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SUMMABILITY IN TOPOLOGICAL GROUPS, IV (CONVERGENCE FIELDS)

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Introduction. In the papers [7], [8], and [9], I introduced the notions and notations of summability theory in the topological group setting. In [7] the groups were assumed Hausdorff and first countable. Some general theorems of the Toeplitz-Schur type were given. In [9] the fact of the existence of a right invariant metric was used and some theorems about the bounded convergence field of a regular limitation method were given. In this paper I shall investigate some (non-metric) properties of the convergence field of a method. The results generalize some theorems obtained by Agnew, Buck, Hill, Keough and Petersen, and the author. The reader may refer to their works listed in the references at the end of this paper. In all that follows G will denote a topological Hausdorff group which satisfies the first axiom of countability.

It is well-known that no regular matrix has all bounded sequences in its convergence field. For generalizations to topological groups see [7, page 265] and [9, Theorem 2, Corollary]. A related result asserts that for each regular matrix there is a strictly stronger matrix. This result generalizes as follows.

THEOREM 1. *Let G be a non-trivial group and let $\{f(m)\}$ be a regular triangular limitation method on G. Then there exists a regular triangular limitation method, $\{g(m)\}$, on G which is consistent with $\{f(m)\}$ and whose convergence field properly contains the convergence field of $\{f(m)\}$.*

PROOF. Let $\{f(m)\}$ be a regular triangular limitation method on G and let x be a non-zero element of G. Let $\{U(i)\}$ be a basic sequence of neighborhoods of 0 with $U(i+1)\subset U(i)$ for $i=1, 2,\cdots$. By condition (2) of the Toeplitz type theorem (Theorem 1, page 261, of [7]) there is a positive

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integer $N(1)$ such that if $m \geqq N(1)$ then

$$f(m : x, x, \cdots, x) \in U(1) + x .$$

By condition (1) of the same theorem there is a positive integer $N(2)$ such that $N(2) > N(1)$ and if $y(n) = x$ for $n = 1, 2, \cdots, N(1)$ then for $m \geqq N(2)$ we have

$$f(m : y(1), y(2), \cdots, y(N(1)), 0, 0, \cdots, 0) \in U(2).$$

It is clear that by using conditions (2) and (1) alternately we can construct a sequence $\{y(n)\}$ of blocks of x's and zeros such that if p is a positive even integer then

$$f(N(p) : y(1), y(2), \cdots, y(N(p))) \in U(p)$$

and is p is a positive odd integer then

$$f(N(p) : y(1), y(2), \cdots, y(N(p))) \in U(p) + x .$$

Thus $\{f(m)\}$ does not limit $\{y(n)\}$.

Let us define a regular triangular limitation method, $\{h(m)\}$, as follows:

$$h(m : z(1), z(2), \cdots, z(m)) = \begin{cases} 0 & \text{for } 1 \leqq m < N(2) \\ z(N(2j)) & \text{for } N(2j) \leqq m < N(2j+2) \end{cases}$$

for all $j = 1, 2, \cdots$ and all sequences, $\{z(n)\}$, of elements of G. If we now define a sequence, $\{g(m)\}$, by setting

$$\begin{aligned} &g(m : z(1), z(2), \cdots, z(m)) \\ &\qquad = h(m : f(1 : z(1)), f(2 : z(1), z(2)), \cdots, f(m : z(1), z(2), \cdots, z(m))) , \end{aligned}$$

for all $m = 1, 2, \cdots$ and all sequences, $\{z(n)\}$, of elements of G, then it is clear that $\{g(m)\}$ is a regular triangular limitation method which is consistent with $\{f(m)\}$. It is also clear that $\{g(m)\}$ limits $\{y(n)\}$ and, hence, that its convergence field properly contains the convergence field of $\{f(m)\}$.

I have not yet been able to prove or disprove generalizations of the theorems of Brudno. The reader may refer to Petersen [6, pages 110-114].

There are many theorems establishing conditions on a matrix to ensure that it be stronger than convergence. Generalizations of some of them follow.

