j is in the class $H_p(D_j)$ for some $p>1$ $(j=1,2)$.*

Then f can be extended analytically to the whole I.

PROOF. First, by the property of I we can make easily a Jordan domain D_0 with the rectifiable boundary corresponding to every point ζ_0 in the whole I such that the following conditions hold:

(1) the same condition as (1) in the proof of Theorem 1;

(2′) $D^j_0 = D_0 \cap D_j$ is a Jordan domain with the smooth boundary $(j=1,2)$;

(3) the same condition as (3) in the proof of Theorem 1.

Here a Jordan curve J: $z=z(t)$, $0\leqq t\leqq 1$ is said to be smooth if any simple open subarc of J is smooth and if we denote by $\theta(t)$ the angle of the tangent vector at the point $z(t)$ $(0\leqq t<1)$ to the real axis, we have $\lim_{t\to 1}\theta(t)=\theta(0)+2\pi$. The existence of D^1_0, for example, is shown by the existence of a smooth curve in D_1 tangent to I at the point near ζ_0.

Next we show

(4′) f_j is in the class $E_1(D^j_0)$ $(j=1,2)$.

Let $z=z(w)$ be a one-to-one conformal map of the unit disc U onto D^1_0. Then by the well-known theorem (cf. Theorem 5, p. 410, [6]) the function $z'(w)$ is in the class $H_q(U)$ for any $q>0$. On the other hand, the function $F_1(w) = f_1(z(w))$ is in the class $H_p(U)$. Hence by Hölder's inequality

$$\int_0^{2\pi} |f_1(z(re^{i\theta}))z'(re^{i\theta})|\, d\theta \leqq \left(\int_0^{2\pi} |F_1(re^{i\theta})|^p\, d\theta\right)^{1/p} \left(\int_0^{2\pi} |z'(re^{i\theta})|^q\, d\theta\right)^{1/q},$$

with $(1/p)+(1/q)=1$, $0 \leqq r < 1$, we know that $f_1(z(w))z'(w)$ is in $H_1(U)$ and hence f_1 is in the class $E_1(D_0^1)$. Similarly f_2 is in the class $E_1(D_0^2)$. The rest of the proof is the same as in the proof of Theorem 1.

4. Under a stronger condition that I is analytic, we obtain

THEOREM 3. *Let D_1, D_2, I, f_1, f_2 and f be as in the problem. Assume that I is a simple open analytic arc and let E be a subset of I of linear measure zero. For every $\zeta \in I-E$, let L_ζ^j be a simple arc in D_j terminating at ζ $(j=1,2)$. Suppose that*

(i) $\lim\limits_{\substack{z\to\zeta \\ z\in L^1_\zeta}} f_1(z) = \lim\limits_{\substack{z\to\zeta \\ z\in L^2_\zeta}} f_2(z)$ $(\neq \infty)$ *for every point ζ of $I-E$;*

(ii) *f_j is in the class $H_1(D_j)$ $(j=1,2)$.*

Then f can be extended analytically to the whole I.

PROOF. We have only to prove the following:

Let G be the open unit disc $|z|<1$, G_1 be the open upper half disc and G_2 be the open lower half disc. Let I be the open interval $-1<x<1$ on the real axis. Let g_j be in $H_1(G_j)$ $(j=1,2)$. Assume that

(iii) $\lim\limits_{\begin{cases} z\to x \\ \Re z = x \\ z\in G_1\end{cases}} g_1(z) = \lim\limits_{\begin{cases} z\to x \\ \Re z = x \\ z\in G_2\end{cases}} g_2(z)$ $(\neq\infty)$ for a.e. point $x\in I$.

Then there exists an analytic function g in G such that $g=g_j$ in G_j $(j=1,2)$.

To prove this, set $\psi(z) = g_1(z)+\overline{g_2(\bar{z})}$ and $\chi(z) = i(g_1(z)-\overline{g_2(\bar{z})})$ for $z\in G_1$, where bar means the complex conjugate. Then both ψ and χ are in $H_1(G_1)$ since the function $g^*(z)=\overline{g_2(\bar{z})}$ for $z\in G_1$ is in $H_1(G_1)$ and the class $H_1(G_1)$ is linear. By the condition (iii) both ψ and χ have real asymptotic values along the vertical lines at a.e. point in I. Let $z=z(w)$ be a one-to-one conformal map of the unit disc U onto G_1. Then we can apply Rudin's lemma (Lemma 4.4., p. 59, [12]) to the functions $\psi(z(w))$ and $\chi(z(w))$. As a consequence we know that both ψ and χ can be continued analytically to the whole G and the Schwarz reflexion principle holds. Let Ψ and X be the resulting functions of ψ and χ respectively and set $g(z)=\frac{1}{2}(\Psi(z)-i\mathrm{X}(z))$. Then $g(z)=\frac{1}{2}(\psi(z) - i\chi(z)) = g_1(z)$ in G_1 and if $z\in G_2$, $g(z)=\frac{1}{2}(\overline{\Psi(\bar{z})}-i\,\overline{\mathrm{X}(\bar{z})})=\frac{1}{2}(\overline{\psi(\bar{z})}-i\,\overline{\chi(\bar{z})}) = g_2(z)$. This completes the proof of our assertion.

REMARK. Noshiro [10] remarked that there exists a function $g_1(z)$ analytic in the upper half plane $\widehat{D}_1$: $\Im z > 0$ with the following properties:

(A) $\Im g_1(z) > 0$ in $\widehat{D}_1$;

(B) $g_1(z)$ has a real vertical limit at a.e. point on the real axis;

(C) $g_1(z)$ has an essential singularity at every point on the real axis.

Now we set $g_2(z)=\overline{g_1(\bar{z})}$ in the lower half plane $\widehat{D}_2$. Then applying Smirnov-Cargo's Theorem (Theorem 2, [4]) we have $g_j \in H_p(\widehat{D}_j)$ for any p, $0 < p < 1$, since g_j takes values in a half plane ($j=1,2$). The vertical limits of g_1 and g_2 coincide at a.e. point on the real axis by the condition (B). Thus we cannot replace the condition (ii) in Theorem 3 by

(ii′) f_j is in the class $H_p(D_j)$ ($j = 1, 2$)

for p, $0 < p < 1$.

5. A totally disconnected compact set E in the plane is said to be null for H_p if any element of $H_p(CE)$ is constant, where CE is the complement of E with respect to the extended plane. It is known that if E is of logarithmic capacity zero, then E is null for any H_p, $0 < p < \infty$ ([11], cf. [12], [13] and [15]). As a direct corollary to Theorem 2 (resp. Theorem 3) we have: *A compact set of linear measure zero lying on a simple open smooth* (*resp. analytic*) *arc is null for* H_p, $p > 1$ (*resp.* H_1).

Obviously, any H_p-null set is an N_B set in the sense of Ahlfors and Beurling [1] ($0 < p < \infty$). On the other hand, if E, lying on a simple open analytic arc, is an N_B set, then E is of linear measure zero ([1]). This shows that the notion of H_p-null sets ($p \geqq 1$) and the notion of N_B sets coincide under the restriction of E lying on a simple open analytic arc. We remark also that Havin and Havinson [7] proved: If E, lying on a smooth Jordan curve of a special type, is an N_B set, then E is of linear measure zero. This shows that the notion of H_p-null sets ($p > 1$) and the notion of N_B sets coincide under their assumption.

It is well known that there exists a compact set of linear measure zero lying on the real axis and of positive logarithmic capacity. This means that Rudin's question (Q_1) (p. 49, [12]) is answered in the *negative* for $p \geqq 1$.

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