THEOREM 2. *Let $\{f(m)\}$ be a triangular limitation method on G with the following two properties*:

(a) *There is a non-zero element x in G such that for all neighborhoods U of 0 there is a positive integer N so that if $m \geqq n \geqq N$ then*

$$f(m:0,0,\cdots,0,x,0,0,\cdots,0) \in U$$

with x in the n^{th} position.

(b) *$\{f(m)\}$ transforms the sequence with the above x in the n^{th} position and zeros elsewhere into a Cauchy sequence for any fixed n.*

Then there is a divergent sequence in the convergence field of $\{f(m)\}$.

PROOF. Let $\{U(j)\}$ be a basic sequence of symmetric neighborhoods of 0 such that for each $j=1,2,\cdots$, it is true that if x, y are in $U(j+1)$ then $x+y$ is in $U(j)$. Let x be the non-zero element of G whose existence is given by (a). Then for each neighborhood $U(j)$ there is a corresponding positive integer $N(j)$ so that if $m \geqq n \geqq N(j)$ then

$$f(m:0,0,\cdots,0,x,0,0,\cdots,0) \in U(j)$$

with x in the n^{th} position. Clearly we may choose the integers $N(j)$ so that $N(j) < N(j+1)+1 \cdots$. Now let us consider the divergent sequence $\{y(n)\}$ for which $y(N(j)) = x$ for $j = 1,2,\cdots$ and $y(n) = 0$ otherwise.

Let V be any neighborhood of 0 and pick a positive integer k for which $U(k) \subset V$. Condition (b) implies that there is a positive integer M, $M > N(k+2)$, so that if p and q are integers with $p, q > M$ then

$$\begin{aligned} &f(p:y(1),y(2),\cdots,y(N(k+2)),0,0,\cdots,0) \\ &\qquad - f(q:y(1),y(2),\cdots,y(N(k+2),0,0,\cdots,0) \in U(k+2)\,. \end{aligned}$$

Then for $p, q > M$ it follows that

$$\begin{aligned} &f(p:y(1),y(2),\cdots,y(p)) - f(q:y(1),y(2),\cdots,y(q)) \\ &\quad = f(p:0,0,\cdots,0,y(N(k+3)),y(N(k+3)+1),\cdots,y(p)) \\ &\qquad + [f(p:y(1),y(2),\cdots,y(N(k+2)),0,0,\cdots,0) \\ &\qquad - f(q:y(1),y(2),\cdots,y(N(k+2)),0,0,\cdots,0)] \\ &\qquad - f(q:0,0,\cdots,0,y(N(k+3)),y(N(k+3)+1),\cdots,y(q))\,. \end{aligned}$$

By the choice of M, the middle bracketed term is in $U(k+2)$. The first term,

$$f(p: 0, 0, \cdots, 0, y(N(k+3)), y(N(k+3)+1), \cdots, y(p)),$$

may be written

$$\sum_{j=k+3}^{m(p)} f(p: 0, 0, \cdots, 0, y(N(j)), 0, 0, \cdots, 0)$$

where $m(p) \leqq p < m(p+1)$. But since

$$f(p: 0, 0, \cdots, 0, y(N(j)), 0, 0, \cdots, 0) \in U(j)$$

for each $j = k+3, k+4, \cdots, m(p)$, it follows that

$$f(p: 0, 0, \cdots, 0, y(N(k+3)), y(N(k+3)+1), \cdots, y(p)) \in U(k+2).$$

Similarly

$$f(q: 0, 0, \cdots, 0, y(N(k+3)), y(N(k+3)+1), \cdots, y(q)) \in U(k+2).$$

Then

$$f(p: y(1), y(2), \cdots, y(p)) - f(q: y(1), y(2), \cdots, y(q)) \in U(k) \subset V.$$

Hence the divergent sequence $\{y(n)\}$ is in the convergence field of $\{f(m)\}$.

COROLLARY. *If $\{f(m)\}$ is a regular triangular limitation method which satisfies condition* (a) *of Theorem* 2 *then $\{f(m)\}$ is stronger than convergence.*

It is quite obvious that if $\{f(m)\}$ is a regular triangular limitation method with an inverse $\{g(m)\}$ which is also a regular triangular limitation method then $\{f(m)\}$ will transform no divergent sequence into a Cauchy sequence. Somewhat less obvious is the following generalization of a theorem to be found in Cooke [2, p. 98].

THEOREM 3. *If $\{f(m)\}$ and $\{g(m)\}$ are regular triangular limitation methods on G then $\{f(m)\}$ and $\{g(m)\}$ are consistent if they commute.*

PROOF. Let $\{x(n)\}$ be a sequence in the convergence field of both $\{f(m)\}$ and $\{g(m)\}$ and let U be a neighborhood of 0. Choose a neighborhood V of 0 such that if x and y are in V then $x+y$ is in U. Since $\{f(m: x(1), x(2), \cdots, x(m))\}$ is Cauchy and $\{g(m)\}$ is regular there is a positive integer $N(1)$ such that if $m, n \geqq N(1)$ then

$$\begin{aligned}&f(m:x(1),x(2),\cdots,x(m))\\&\qquad -g(n:f(1:x(1)),f(2:x(1),x(2)),\cdots,f(n:x(1),x(2),\cdots,x(n)))\in V\,.\end{aligned}$$

Similarly there is an integer $N(2)$ such that if $m, n \geqq N(2)$ then

$$\begin{aligned}&g(m:x(1),x(2),\cdots,x(m))\\&\qquad -f(n:g(1:x(1)),f(2:x(1),x(2)),\cdots,f(n:x(1),x(2),\cdots,x(n)))\in V\,.\end{aligned}$$

Then if $m, n \geqq \max[N(1), N(2)]$

$$\begin{aligned}&f(m:x(1),x(2),\cdots,x(m))-g(n:x(1),x(2),\cdots,x(n))\\&\qquad=[f(m:x(1),x(2),\cdots,x(m))\\&\qquad\qquad-g(p:f(1:x(1)),f(2:x(1),x(2),\cdots,f(p:x(1),x(2),\cdots,x(p)))]\\&\qquad\qquad+[f(p:g(1:x(1)),g(2:x(1),x(2)),\cdots,g(p:x(1),x(2),\cdots,x(p)))\\&\qquad\qquad-g(n:x(1),x(2),\cdots,x(n))]\in U\end{aligned}$$

for any integer $p \geqq \max[N(1), N(2)]$. Thus $\{f(m)\}$ and $\{g(m)\}$ are consistent.

The following theorem again restricts the convergence field of a regular method. The result is certainly stronger than the Corollary to Theorem 2 of [7] and Theorem 1 of this paper. It is closely related to but sharper than Theorem 5 of [9] on metric groups and generalizes a theorem of Hill [4, page 560].

Let G be a non-trivial group and let x be a non-zero element of G. Let T be the set of all sequences of zeros and x's. It is clear that we can correspond the elements of T with the real numbers in [0, 1] in the form of dyadic numbers. Let T have the weak topology induced by this correspondence. Now we can state the theorem.

THEOREM 4. *Let G be a non-trivial group, x a non-zero element of G, and $\{f(m)\}$ a regular triangular limitation method on G. Then the subset of T which is not in the convergence field of $\{f(m)\}$ is of the second category in T and its complement relative to T is of the first category in T.*

We may remark here, as Hill does, that the measure of the subset of [0, 1] corresponding to the subset of T in the convergence field of $\{f(m)\}$ is either 0 or 1. For, clearly, changing any finite number of the terms of a sequence will not affect the convergence or divergence of the sequence's $\{f(m)\}$ transform and so the subset of $[0, 1]$ is homogeneous, hence of measure 0 or 1.

PROOF. Let T' be the subspace of T obtained by deleting all eventually constant sequences. All such sequences are in the convergence field of $\{f(m)\}$ and there are countably many of them. Thus it is sufficient to prove that the subset of T' in the convergence field of $\{f(m)\}$ is of the first category in T'. As in the proof of Theorem 1 above we can choose a basic sequence $\{U(i)\}$ of symmetric neighborhoods of 0 which decreases monotonically, an increasing sequence of positive integers $\{N(p)\}$, and a sequence $\{y(n)\}$, consisting of alternate blocks of 0's and x's, for which

$$f(N(p): y(1), y(2), \cdots, y(N(p))) \in U(p)$$

if p is even and

$$f(N(p): y(1), y(2), \cdots, y(N(p))) \in U(p) + x$$

if p is odd. Let the positive integer P be chosen such that $x \notin U(P)$, $0 \in U(P)$, and $U(P) \cap (U(P)+x) = \phi$. Now let Q and R be integers with $R > Q > P$ for which $y \in U(R)$ implies that $x+y-x \in U(Q)$ and y, $z \in U(Q)$ implies that $y+z \in U(P)$.

For each positive integer i, let $T(i)$ be the subset of T'(prime) which consists of those sequences, $\{z(n)\}$, with the property that there are integers $p, q > i$ with

$$f(p: z(1), z(2), \cdots, z(p)) \in U(R)$$

and

$$f(q: z(1), z(2), \cdots, z(q)) \in U(R) + x\,\cdot$$

Let $\{x(n)\}$ be any sequence of 0's and x's. For each positive integer j define the sequence $\{w(j:n)\}$ to be

$$x(1), x(2), \cdots, x(j), y(j+1), y(j+2), \cdots.$$

Clearly $\{w(j:n)\}$ is in $T(i)$ for each $i=1, 2, \cdots$ and $j=1, 2, \cdots$. Since $\{w(j:n)\}$ converges to $\{x(n)\}$ in the topology of T' it follows that each $T(i)$ is dense in T'.

Let $\{x(n)\}$ be a sequence in $T(i)$ and let p and q be integers greater than i for which

$$f(p: x(1), x(2), \cdots, x(p)) \in U(R)$$

and

$$f(q: x(1), x(2), \cdots, x(q)) \in U(R) + x .$$

We may assume that $q > p$. Now since $\{x(n)\}$ is not eventually constant we may choose integers r, $s > q$ such that $x(r) = x$, $x(s) = 0$. Let S be the subset of T' which corresponds to the open disc of radius exp [max $(r,s) \cdot \log 2$] and with center the real number corresponding to $\{x(n)\}$. Any suquence in S must have the same first q terms as $\{x(n)\}$. But the sequence

$$x(1), x(2), \cdots, x(q), z(q+1), z(q+2), \cdots$$

is in $T(i)$ for any sequence $z(q+1), z(q+2), \cdots$ of 0's and x's. Thus each $T(i)$ is open in T'.

Let $\{z(n)\}$ be any sequence which is in each $T(i)$, $i=1, 2, \cdots$. Then for any i, there are integers $p, q > i$ such that

$$f(p: z(1), z(2), \cdots, z(p)) \in U(R)$$

and

$$f(q: z(1), z(2), \cdots, z(q)) \in U(R) + x .$$

Thus

$$\begin{aligned} &f(q: z(1), z(2), \cdots z(q)) - f(p: z(1), z(2), \cdots, z(p)) \\ &\in U(R) + x - U(R) \subset U(Q) + x + U(R) - x + x \subset U(P) + x. \end{aligned}$$

and hence is not in $U(P)$. Thus $\{z(n)\}$ is not in the convergence field of $\{f(m)\}$. So the part of T' in the convergence field of $\{f(m)\}$ is contained in the union of the complements of $T(i)$. Each $T(i)$ is open and dense in T', so each complement is nowhere dense, and this fact completes the proof of Theorem 4.

COROLLARY. *If G is a non-trivial group and if $\{f(k:m)\}$ is a regular triangular limitation method for each $k = 1, 2, \cdots$, then there is a totally bounded sequence which is not in the convergence field of any of the methods.*

The proof of this corollary is trivial and so is omitted.

Let us now consider a group G and a sequence $\{x(n)\}$ of elements of G. Given any subsequence $\{x(i(n))\}$ of $\{x(n)\}$ we may correspond to it a unique real number in [0, 1] given in the form of the dyadic number

$$a(1)\cdot 2^{-1} + a(2)\cdot 2^{-2} + \cdots,$$

by putting $a(n)=0$ if $x(n)$ is not in $\{x(i(n))\}$ and $a(n)=1$ if $x(n)$ is in $\{x(i(n))\}$. Using the weak topology induced by this correspondence on the class of subsequences of $\{x(n)\}$, we can now state the following theorem.

THEOREM 5. *If $\{x(n)\}$ is a totally bounded sequence of elements of G then $\{x(n)\}$ is Cauchy if and only if there is a regular triangular limitation method on G which transforms a set of subsequences of $\{x(n)\}$ of the second category into Cauchy sequences.*

PROOF. We shall not consider finite sequences but for simplicity let us consider T', the subspace of T obtained by deleting all subsequences which omit only finitely many terms of $\{x(n)\}$. If $\{x(n)\}$ is Cauchy then any regular triangular limitation methodtransforms all subsequences of $\{x(n)\}$ into Cauchy sequences.

If $\{x(n)\}$ is not Cauchy but is totally bounded it has a Cauchy subsequence $\{x(i(n))\}$. The subsequence of $\{x(n)\}$ obtained by deleting the terms of the sequence $\{x(i(n))\}$ is also totally bounded and so has a Cauchy subsequence $\{x(j(n))\}$. We may choose the sequence $\{x(j(n))\}$ so that the subsequence of $\{x(n)\}$ which consists of the terms of $\{x(i(n))\}$ and $\{x(j(n))\}$ and no other is not Cauchy. Let $\{U(i)\}$ be a basic sequence of symmetric neighborhoods of 0 with $U(i+1)\subset U(i)$ for $i=1,2,\cdots$.

Since $\{f(m)\}$ is regular there is an integer $N(1)$ such that if $m,n \geqq N(1)$ then

$$f(m: x(i(1)), x(i(2)), \cdots, x(i(m))) - x(i(n)) \in U(1).$$

Also there is an integer $N(2) > N(1)$ such that if $m,n \geqq N(2)$ and if $M(1)$ is an integer such that $j(M(1)) \geqq i(N(1))$ then

$$f(m: x(i(1)), x(i(2)), \cdots, x(i(N(1))), x(j(M(1)+1)), x(j(M(1)+2)), \\ \cdots, x(j(m))) - x(j(n)) \in U(2).$$

It is clear that we can continue to choose integers $N(2) < N(3) < N(4) < \cdots$ and blocks from $\{x(i(n))\}$ and $\{x(j(n))\}$ alternately such that if p is even then

$$f(m: x(i(1)), x(i(2)), \cdots, x(j(m))) - x(j(n)) \in U(p)$$

for all $m,n \geqq N(p)$ and if p is odd then

$$f(m : x(i(1)), x((i(2)), \cdots, x(i(m))) - x(i(n)) \in U(p)$$

for all $m, n \geqq N(p)$. If we choose $i(M(p)) \geqq j(N(p))$ for each even p and $j(M(p)) \geqq i(N(p))$ for each odd p then

$$x(i(1)), x(i(2)), \cdots, x(i(N(1))), x(j(M(1)+1)), x(j(M(1)+2)), \cdots,$$

$$x(j(N(2))), x(i(M(2)+1)), x(i(M(2)+2)), \cdots$$

is a subsequence of $\{x(n)\}$. It is clear that this subsequence is not in the convergence field of $\{f(m)\}$. Let us call this subsequence $\{x(k(n))\}$ for convenience. Since $\{x(k(n))\}$ is not Cauchy there is an integer P such that for any integer M there are integers $p, q > M$ such that $x(k(p)) - x(k(q)) \notin U(P)$. Pick $Q > P$ such that if $z(1), z(2), \cdots, z(5) \in U(Q)$ then $z(1)+z(2) + \cdots + z(5) \in U(P)$.

For each positive integer i, let $T(i)$ be the set of subsequences, $\{x(l(m))\}$, in T' such that there are integers $p, q > i$ with

$$f(p: x(l(1)), x(l(2)), \cdots, x(l(p))) - f(p: x(i(1)), x(i(2)), \cdots, x(i(p))) \in U(Q)$$

and

$$f(q : x(l(1)), x(l(2)), \cdots, x(l(q))) - f(q : x(j(1)), x(j(2)), \cdots, x(j(q))) \in U(Q).$$

Let $\{x(l(n))\}$ be any subsequence of $\{x(n)\}$ and let N be a positive integer. Similar to our construction of $\{x(k(n))\}$ we may construct of subsequence of $\{x(n)\}$ which begins with $x(l(1))$, $x(l(2)),\cdots$, $x(l(N))$, then continues with alternating blocks from the sequences $\{x(i(n))\}$ and $\{x(j(n))\}$, and is in each $T(i)$. Thus each $T(i)$ is dense in T'.

Let $\{x(l(n))\}$ be any sequence in $T(i)$. Then there are integers $p, q > i$ with

$$f(p: x(l(1)), x(l(2)), \cdots, x(l(p))) - f(p: x(i(1)), x(i(2)), \cdots, x(i(p))) \in U(Q)$$

and

$$f(q : x(l(1)), x(l(2)), \cdots, x(l(q))) - f(q : x(j(1)), x(j(2)), \cdots, x(j(q))) \in U(Q).$$

We may assume that $q > p$. As in the proof of Theorem 4 above we may choose a neighborhood of $\{x(l(n))\}$ such that if $\{x(r(n))\}$ is in this neighborhood then $l(n)=r(n)$ for $n=1,2,\cdots, q$. But the sequence

$$x(l(1)), x(l(2)), \cdots, x(l(q)), x(r(q+1)), x(r(q+2)), \cdots$$

is in $T(i)$ Thus each $T(i)$ is open in T'.

Let $\{x(l(n))\}$ be any subsequence of $\{x(n)\}$ which is in each $T(i)$, $i=1, 2, \cdots$. Let K be a positive integer for which $m, n > K$ implies that

$$x(i(m)) - f(n : x(i(1)), x(i(2)), \cdots, x(i(n))) \in U(Q),$$

and

$$x(j(m)) - f(n : x(j(1)), x(j(2)), \cdots, x(j(n))) \in U(Q).$$

Let $p(1)$ and $q(1)$ be two integers, $p(1), q(1) > K$, for which $x(i(p(1))) - x(j(q(1))) \notin U(P)$. Let $p(2)$ and $q(2)$ be two integers, $p(2), q(2) > K$, for which

$$\begin{aligned} &f(p(2) : x(i(1)), x(i(2)), \cdots, x(i(p(2)))) \\ &\qquad - f(p(2) : x(l(1)), x(l(2)), \cdots, x(l(p(2)))) \in U(Q) \end{aligned}$$

and

$$\begin{aligned} &f(q(2) : x(l(1)), x(l(2)), \cdots, x(l(q(2)))) \\ &\qquad - f(q(2) : x(j(1)), x(j(2)), \cdots, x(j(q(2)))) \in U(Q). \end{aligned}$$

Then

$$\begin{aligned} x(i(p(1))) - x(j(q(1))) &= [x(i(p(1))) - f(p(2) : x(i(1)), x(i(2)), \cdots, x(i(p(2))))] \\ &+ [f(p(2) : x(i(1)), x(i(2)), \cdots, x(i(p(2)))) - f(p(2) : x(l(1)), x(l(2)), \cdots, x(l(p(2))))] \\ &+ [f(p(2) : x(l(1)), x(l(2)), \cdots, x(l(p(2)))) - f(q(2) : x(l(1)), x(l(2)), \cdots, x(l(q(2))))] \\ &+ [f(q(2) : x(l(1)), x(l(2)), \cdots, x(l(q(2)))) - f(q(2) : x(j(1)), x(j(2)), \cdots, x(j(q(2))))] \\ &+ [f(q(2) : x(j(1)), x(j(2)), \cdots, x(j(q(2)))) - x(j(q(1)))] \notin U(P). \end{aligned}$$

Thus the middle bracketed term does not belong to $U(Q)$. Hence $\{x(l(m))\}$ is not in the convergence field of $\{f(m)\}$. So the subsequences in T' which are in the convergence field of $\{f(m)\}$ are in the union of the complements of the dense, open sets $T(i)$. Hence the theorem follows.

This theorem is a generalization of theorems of Buck [2], Keough and Petersen [5], and the author [8, page 134].

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